**MATH 195: 11/24**
**POLYNOMIALS**

**Polynomials.**

A *polynomial* is a function which can be written in the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0,$$

where the *degree* $n$ is a nonnegative integer and each *coefficient* $a_i$ is a number. We assume that the *leading coefficient* $a_n$ is not zero; if it were we simply wouldn't write it.

Here's a couple other ways to think about polynomials.

- A polynomial is a function you can compute just by adding, subtracting and multiplying.
- The *factored form* of a polynomial writes it as a product of terms which can't be simplified even further. For instance,
$$x^4 - 3x^2 - 4 = (x-2)(x+2)(x^2+1)$$
Note that linear roots correspond to *roots* or *zeroes* of the polynomial.

**Theoretical results.**

- The *fundamental theorem of algebra* (circa 1800) says that any polynomial can be factored down to linear (= degree 1) and irreducible quadratic (= degree 2) terms.
- For degree $\leq 4$ polynomials there are formulas to find the roots: the quadratic formula, the cubic formula, and the quartic formula.
- But this is not a computationally feasible process in general. The *Abel–Ruffini theorem* (1813 and 1824) gives a degree 5 polynomial whose roots cannot be written using radicals:
$$x^5 - x - 1.$$
- Évariste Galois (1830) starts a new branch of mathematics to answer the question of when a polynomial is solvable using radicals.

The upshot of all this: we can do a lot given the factored form of a polynomial, but in general it is hard to find.

**Polynomials and calculating other functions.**

Many functions can be expressed as an infinite sum of polynomials—in calculus speak, as a *power series*. Here, $n! = n(n-1)\cdots 2 \cdot 1$ is the *factorial* of $n$.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$
$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} \pm \cdots$$
$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \pm \cdots$$

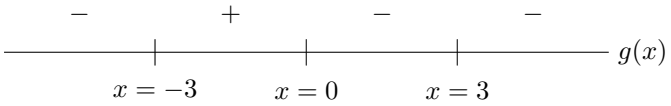
This can be used to approximate these functions. If you compute the polynomial consisting of the first several terms, you get a good approximation. (How good of an approximation? A result in calculus, known as *Taylor's theorem*, tells you a bound for how bad the error can be. But it's beyond the scope of this class to look at it.)

**Sign diagrams and graphing polynomials.**

In general, this process requires calculus. That is how we can pinpoint where the maximums and minimums are. But without calculus we can still determine where a polynomial is positive or negative, and from that produce a rough sketch.

- The key observation is, the only place a function change sign is across a zero, a hole in the graph, or an asymptote.
- So we need only to find the zeros, holes, and asymptotes, and determine the signs in between those.
- You can do this for any function by checking the sign for a single input in the interval.
- For polynomials, there's a shortcut available.
  - There are only zeroes, no holes nor asymptotes.
  - The signs of the outside two regions is determined by the end behavior, which is determined by the *leading term* (= the term with the largest power).
  - You determine whether the sign flips across a zero by looking at the *multiplicity* of the zero (= the power of the corresponding term). Even means keep the same sign, odd means flip the sign.

A *sign diagram* is a diagram to summarize where a function is positive versus negative. To sketch a graph from a sign diagram, determine the end behavior and draw the function above/below the $x$-axis as appropriate between the zeroes.




A sign diagram for $g(x) = -2(x+3)x^3(x-3)^2$.

The leading term is $-2x^6$ so the two outside regions are negative, since $g(x)$ goes off to $-\infty$ on both sides.

**Inequalities.**

Sign diagrams can also be used to solve inequalities. To solve, for instance, $f(x) > 0$ you look at the sign diagram to see where $f(x)$ is positive, and then write it in interval notation.

## Exploratory exercises

For some of these, use a graphing calculator such as the one found at `desmos.com/calculator`.

(1) Graph $e^x$. Compare to the graphs you get of the polynomials you get by taking the first few terms of the power series for $e^x$. What do you observe as you add more terms to your polynomial?
(2) Do the same for $\sin x$.
(3) And $\cos x$.
(4) Use the approximation $e^x \approx 1 + x + \frac{x^2}{2}$ to approximate $e^{0.1}$. That is, compute $1 + 0.1 + \frac{0.1^2}{2}$ and compare it to what your calculator gives for $e^{0.1}$. Do the same at $x = 0.01$ and $x = 1$. What do you notice?
(5) Use the approximation $\sin(x) \approx x - \frac{x^3}{6}$ to approximate $\sin(0.1)$ and compare to what your calculator gives. How would you approximate $\sin(2\pi + 0.1)$? Explain.

## Graphing exercises

For these polynomials, determine the leading term and the roots and their multiplicities, use that to create a sign diagram, then use the sign diagram to sketch a graph. *[Warning! Some of these are not fully factored.]*

(1) $a(x) = x(x-1)(x-2)(x-4)$
(2) $b(x) = x^2(x-1)^2(x-2)^2(x-4)^2$
(3) $c(x) = x(x-1)^2(x-2)^2(x-4)$
(4) $d(x) = (x^2-1)^2(x^2+4)x^3$
(5) $f(x) = (x^2-3x+2)(x^2-2x+3)$
(6) $g(x) = (x-1)(x+2)^2(x-3)^3(x+4)^4(x-5)^5(x+6)^6(x-7)^7(x+8)^8$
(7) $h(x) = -2x^3(x-2)(x+2)^2$
(8) $j(x) = (4x+1)(1-2x)^2(6-x)^3$
(9) $k(x) = (x^2+1)(x^2+4)(x^2+9)$
(10) $\ell(x) = 3(x-4)^3(x+2)^3(x+7)(x-1)^2$

## Inequality exercises

(1) Solve the inequality $x^2(x-2)(x+2) > 0$.
(2) Solve the inequality $x^2(x-2)(x+2) \leq 0$.
(3) Solve the inequality $x^2 - 4x + 12 < 0$.
(4) Solve the inequality $e^x(x^2-9) \geq 0$. [Hint: first find the zeroes, then for each interval check a single point to determine whether it's positive or negative in the region.]
(5) What is the domain of $m(x) = \sqrt{(2x-1)(x+3)^2(x+5)}$? [Hint: inputs to square roots must be $\geq 0$.]
(6) What is the domain of $n(x) = \ln(x^3 - 4x)$?
(7) What is the domain of $p(x) = \sqrt[3]{e^x(x^4 - 4x^2)}$?