# MATH 130: 3/19 WORKSHEET
# PROBABILITY: COMBINING EVENTS

*Probability* is the mathematics of quantifying uncertainty. Today we continue last class's talk of events to look at multiple events simultaneously.

**Independence.**

Intuitively, two events are *independent* if they don't affect each other. For example, if you roll two dice, the events "the first die rolls 6" and "the second die rolls 6" should be independent. One die's roll shouldn't affect the other's.

Giving a mathematical definition of independence is harder. We will revisit this later when we have more concepts to justify it. Events $E$ and $F$ are *independent* if the probability they both occur is the product of their probabilities. In symbols:

$$P(E \cap F) = P(E) \cdot P(F).$$

If two events are not independent we call them *dependent*.

- Let's justify the intuition that die rolls are independent. Let $E$ be the event "the first die rolls 6" and $F$ be the event "the second die rolls 6". Then $P(E) = P(F) = \frac{6}{36} = \frac{1}{6}$. This is because there are six outcomes where the first die rolls 6, namely the outcomes
$$(6,1), \quad (6,2), \quad (6,3), \quad (6,4), \quad (6,5), \quad (6,6)$$
Also, $P(E \cap F) = \frac{1}{36}$, since there is only one outcome with both sixes. And $\frac{1}{36} = \frac{1}{6} \cdot \frac{1}{6}$, so the two events are independent.
- Independence can be weird, as this example demonstrates. Consider the random trial of rolling a single six-sided die. Let $E$ be the event "the number rolled is even" and $F$ be the event "the number rolled is 5 or 6". Check that $P(E) = \frac{1}{2}$, $P(F) = \frac{1}{3}$, and $P(E \cap F) = \frac{1}{6}$ to confirm these two events are independent.
- With the same random trial, consider the events $E$ of rolling an even number and $F$ of rolling an odd number. Intuitively, these are not independent; if you know whether you rolled an even you know whether you rolled an odd, and vice versa. Compute the probabilities $P(E)$, $P(F)$, and $P(E \cap F)$ to confirm this.

The next concept we've already used without remarking on it.

**Complementary events**

For an event $E$, the *complement* of $E$, written $\bar{E}$, is the event "$E$ doesn't happen". Using notation for sets, $\bar{E} = S \setminus E$. Their probabilities are related:

$$P(\bar{E}) = 1 - P(E).$$

This is because the total probability across the whole sample space needs to sum to 1, so $P(E) + P(\bar{E}) = 1$.

Often it's easy to compute the probability of a complementary event, then use this relationship to get the probability you care about.

- There are twenty people in a room, how likely is it that two of them share a bithday? It's easier to instead compute the probability of the complementary event that they all have different birthdays.
- A PIN consists of four randomly chosen digits. How likely is it that a digit occurs twice? It's easier to insead compute the probabily of the complementary event that they are all different.

**Disjoint events**

Two events are *disjoint* or *mutually exclusive* if one happening means the other didn't happen. In the language of sets, $E$ and $F$ are disjoint if $E \cap F = \emptyset$.

If two events are disjoint, it is easy to compute the probability at least one of them happens:

$$P(E \cup F) = P(E) + P(F). \qquad \text{(disjoint events)}$$

If they are not disjoint, some atomic outcomes are counted twice by the above formula. To get the correct count you need to subtract out the over count.

$$P(E \cup F) = P(E) + P(F) - P(E \cap F).$$

Note that these formulas are just like the formulas we saw in math 013 about counting.

## Practice Problems

(1) Consider the random trial of rolling two six-sided dice. Compute the probabilities of the following events.
- Both dice roll 6s.
- Both dice roll even numbers.
- At least one dice rolls an odd number.

(2) Consider the random trial of a coin 8 times in a row. Compute the probabilities of the following events, where you assume the coin has equal odds of coming up heads versus tails.
- All flips come up heads.
- At least one flip comes up tails.
- The first flip comes up heads.
- The first and last flips come up heads.

(3) Consider the random trial of rolling a single six-sided die. For the following pairs of events, determine whether they are independent.
- "The result is even" and "the result is a prime number".
- "The result is even" and "the result is a multiple of 3".
- "The result is even" and "the result is a multiple of 4".
- "The result is even" and "the result is at most 2".

(4) Consider the random trial of rolling a single eight-sided die. For the following pairs of events, determine whether they are independent.
- "The result is even" and "the result is a prime number".
- "The result is even" and "the result is a multiple of 3".
- "The result is even" and "the result is a multiple of 4".

- "The result is even" and "the result is at most 2".

(5) Consider the random trial of flipping a coin 8 times in a row. For the following pair of events, determine whether they are disjoint.
   - "The first flip is heads" and "the last flip is heads".
   - "The first flip is heads" and "the first flip is tails".
   - "There are at least 5 heads" and "there are at least 5 tails".
   - "There are at least 3 heads" and "there are at least 3 tails".

(6) Your friend says that two disjoint events can each have probability $> \frac{1}{2}$. Explain why they are wrong.

(7) Are disjoint events independent or dependent (or could it go either way)? Explain.

(8) Can an event $A$ be independent from itself? Explain.