MATH 130: 4/2 WORKSHEET
PROBABILITY: BAYES'S THEOREM

Last time we talked about *conditional probability*. $P(A \mid B)$ is the likelihood that A happened given the info that B happened.

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

You might wonder how $P(A \mid B)$ and $P(B \mid A)$ are related.

Bayes's theorem.
Bayes's theorem says how the two directions of conditional probability are related.

$$P(A \mid B) = \frac{P(B \mid A) \cdot P(A)}{P(B)}$$

Law of total probability.
If A and B are events and $\bar{B}$ is the complementary event "B doesn't happen", then

$$\begin{aligned} P(A) &= P(A \cap B) + P(A \cap \bar{B}) \\ &= P(A \mid B) \cdot P(B) + P(A \mid \bar{B}) \cdot P(\bar{B}) \end{aligned}$$

Example: your doctor runs a test that says you have a high chance of having a rare disease. How worried should you be? Let D be the event "you have disease X" and T be the event "the test came back positive".

- The probability a random person has disease X is $P(D) = 0.001$.
- The *true positive rate* for the test is 99%. That is, $P(T \mid D) = 0.99$.
- The *false positive rate* for the test is 1%. That is, $P(T \mid \bar{D}) = 0.01$.
- The missing datum needed is $P(T)$, the probability that if you test is positive on a random person. This can be calculated used the law of total probability:

$$\begin{aligned} P(T) &= P(T \mid D) \cdot P(D) + P(T \mid \bar{D}) \cdot P(\bar{D}) \\ &= 0.99 \cdot 0.001 + 0.01 \cdot 0.999 \\ &= 0.01098. \end{aligned}$$

- Finally we put these together using Bayes's theorem:

$$\begin{aligned} P(D \mid T) &= \frac{P(T \mid D) \cdot P(D)}{P(T)} \\ &= \frac{0.99 \cdot 0.001}{0.01098} \\ &\approx 0.09 \end{aligned}$$

The conclusion is, you only have about a 9% chance of having disease X!

Probability squares.

Here is a device for organizing the data used with Bayes's theorem. A 2×2 table represents all possible combinations of whether two events A and B happen.

	A	$\bar{A}$
B	20%	40%
$\bar{B}$	30%	10%

This table summarizes the following data:

$$P(A \cap B) = 20\% \quad P(\bar{A} \cap B) = 40\% \quad P(A \cap \bar{B}) = 30\% \quad P(\bar{A} \cap \bar{B}) = 10\%$$

From this you can also determine $P(A) = 50\%$ and $P(B) = 60\%$, as well as the conditional probabilities. For example,

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{.2}{.6} = \frac{1}{3}.$$

Practice Problems

(1) Suppose A and B are events and you know $P(A) = 0.4$, $P(B) = 0.7$, and $P(B \mid A) = 0.3$. Calculate $P(A \mid B)$ using Bayes's theorem.

(2) Two companies supply blankets to an emergency relief organization. Company A supplies $3,000$ blankets, and 4% of them are of unusable quality. Company B supplies $4,000$ blankets, and 6% of them are of unusable quality. If a randomly grabbed blanket is of unusable quality what is the probability it came from Company B?

(3) Two unlabeled bags are filled with marbles. Bag A has 13 red marbles and 12 blue marbles and Bag B has 5 red marbles and 20 blue marbles. You grab a bag at random and pull out a blue marble. What is the probability you grabbed bag A?

(4) Your classmate walks to school 20% of the time and bikes the other 40%. She is late 10% of the time when she walks and 5% of the time when she bikes. (a) If she showed up late today, how likely is it that she walked? (b) If she showed up on time today, how likely is it that she biked?