MATH 195: 4/8 WORKSHEET

Sign diagrams and other functions.

Sign diagrams can be used for any function,[a] not just polynomials. The key point is, the sign can only change at a zero of the function or at a *discontinuity*—a point where there is a hole, jump, or asymptote on the graph.

To make a sign diagram for a function $f(x)$:

- Find all zeroes and discontinuities.
- Mark the zeroes and discontinuities on a number line, dividing it into regions.
- For each region, determine whether $f(x)$ is positive, negative, or undefined there.
  - Sometimes, like with polynomials, there's a property or rule you can use to determine the signs.
  - If you know the shape of the graph that can inform your answer.
  - If no other option is available, you can always plug in a single input from the region and checking its sign.

[a] "Any" needs an asterisk. Mathematicians will sometimes consider weird functions, such as this one:

$$\chi_{\mathbb{Q}}(x) = \begin{cases} 1 & \text{if } x \text{ is a rational number} \\ 0 & \text{if } x \text{ is an irrational number.} \end{cases}$$

This function is discontinuous everywhere and so you can't divide its domain into intervals where it is positive versus negative. But the functions considered by non-perverts will be continuous almost everywhere and so all is good.

Sign diagrams, inequalities, and domains.

One use of sign diagrams is solving inequalities. A sign diagram for $f(x)$ lets you solve inequalities like $f(x) \geq 0$ or $f(x) < 0$.

- For a strict inequality $<$ or $>$ you only want the interior of the regions.
- For a nonstrict inequality $\leq$ or $\geq$ you want to include the endpoints, but only if they are zeroes. If there is a gap in the $\operatorname{dom} f$ it cannot be part of a solution to an inequality.
- In general the answer will be a union of intervals, not a single interval. We write e.g. $(1, 3) \cup [2, \infty)$ to represent a union of disjoint intervals.

One application is determining domains. For example, the domain of $f(x) = \sqrt{g(x)}$ is the solution to the inequality $g(x) \geq 0$.

Practice Problems

(1) Create a sign diagram for $4 - 2^x$, using what you know about graphs of exponentials to determine the signs in the regions. Use this sign diagram to find the domain of $a(x) = \sqrt{4 - 2^x}$.

(2) Create a sign diagram for $x^2 - 4x - 12$. Either use the methods for polynomials or use what you know about the graphs of quadratics.

(3) Create a sign diagram for $x^2 + 2x$. Use that to determine the domain of $b(x) = \ln(x^2 + 2x)$. Knowing the domain, then create a sign diagram for $b(x)$.

(4) Go yet one level deeper. Use the work from the previous problem to determine the domain of $c(x) = \sqrt{b(x)}$.

(5) Create a sign diagram for $d(x) = x^3 - 9x$. Then create a sign diagram for $d'(x) = 3x^2 - 9$ and $d''(x) = 6x$.

(6) Bringing in a little calculus, the function $d'(x)$ says where $d(x)$ is increasing or decreasing. Where $d'(x)$ is positive $d(x)$ is increasing and where $d'(x)$ is negative $d(x)$ is decreasing. Similarly, $d''(x)$ has the info of $d(x)$'s concavity: where $d''(x)$ is positive $d(x)$ is concave up and where $d''(x)$ is negative $d(x)$ is concave down. Use your sign diagrams from the previous problem to determine where $d(x)$'s zeroes, maximums, minimums, and inflection points are. Use this fuller information to sketch a better graph of $d(x)$.

We call $d'(x)$ the *(first) derivative* of $d(x)$ and $d''(x)$ the *second derivative* of $d(x)$. In general, the first derivative of a function tells you about where it's increasing or decreasing and the second tells you about its concavity. One of the main tasks of calculus I is learning how to compute derivatives.

(7) Create a sign diagram for $f(x) = e^x(x^3 - 3x^2)$. [Hint: completely factor the polynomial to write $f(x)$ as a product of simple functions. One of the multiplicands in e^x, but you know it's always positive so it won't contribute any roots.]

(8) What is the domain of $g(x) = \sqrt{x^3 - x} - \ln(16 - \sqrt{x})$. [Hint: separately determine the domains of $\sqrt{x^3 - x}$ and $\ln(16 - \sqrt{x})$. The domain of $g(x)$ is then the *intersection* or overlap of those two domains.]