MATH 195: 4/29 WORKSHEET

Poles and multiplicity.

- The *multiplicity* of a pole of a rational function is the multiplicity of the corresponding zero in the denominator.
- If $f(x)$ has a pole at $x = a$ corresponding to the term $(x-a)^n$ in the denominator, then the graph of $f(x)$ locally looks like $\pm\frac{1}{(x-a)^n}$ near the pole.
- That is, if n is odd then the sign changes across the asymptote, and if n is even then the sign stays the same across the asymptote.

Holes in rational functions.

- A rational function $f(x)$ has a hole at $x = a$ if the term $x - a$ appears in both the numerator and denominator. At this point, the function is undefined.
- The graph of $f(x)$ looks like the graph of what you get by cancelling out like terms from the numerator and denominator, except with the holes.
- If the multiplicity at a is the same in both the numerator and denominator, then the hole appears at a height $\neq 0$. Thus the sign is the same on both sides of the hole.
- If the multiplicity at a is larger in the numerator than the denominator, the hole appears at height 0. Use the cancelled form to determine whether sign changes across this hole at 0.
- If the multiplicity at a is larger in the denominator than the numerator, there is an asymptote at $x = a$.

Graphing rational functions.

Putting everything together to graph $f(x) = \frac{n(x)}{d(x)}$.

- The end behavior of $f(x)$ is determined by the fraction of the leading terms.
- Identify the location of all holes, then cancel out like terms.
- Create a sign diagram. The sign can only change at zeros, poles, or holes of height 0. Use the rule of multiplicities to fill out the sign diagram.
  - The end behavior lets you determine the sign in the outermost regions.
- To sketch a graph, locate all zeroes and multiplicities. Use the end behavior and the sign diagram to sketch a graph. Then go back and add in the holes at the proper locations.

Warning! Graphing calculators often fail to properly show holes. Be careful when using them.

Practice Problems

Make sign diagrams for and sketch graphs of the following rational functions.

(1) $a(x) = \dfrac{x^3(x-1)^2}{3(x-2)^2(x+2)^2(x-4)}$

(2) $b(x) = \dfrac{x-1}{x^2(x-4)}$

(3) $c(x) = -2 \cdot \dfrac{(x-2)^2(x+3)^2}{(x+1)^3}$

(4) $d(x) = \dfrac{(x+1)(x+2)^2(x+3)^3}{(x+1)^4(x+2)^2(x+3)}$

(5) $f(x) = \dfrac{-3(x+2)^2(x-2)^3}{-2(x^2-4)^2}$

(6) Determine the domain of the following function.

$$h(x) = \ln\left(\frac{x^2(x-2)(x+2)^2}{(x-2)(x+2)(x-4)^2}\right)$$

(7) Determine the domain of the following function. [Hint: at holes and poles a rational function is undefined, and so shouldn't be included for where it is ≥ 0.]

$$g(x) = \sqrt{\frac{x^2(x-2)(x+2)^2}{(x-2)(x+2)(x-4)^2}}$$