MATH 195: 4/22 WORKSHEET

Sigma notation.

One often finds oneself adding up lots of things, and it's convenient to have a shorthand for this.

$$\sum_{i=1}^{n} f(i) = f(1) + f(2) + \cdots + f(n)$$

The symbol $\sum$ is the Greek letter Sigma. Think Sigma = S = Sum. The variable i is the *index variable*. The *upper limit* n is the largest value for the index i. The *lower limit* here is 1, but it can be any integer $\leq n$.

This notation is also used for *series*, also called *infinite sums*.

$$\sum_{i=1}^{\infty} f(i) = f(1) + f(2) + \cdots + f(i) + f(i+1) + \cdots$$

Making precise what this means and when it can be sensibly be said to have a value requires ideas from calculus. Briefly, $\sum_{i=1}^{\infty} f(i) = S$ if the partial sums $\sum_{i=1}^{n} f(i)$ get asymptotically closer and closer to S as n goes to ∞. When this happens, we say the series *converges*, otherwise it *diverges*.

Geometric series.

A *geometric series* is one of the form

$$\sum_{i=0}^{\infty} r^i = 1 + r + r^2 + r^3 + \cdots$$

Here r is called the *radix* of the series.

- A geometric series converges if and only if $|r| < 1$.
- When this happens, $\sum_{i=0}^{\infty} r^i = \frac{1}{1-r}$.

(1) Calculate $\sum_{i=0}^{\infty} \frac{1}{3^i}$.

(2) Calculate $\sum_{i=1}^{\infty} \frac{2^i}{3^i}$. [Hint: $\sum_{i=1}^{\infty} f(i) = (\sum_{i=0}^{\infty} f(i)) - f(0)$ (why?)]

(3) Calculate $\sum_{i=0}^{\infty} \frac{2}{3^i}$. [Hint: factor out the 2 from the sum.]

(4) Writing a number as a decimal expansion is a shorthand for an infinite sum. For example,

$$0.333\ldots = \frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \cdots + \frac{3}{10^i} + \cdots = \sum_{i=1}^{\infty} \frac{3}{10^i}.$$

Find the value of this sum.

(5) Write $0.9999\ldots$ (i.e. all 9s repeating) as an infinite sum, then use what you know about geometric series to calculate its value.

Power series.

Many important functions can be defined using series. A particularly useful class is series of polynomials, called *power series* or *Taylor series*. Namely, a power series (centered at 0) is of the form

$$f(x) = \sum_{i=0}^{\infty} a_i x^i,$$

where the coefficients a_i are numbers.

- $e^x = \sum_{i=0}^{\infty} \frac{x^i}{i!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$
- $\cos(x) = \sum_{i=0}^{\infty} (-1)^i \frac{x^{2i}}{(2i)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} \mp \cdots$
- $\sin(x) = \sum_{i=0}^{\infty} (-1)^i \frac{x^{2i+1}}{(2i+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} \mp \cdots$

Here, $i! = i(i-1)(i-2)\cdots 1$ is the product from 1 to i and $0! =!$.

Taylor polynomials.

If a function can be represented as a Taylor series, then it can be approximated with *Taylor polynomials*—those polynomials obtained by looking at the first n terms of the series. This means these functions can be approximately calculated just using addition, multiplication, and subtraction.

Use a graphing calculator such as the one at desmos.com for the graphing questions.

(6) Graph e^x. Compare to the graphs you get of the Taylor polynomials for e^x by adding more and more terms of the series. What do you observe?

(7) Do the same for $\sin x$.

(8) And $\cos x$.

(9) Use the approximation $e^x \approx 1 + x + \frac{x^2}{2}$ to approximate $e^{0.1}$. Compare to what a calculator gives you.

(10) Approximate e^1. Compare to what a calculator gives you. [Hint: plugging in 1 to the Taylor polynomial will give a quite bad approximation—think of what you saw with the graphs. Instead, use your approximation for $e^{0.1}$ plus the rule of exponentiation $a^{xy} = (a^x)^y$.]

(11) Approximate e^3. Compare to what a calculator gives you.

(12) Use the approximation $\sin x \approx x - \frac{x^3}{6}$ to approximate $\sin(0.1)$. Compare to what a calculator gives you for $\sin(0.1)$.

(13) Approximate $\sin(2\pi + 0.1)$. Compare to what a calculator gives you. Can you get an equally accurate approximation as the last one? [Hint: yes. Think about what you know about trig functions.]

(14) Approximate $\sin(\pi - 0.1)$. Compare to what a calculator gives you.

(15) Approximate $\sin(\pi + 0.1)$. Compare to what a calculator gives you.