MATH 195: 2/4 WORKSHEET

One-to-one.

A function $f(x)$ is *one-to-one* (synonymously, *injective*) if each output comes from exactly one input. Said another way, one-to-one means different inputs go to different outputs. You can write this in symbols:

$$\text{if } f(x_1) = f(x_2) \text{ then } x_1 = x_2$$
$$\text{if } x_1 \neq x_2 \text{ then } f(x_1) \neq f(x_2)$$

Geometrically, being one-to-one means passing the *horizontal line test*: any horizontal line crosses the graph in at most one point.

Inverses of functions.

If $f(x)$ is one-to-one then you can define its *inverse function* $f^{-1}(y)$ as $f^{-1}(y) = x$ if and only if $f(x) = y$. The reason $f^{-1}(y)$ is a function—each input goes to exactly one output—is because $f(x)$ passes the horizontal line test.

- To find a formula for $f^{-1}(y)$, solve the equation $f(x) = y$ for x, this then gives a formula for $f^{-1}(y)$. (And if you want to call the input x you can then swap out the variables at the end.)
- Graphically, the graph of $f^{-1}(x)$ is obtained by reflecting the graph of $f(x)$ across the diagonal line $y = x$.
- If $f(x)$ is not one-to-one, you might be able to restrict its domain down to make it one-to-one, and then you can take the inverse.

Radical functions

If n is a positive integer, then $\sqrt[n]{x}$ is the inverse of x^n.

- When n is odd, x^n is one-to-one and so this can be done directly.
- When n is even, x^n is not one-to-one. To make it one-to-one restrict its domain to $[0, \infty)$ and then take the inverse.

A *radical function* is one of the form $f(x) = \sqrt[n]{x}$. We will sometimes use *radical function* to refer more generally to any function which is a geometric transformation of some $\sqrt[n]{x}$.

Desmos exercises

For these use the desmos graphing calculator `https://desmos.com/calculator` or another graphing calculator.

(1) In desmos you can graph the inverse of $f(x)$ without having a formula for $y = f^{-1}(x)$ by plotting `x = f(y)`. Plot a few different functions and their inverses. What happens if $f(x)$ isn't one-to-one?

(2) Based on your observations from the previous problem, explain why the horizontal line test for $f(x)$ guarantees that $f^{-1}(x)$ is a function.

(3) To restrict the domain for a function in desmos, the syntax is like `y = x^2 { x >= 0 }`. Plot the following functions with the following domains.

$$\begin{aligned} a(x) &= x^2, \text{ where } x \geq 0 \\ b(x) &= \frac{1}{x}, \text{ where } x < -2 \\ c(x) &= 2^x, \text{ where } 2^x < 4 \\ d(x) &= 2 + 4x - 5x^2, \text{ where } 0 \leq x \leq 1 \end{aligned}$$

Looking at their graphs, which of these are one-to-one on that domain?

(4) For the following functions, plot their inverses using desmos as in problem 1. Use this information to help you find formulas for their inverses and the domains of their inverses. [In desmos, type `sqrt` to get the square root symbol $\sqrt{\bigcirc}$ and type `nthroot` to get an nth root $\sqrt[\bigcirc]{\bigcirc}$.]

$$\begin{aligned} a(x) &= \sqrt{x} \\ b(x) &= \sqrt{x-2} + 3 \\ c(x) &= -\sqrt[4]{x+1} - 2 \end{aligned}$$

(5) Compare the domains and ranges of the functions and their inverses from their previous problems. Formulate a hypothesis about how dom f and ran f^{-1} relate, and how ran f and dom f^{-1} relate. Explain why you think your hypothesis is true.

(6) Plot the following functions and their inverses, as in problem 1.

$$\begin{aligned} a(x) &= x^3 \\ b(x) &= 2^x \\ c(x) &= x^4, \text{ where } x \geq 0 \end{aligned}$$

Based on your observations of these graphs, form a hypothesis about whether the inverse of an increasing function is increasing or decreasing. Explain why you think your hypothesis is true.

(7) Make a similar hypothesis about the inverse of a decreasing function, plotting some decreasing functions and their inverses to check it.

(8) Make a similar hypothesis about concavity, plotting some functions and their inverses to check.