MATH 130: 3/31 WORKSHEET
PROBABILITY: CONDITIONAL PROBABILITY

Being told information can affect the calculation of probabilities. For example, if I secretly roll a die there's a $\frac{1}{6}$ chance the outcome is a 6. But if I tell you that I rolled an even number then you should adjust those odds to a $\frac{1}{3}$ chance.

Conditional probability.
Conditional probability measures how likely an event B is if you know that event A happened. We write $P(B \mid A)$ to express the probability of B knowing that A happened, and we calculate

$$P(B \mid A) = \frac{P(B \cap A)}{P(A)}.$$

Think: $P(A)$ is the overall likelihood of A, and we want to know what proportion of that has B also happen. That proportion should be the conditional probability.

Example: you roll two dice and want to know whether the sum is at least 8. If you see that your first roll is 4, how likely is it that the sum is at least 8?

Conditional probability and independence.
Remember that two events are *independent* if, intuitively, knowing one happens gives you no info about whether the other happens. Formally, this means $P(A \cap B) = P(A) \cdot P(B)$.

Using the idea of conditional probability, we can phrase this in a new way: A and B are independent if

$$P(B \mid A) = P(B).$$

That is, knowing that A happened doesn't change the likelihood of whether B happened.

Example: you flip two coins. The events A = "the first coin is heads" and B = "the second coin is heads" are independent because knowing that one coin is heads doesn't affect the outcome of the other.

Conditional probability and disjoint events.
Remember that two events A and B are disjoint if there are no outcomes common to both events. In symbols: $A \cap B = \emptyset$. If two events are disjoint then their conditional probabilities are always 0:

$$P(A \mid B) = P(B \mid A) = 0.$$

Example: you roll a die. If you know that you rolled an even number, then you know there is 0% chance you rolled an odd number.

Practice Problems

(1) Consider the random trial of rolling one six-sided die. Let A be the event "the result is even" and B be the event "the result is odd". Calculate $P(A \mid B)$.

(2) Consider the same trial. Now let A be the event "the result is even" and B be the event "the result is at least 5". Calculate $P(A \mid B)$, and use this as part of explaining whether or not A and B are independent.

(3) Consider the random trial of rolling two six-sided dice. Let A be the event "the sum is even" and B be the event "the first die rolled 5". Calculate $P(A \mid B)$.

(4) Consider the same trial. Now let A be the event "the sum is at least 5" and B be the event "the first die rolled a 1". Calculate $P(A \mid B)$.

(5) Sometimes $P(A \mid B)$ and $P(B \mid A)$ are the same. Calculate these conditional probabilities for the following events to see an example of this. For the random trial of flipping two coins, A is the event "the two coins had different results" and B is the event "the first coin came up heads".

(6) Sometimes $P(A \mid B)$ and $P(B \mid A)$ are different. Calculate these conditional probabilities for the following events to see an example of this. For the random trial of flipping two coins, A is the event "both coins came up heads" and B is the event "the first coin came up heads".

(7) Consider the random trial of flipping two coins. Let A be the event "the first coin came up heads" and B be the event "at least one coin came up heads". Do you expect $P(A)$ and $P(A \mid B)$ to be the same or different? Confirm your guess by calculating them.

(8) Consider this variant of the Monty Hall problem. The host knows which of ten doors has a car behind it. You as contestant pick a door, with a $\frac{1}{10}$ chance of selecting the car. The host then opens eight of the remaining nine doors—careful not to pick the one she knows has the car—to reveal they are empty, and gives you the option to switch your choice to the other unopened door. Calculate the probability you get the car if you switch.