MATH 130: 3/24 WORKSHEET
PROBABILITY: AVERAGES

Today we study one of the most important concepts in probability, namely the notion of an average. An *average* is a measure of the "expected" or "center" of a random measurement.

Warning! Everything today is about uniform probability distributions. These concepts make sense in a more general context but the calculations are more intricate.

Random variables.
A *random variable* is any way of associating an output to ecah possible outcome of a sample space. Typically, the outputs are numbers, though in some circumstances you want random variables which aren't numerical. We use captial letters $X, Y, \ldots$ to refer to random variables.

Examples:

- If you flip a coin 20 times and count the number of heads that is a random variable.
- If you roll two dice and add the results that is a random variable.
- If you draw a marble out of a bag and record its color that is a random variable.

Many events can be described using random variables. For example, if you roll two dice and want to know whether the sum is 7 you can represent that as $P(X = 7)$ where X is the random variable representing the sum.

Intuitively, a random variable is something you can measure from a random trial. With these measurements you want to have a way to describe what you expect to see. Different notions of an average give different ways to do this.

Visualizing random variables.
One way to visualize a random variable is via a *bar chart*. For each possible value of X, plot a bar with height equal to the number of outcomes which give that value.

Mode.
The *mode* of a random variable X is the value that occurs most often. In symbols, it is the value m so that $P(X = m)$ is as large as possible. It is possible for a random variable to have multiple modes.

- To calculate a mode, list out all possible values of X for all outcomes and count which value(s) is most common.

This notion of average makes sense for random variables taking on any kind of value, numerical or otherwise.

Example: if X is the sum of two random dice rolls then the mode of X is 7.

Median.

The *median* of a random variable X is a value m so that it's equally likely that X is larger or smaller than X.

- To calculate a median , list out in order all possible values for X and look at the value at the middle position. If there are an even number of possible values, then there is no middle position; by convention take the average on the two closest.

This notion of average only makes sense when the values can be ordered.

Example: if X is the sum of two random dice rolls then the median of X is 7.

Mean or expected value.

The *mean*, also called the expected value, is a notion of average which has many nice mathematical properties. As such, it has special importance. The expected value of a random variable is denoted $E[X]$.

- To calculate a mean, add up all possible values for X then divide by the number of outcomes.

This notion of average only makes sense for random variables which take numerical values.

Example: if X is the sum of two random dice rolls then the mean of X is 7.

The additivity of the mean.

If X and Y are random variables taking on numerical value then $E[X+Y] = E[X] + E[Y]$.

Example: if X is the sum of one random dice roll then $E[X] = 3.5$. To calculate the expected value of rolling three random dice, calculate $E[X + X + X] = 3E[X] = 10.5$.

Practice Problems

(1) Suppose the possible values of a random variable X are

$$0, 1, 1, 2, 3, 5, 8, 13.$$

Determine the mean, median, and mode of X.

(2) Let X be the random variable representing the sum of rolling two 6-sided dice. Draw a bar graph representing the values of X.

(3) Draw a bar graph representing the values of a random variable X whose mean, median, and mode are all different. Explain why they are different.

(4) Suppose you flip a fair coin 40 times. Calculate the expected value of the number of heads. [Hint: it for one flip then use the additivity of the mean.]

(5) Suppose you roll a 6-sided die 40 times. Calculated the expected value of the number of 1s. [Hint: calculate it for one roll, then use additivity of the mean.]

(6) Suppose you roll a 6-sided die once. Determine the expected value of the roll. Do the same for an 8-sided, 10-sided, and 12-sided die.

(7) Suppose you roll a 6-sided die 8 times and sum up the results. Determine the expected value of this sum. [Hint: same hint as before.]