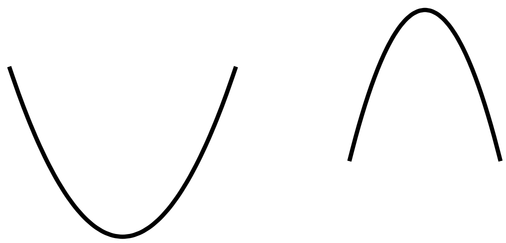
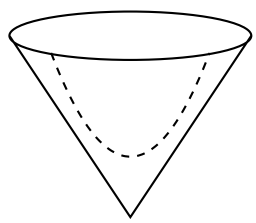
MATH 195: 9/10
PARABOLAS AND QUADRATIC FUNCTIONS

There are multiple lens from which to view *parabolas* and *quadratic functions*.

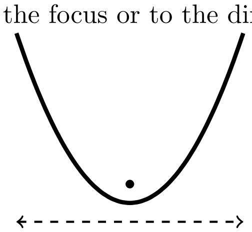
Geometric. A parabola is an arch-shaped curve.



A parabola is a *conic section*: cut a diagonal slice along a cone parallel to its slope, the curve on the edge of the cut is a parabola.



A parabola is determined by a *focus* point and a *directrix* line: the parabola is the curve of all points with the same distance to the focus or to the directrix.

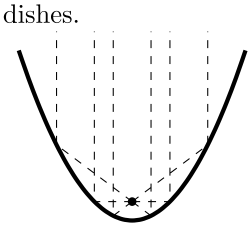


(1) Which of these geometric descriptions do you find most useful?
(2) Draw lines from the parabola to its focus and directrix and convince yourself that the distances to the focus and the directrix are the same.

Physical. An object thrown diagonally through the air, assuming no friction, traces a downward-facing parabola. The height of an object thrown vertically plotted over time is a downwarf-facing parabola, again assuming no friction.

(1) Toss (gently) an object and try to convince yourself it traces a parabola.
(2) Are there objects you can throw whose trajectories are obvious not parabolas?
(3) What does the *vertex*, the maximum or minimum point on the parabola, mean in this context?
(4) If you've taken some physics, what do you think explains why this shape is a parabola?

A light source at the focus of a parabolic mirror gets reflected out all in the same parallel direction. This property of parabolas is used for car headlights. The opposite direction—parallel rays in get reflected to the focus—is used for satellite dishes.



(1) Can you think of any other possible applications for the reflective property of parabolas?
(2) Are there any physical phenomena that have the shape of a parabola?

Algebraic. A *quadratic function* is a function whose graph is a parabola. The basic quadratic function is $f(x) = x^2$.

There are multiple ways to write a quadratic function.

Standard form

$f(x) = ax^2 + bx + c \qquad (a \neq 0)$

Vertex form

$f(x) = a(x-h)^2 + k \qquad (a \neq 0)$

Has *vertex* at (h, k).

Factored form

$f(x) = a(x-r)(x-s) \qquad (a \neq 0)$

Has *zeroes* (= *roots*) at $x = r, s$.

Translating between algebraic representations

Factored form to standard form OR vertex form to standard form. Distribute the multiplications and simplify.

(1) Rewrite $a(x) = (x-2)^2+3$ in standard form.
(2) Rewrite $b(x) = -2(x-1)(x+2)$ in standard form.

Standard form to factored form. Either factor or use the quadratic formula. The zeroes given by the quadratic formula are the zeroes in the factored form. And a is the same number in either form.

(1) Use factoring to rewrite $c(x) = x^2 - 6x + 8$ in factored form.
(2) Write down the quadratic formula.
(3) Use the quadratic formula to rewrite $d(x) = x^2 + 2x - 5$ in factored form. *Hint: expect to see square roots!*
(4) Use the quadratic formula to explain why some quadratic functions can't be written in factored form.
(5) Can you give an alternate explanation for this based on graphs?

Standard form to vertex form. Complete the square to do this translation.

Remember: think of $x^2 + Ax$ as two thirds of the perfect square

$$x^2 + Ax + B^2 = (x + B)^2.$$

You need to determine what value for B makes this work. Work backwards from the pattern of the square of a sum: $(x + B)^2 = x^2 + 2Bx + B^2$. Now solve for B.

Once you know B, you need to add in the missing B^2 to complete the square. To not change the function you have to also subtract B^2 and keep it balanced. Then you collapse x^2+Ax+B^2 to the perfect square $(x+B)^2$, and simplify what remains.

(1) Rewrite $f(x) = x^2 + 4x$ in vertex form.
(2) Rewrite $g(x) = x^2 - 3x + 1$ in vertex form.
(3) Rewrite $h(x) = 2x^2 - 8x + 3$ in vertex form.
(4) If you have time after completing the other problems, come back to this question. Complete the square with the equation $0 = ax^2 + bx + c$ to solve for its zeroes. Can you write what you get so that it looks exactly like the quadratic formula? *Hint: yes, you can. This is where the quadratic formula comes from.*

Algebraic representations to geometric

For any form, the *sign* of a tells you the orientation: positive means face upward, negative means face downward. The main remaining info to determine is the location of the vertex.

Vertex form. This form directly tells you the vertex. Plot the vertex at (h, k). Then draw the parabola from that vertex according to its orientation.

Factored form. The vertex always has x-coordinate halfway between the zeroes. To find the y-coordinate plug in the x-coordinate into the function. Then you can plot the vertex and draw the parabola according to its orientation.

Standard form. The vertex is always at $x = -\frac{b}{2a}$. To find the y-coordinate plug the x-coordinate into the function. Then plot the vertex and draw the parabola according to its orientation.

(1) Sketch a graph of $k(x) = (x - 2)^2 + 3$.
(2) Sketch a graph of $\ell(x) = -3(x + 1)^2 + 1$. Use the graph to determine its domain and range.
(3) Sketch a graph of $m(x) = -2(x - 3)(x + 1)$.
(4) Sketch a graph of $n(x) = 3(x + 4)^2$. Use the graph to determine its domain and range.
(5) Sketch a graph of $p(x) = x^2 - 4x + 3$.
(6) Sketch a graph of $q(x) = 2x^2 + 6x - 3$. Use the graph to determine its domain and range. Where is $q(x)$ decreasing? Increasing?
(7) Generalize what you've seen and come up with a general rule for the domain of a quadratic function.
(8) Your friend claims a quadratic function can have $\mathbb{R}$, the entire real line, as its range. Are they correct? If yes, provide an example. If no, explain why not.
(9) Can you determine the range of a quadratic function in vertex form without first drawing its graph? Explain your process.
(10) Can you determine where a quadratic function in vertex form is increasing/decreasing without first drawing its graph? Explain your process.