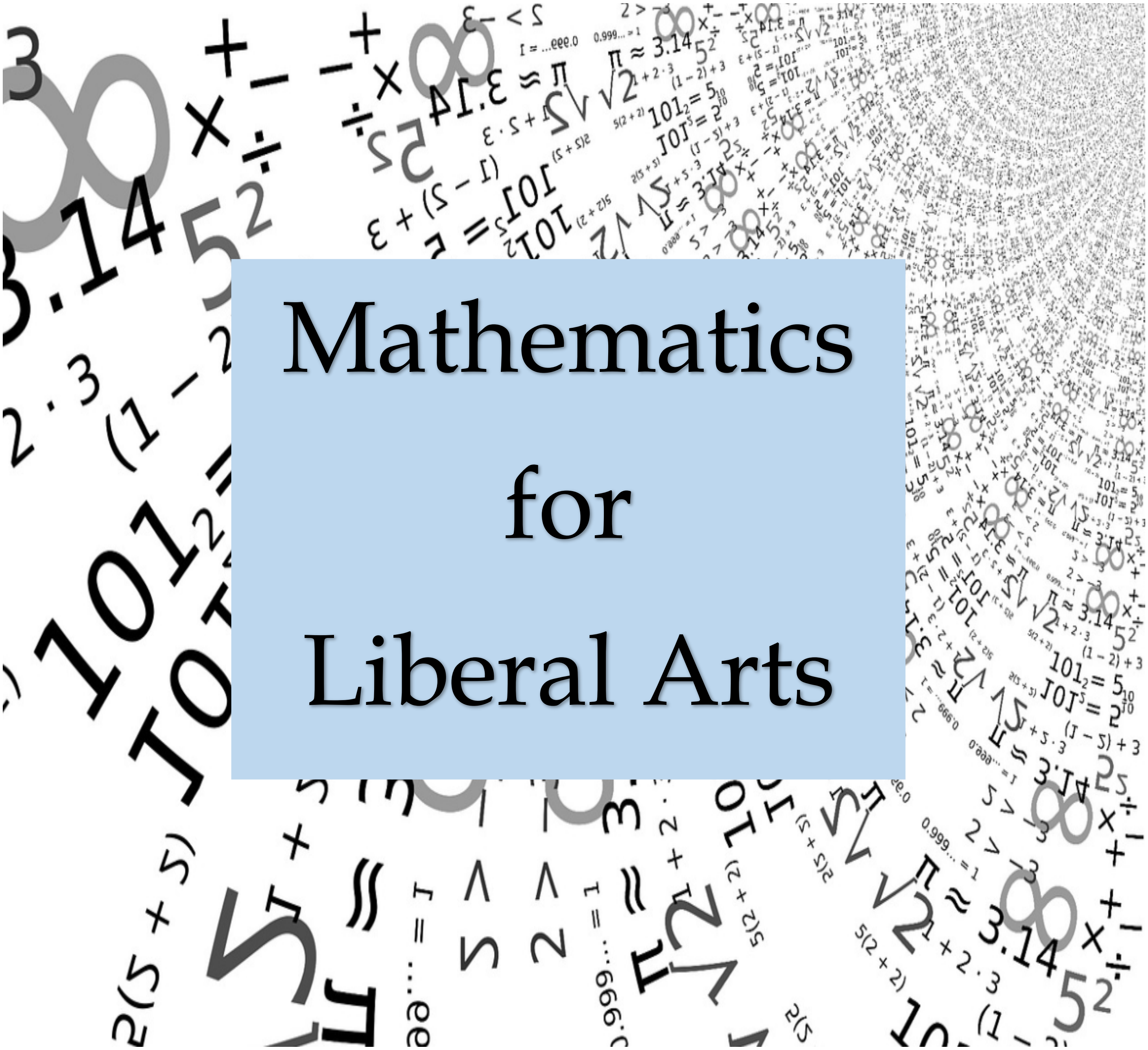

Mathematics for Liberal Arts

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College of DuPage

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Set Theory

Section 1.1 | Basic Set Concepts

Objectives

- Define a set
- Write a set using roster method, word description, and set-builder notation
- Define the empty set
- Define finite and infinite sets
- Determine the cardinality of a set

In math and in everyday life, objects with common characteristics are often grouped together to represent one entity. We do this without realizing it, whether it be grouping items to purchase at the grocery store, courses that must be taken in a given semester, or people who will be invited to a party. In this section it will be shown how to group objects into sets and represent sets using mathematical notation. Additionally, properties of sets will be analyzed.

Sets

DEFINITIONS: A **set** is a collection of objects whose contents can be clearly determined. The individual objects are called **elements** of the set.

A set must be well-defined, meaning that its contents must be clearly determined.

➢ **EXAMPLE 1.1.1:** Determine if the following sets are well-defined.

a. The collection of all Nobel Peace Prize winners
b. The collection of all even numbers that are divisible by 7
c. The collection of the most talented musicians in history

SOLUTION:

a. The collection of all Nobel Peace Prize winners is a **well-defined set**, since the contents are clearly determined.

b. The collection of all even numbers that are divisible by 7 is a **well-defined set**, since the contents are clearly determined.

c. The collection of the most talented musicians in history is **not a well-defined set**, since the contents are a matter of opinion and are not clearly determined.

Using Mathematical Notation to Represent Sets

Capital letters are usually used to identify sets. There are three common ways to denote a set: **roster method**, **word description**, and **set-builder notation**. See Figure 1.1.1 for descriptions and examples.

In **roster method**, elements are placed in braces and are separated by commas. In some cases, it is easy to represent a set using this method. However, if a set contains many elements, it may be cumbersome to list all of the elements. For example, the set of all counting numbers from 1 to 100 can be written as $\{1, 2, 3, \dots, 100\}$. In this representation, the first few elements and the last element are listed. The dots are called an **ellipsis** and replace the elements of the pattern that are not listed. Similarly, the list of all counting numbers can be written as $\{1, 2, 3, \dots\}$. Since a final element is not included, it is implied that the pattern does not end.

FIGURE 1.1.1

Common Ways to Denote a Set		
Roster Method	Elements are placed in braces and are separated by commas	**Ex:** $F = \{red, white, blue\}$
Word Description	Words are used to describe a set	**Ex:** $F\ is\ the\ set\ containing\ the\ colors\ of\ the\ American\ flag$
Set-builder Notation	Mathematical notation is used to describe a set and contains: • a variable • a vertical bar or colon separator which is read as "*such that*" • a statement indicating the properties of the set $\{x \mid x\ must\ satisfy\ these\ conditions\}$	**Ex:** $F = \{x \vert x\ is\ a\ color\ of\ the\ Amerian\ flag\}$ This is read as, "F is *the set of all* x *such that* x *is a color of the American flag.*"

Elements

The symbol ∈ denotes that an element is a member of a set. The symbol ∉ denotes that an element is not a member of a set.

➢ **EXAMPLE 1.1.2:** Determine if the following are true or false.

a. $Ellen\ Johnson\ Sirleaf \in \{x \mid x\ is\ a\ Nobel\ Peace\ Prize\ winner\}$
b. $296 \notin \{3, 6, 9, \ldots, 300\}$
c. $302 \in \{y \mid y\ is\ even\ and\ divisible\ by\ 7\}$

SOLUTION:

a. True. Ellen Johnson Sirleaf was awarded the Nobel Peace Prize in 2011. (She was Liberia's President from 2006 to 2018 and was Africa's first elected female head of state.)

b. True. 296 is not an element of the set containing all multiples of 3 up to 300.

c. False. 302 is not an element of the set of all even numbers that are divisible by 7. Even though 302 is even, it is not divisible by 7.

Common Number Sets

Numbers can be classified into different sets, such as natural numbers, whole numbers, integers, rational numbers, irrational numbers, and real numbers. Not only are these sets used often by mathematicians, they also appear in everyday life. For example, whole numbers can be used when reporting the score of a football game or the number of animals that were adopted on a given day. Integers can be used when indicating a temperature or the performance of a stock.

The set of **natural numbers**, represented by $\mathbb{N}$, is the set of all counting numbers.

$$\mathbb{N} = \{1,\ 2,\ 3, \ldots\}$$

The set of **whole numbers**, represented by $\mathbb{W}$, is the set of all counting numbers and 0.

$$\mathbb{W} = \{0, 1,\ 2,\ 3, \dots\}$$

The set of **integers**, represented by $\mathbb{Z}$, contains 0 and all counting numbers along with their opposites.

$$\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1,\ 2,\ 3, \dots\}$$

See Figure 1.1.2 for a list of these and other common number sets. In Section 1.2, we will refer back to these sets when using Venn diagrams to visually represent relationships among them.

FIGURE 1.1.2

Common Number Sets		
Natural Numbers ($\mathbb{N}$)	The set of counting numbers	$\mathbb{N} = \{1, 2, 3, \dots\}$
Whole Numbers ($\mathbb{W}$)	The set of counting numbers and zero	$\mathbb{W} = \{0, 1, 2, 3, \dots\}$
Integers ($\mathbb{Z}$)	The set of counting numbers and their opposites and zero	$\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$
Rational Numbers ($\mathbb{Q}$)	The set of numbers that can be written as a quotient of integers	**Ex:** $-100,\ -\frac{1}{2},\ 0,\ 4.5,\ 9\frac{3}{4},\ 112.\overline{6}$
Irrational Numbers	The set of numbers that cannot be written as a quotient of integers	**Ex:** $\pi,\ e,\ \sqrt{2},\ \sqrt[3]{10}$
Real Numbers ($\mathbb{R}$)	The set of all rational and irrational numbers	Every number listed above is a real number!

Empty Set

DEFINITION: The **empty set**, or **null set**, is a set that does not contain any elements. The empty set is represented symbolically by { } or Ø.

- **EXAMPLE 1.1.3:** Determine if the following sets are empty.

  a. $A = \{x \mid x \in \mathbb{N}\ and\ 4 < x \leq 10\}$
  b. $B = \{x \mid x \in \mathbb{N}\ and\ 4 < x < 5\}$
  c. $C = \{\ 0\ \}$

  SOLUTION:

  a. Set A is **not** empty, as it contains the natural numbers 5, 6, 7, 8, 9, and 10.

  b. Set B is empty, as there does not exist a natural number between 4 and 5.

  c. Set C is **not** empty, as it has the number 0 as an element.

Finite and Infinite Sets

DEFINITIONS: A **finite** set is one in which the number of distinct elements is a whole number. A set that is not finite is **infinite**.

If the elements of a set can be numbered from 1 to n, for some natural number n, then the set is finite. This is called a **one-to-one correspondence**.

For example, the set $\{a, b, c, \dots, z\}$ is a finite set, because the elements are fixed and countable. The elements of this set can be placed in a one-to-one correspondence with the set containing numbers 1 to n, where $n = 26$.

$$\begin{array}{ccccc}
\{a, & b, & c, & \dots, & z\} \\
\updownarrow & \updownarrow & \updownarrow & & \updownarrow \\
\{1, & 2, & 3, & \dots, & 26\}
\end{array}$$

If this type of one-to-one correspondence cannot be created, then the set is infinite. When written in roster form, an infinite set will usually begin or end with an ellipsis, which contains three dots in succession.

- **EXAMPLE 1.1.4:** Determine if the following sets are finite or infinite.

  a. $A = \{x \mid x \in \mathbb{N} \text{ } and \text{ } x < 100\}$
  b. $B = \{100, 101, 102, \dots \}$
  c. $C = The\ set\ of\ all\ prime\ numbers\ greater\ than\ 2\ that\ are\ divisible\ by\ 2.$
  d. $D = \{x \mid x\ is\ a\ word\ used\ in\ Harry\ Potter\ and\ the\ Order\ of\ the\ Phoenix\}$

  SOLUTION:

  a. This set is **finite**, because its elements are countable. There are 99 natural numbers that are less than 100.

  b. This set is **infinite**, because its elements cannot be counted. There are infinitely many natural numbers that are greater than or equal to 100.

  c. This set is **finite**, because it does not contain any elements. Recall that a prime number is a natural number greater than 1 that is divisible by only 1 and itself. All prime numbers greater than 2 are odd and are not divisible by 2. Thus, this set is empty.

  d. This set is **finite**, because it is possible to count all of the words used in this book. Even though it would be cumbersome to count the large number of elements of this set, it is still countable.

Cardinal Number

DEFINITION: The **cardinal number** (or **cardinality**) of set A, denoted $n(A)$, is the number of distinct elements in set A. The symbol $n(A)$ is read "*n of A.*"

The cardinal number of any finite set will be a whole number. Considering the cardinality of an infinite set is beyond the scope of this text.

➢ **EXAMPLE 1.1.5:** Find the cardinality of the following sets.

a. $A = \{x \mid x \in \mathbb{N}\ and\ 4 < x \leq 10\}$
b. $B = \{x \mid x \in \mathbb{N}\ and\ 4 < x < 5\}$
c. $C = \{\,0\,\}$
d. $D = \{3, 6, 9, \dots, 300\}$

SOLUTION:

a. Set A can be written as $\{5, 6, 7, 8, 9, 10\}$. Since there are six elements, $n(A) = 6$.

b. Since there is no natural number between 4 and 5, set B is empty. Thus, $n(B) = 0$.

c. Set C contains the number 0. Since there is one element, $n(C) = 1$.

d. Set D contains all multiples of 3 up to 300. Since there are 100 elements, $n(D) = 100$.

❖ **THINK ABOUT IT:** Is the set { ∅ } empty? What is its cardinality?

ANSWER: This set is not empty; it contains one element, the empty set. So, the cardinality of this set is 1. Note that this set itself is not empty, because it is not represented as {   } or ∅.

Quick Review

- A **set** is a collection of objects whose contents can be clearly determined.
- The individual objects are called **elements** of the set.
- The symbol ∈ denotes that an element is a member of a set, whereas ∉ denotes that an element is not a member of a set.
- Three common ways to denote a set are **roster method**, **word description**, and **set-builder notation**.
- The **empty set**, or **null set**, is a set that does not contain any elements and is symbolized by { } or ∅.
- A **finite set** is one in which the elements of the set are fixed and countable. A set that is not finite is **infinite**.
- The **cardinal number**, or **cardinality**, of set A, denoted $n(A)$, is the number of distinct elements in set A.

Section 1.1 | Exercises

In Exercises 1 – 7, determine which collections are not well defined and are therefore not sets.

1. The collection of all community colleges in Illinois

2. The collection of the most talented WNBA players of all time

3. The collection of all artwork in the Art Institute of Chicago

4. The collection of the best beaches in Florida

5. The collection of math courses offered at College of DuPage

6. The collection of the best holidays

7. The collection of months that start with the letter H

In Exercises 8 – 13, write the set using roster method.

8. The set of months of the year that begin with the letter J

9. The set of players in the starting lineup for the Chicago Bulls on December 9, 1997 (those players are R. Harper, M. Jordan, T. Kukoc, L. Longley, and D. Rodman)

10. The set of days of the week that start with the letter H

11. $\{x \mid x \in \mathbb{N}\ and\ x < 10\}$

12. $\{x \mid x \in \mathbb{N}\ and\ \ 41 < x < 53\}$

13. $\{2x \mid x \in \mathbb{N}\ and\ x \leq 10\}$

In Exercises 14 – 20, write the set using set-builder notation. For some, more than one correct answer is possible.

14. The set of the four seasons of a year in Chicago

15. The set of presidents on Mount Rushmore

16. $\{a, e, i, o, u\}$

17. $\{1, 2, 3, \dots, 100\}$

18. $\{281, 282, 283, \dots, 300\}$

19. $\{3, 6, 9, 12, 15, \dots\}$

20. $\{4, 8, 12, 16, 20, \dots\}$

In Exercises 21 – 27, determine if the statement is true or false.

21. $m \notin \{x \mid x\ is\ a\ letter\ in\ the\ word\ mathematics\}$

22. $50 \in \{1, 2, 3, \dots, 100\}$

23. $37 \in \{30, 32, 34, \dots, 60\}$

24. $-5 \notin \{x \mid x \in \mathbb{N}\ and\ x < 10\}$

25. $7.5 \notin \{x \mid x \in \mathbb{N}\ and\ 5 < x < 15\}$

26. $7.5 \in \{x \mid x \in \mathbb{Q}\ and\ 5 < x < 15\}$

27. $2 \in \{x \mid x \in \mathbb{R}\ and\ 2 < x \leq 12\}$

In Exercises 28 – 35, determine which sets are empty.

28. $\{x \mid x\ is\ a\ woman\ who\ served\ as\ U.S. president\ before\ 2022\}$

29. $\{x \mid x\ is\ a\ spacecraft\ lanched\ by\ NASA\ before\ 1960\}$

30. $\{\ \emptyset\ \}$

31. $\emptyset$

32. $\{x \mid x \in \mathbb{N}\ and\ 99 < x < 100\}$

33. $\{x \mid x\ is\ a\ fish\ with\ five\ eyes\}$

34. The set of days of the week that start with the letter H

35. $\{y \mid y \in \mathbb{Q}\ and\ 0 < y < 1\}$

In Exercises 36 – 45, determine which sets are finite and which are infinite.

36. The set of bones in the human body

37. The set of all points on a line

38. The set of seconds in one year

39. $\{1, 1, 2, 3, 5, 8, 13, \dots\}$

40. $\{1, 2, 4, 6, 8, 10, 12, \dots\}$

41. $\{2, 4, 6, 8, \dots, 200\}$

42. $\{x \mid x \in \mathbb{N}\ and\ x > 1{,}000{,}000\}$

43. $\{x \mid x \in \mathbb{N}\ and\ x < 1\}$

44. $\{x \mid x \in \mathbb{R}\ and\ x > 1{,}000{,}000\}$

45. $\{x \mid x \in \mathbb{R}\ and\ x < 1\}$

In Exercises 46 – 52, find the cardinal number of each set.

46. $A = \{21, 22, 23, \dots, 78\}$

47. $G = \{85, 87, 89, \dots, 99\}$

48. $B = \{x \mid x\ is\ even\ and\ 24 \leq x < 52\}$

49. $P = \{x \mid x \text{ is divisible by } 3 \text{ and } 3 \leq x < 33\}$

50. $A = \{y \mid y \in \mathbb{N} \text{ and} -1 \leq y < 0\}$

51. $W = \{y \mid y \in \mathbb{Z} \text{ and} -1 \leq y < 0\}$

52. $Y = \{1, 1, 1, 1, 1\}$

In Exercises 53 – 58, determine if each statement is true or false.

53. If a set can be written using set-builder notation, then it can be written in roster method.

54. $n(\emptyset) = 1$

55. $n(\{\emptyset\}) = 1$

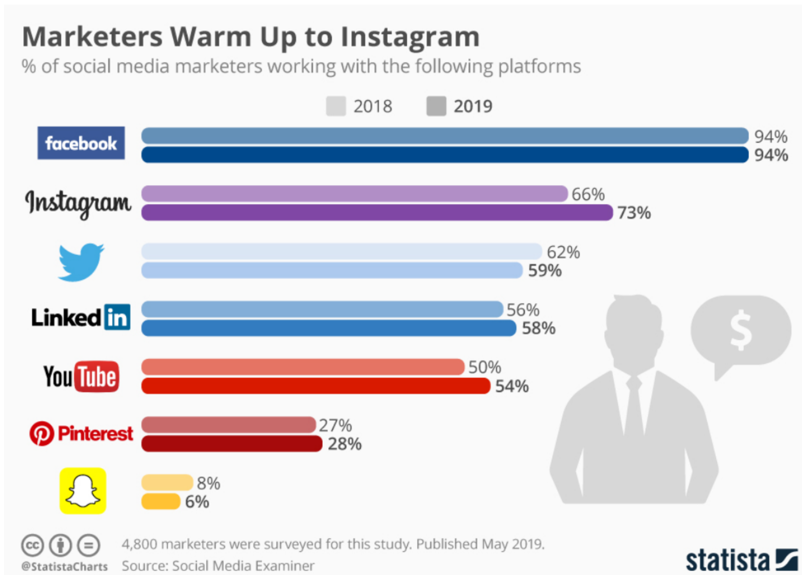
56. If a set contains an ellipses, then it must be infinite.

57. Elements that are repeated are only counted once when determining the cardinal number of a set.

58. If the elements of a set cannot be counted in 100 years, then the set is infinite.

Applications

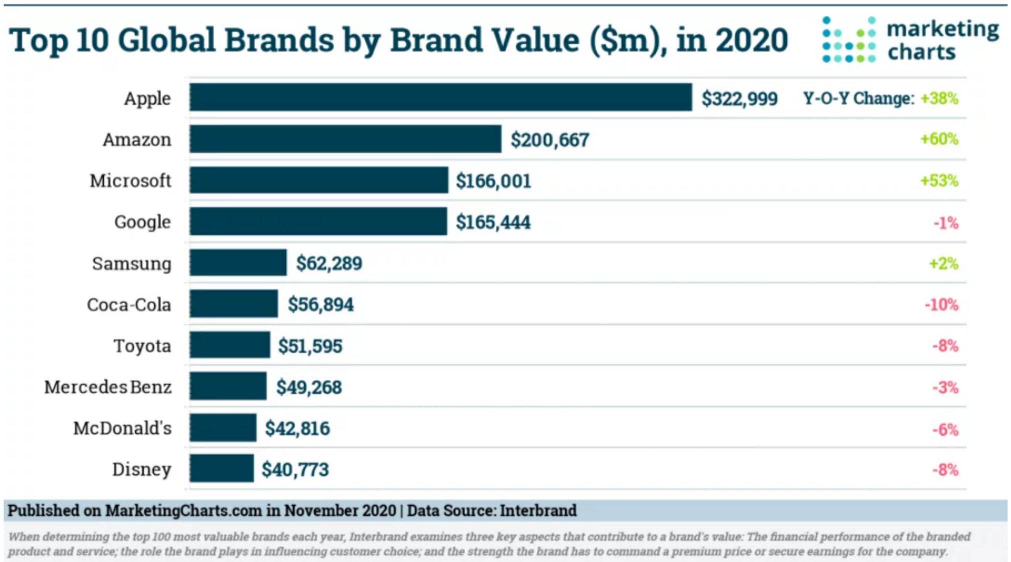
The percentage of marketers using various social media platforms in 2018 and 2019 are displayed in the following graph that uses data from the Social Media Examiner. The platforms included are Facebook, Instagram, Twitter, LinkedIn, YouTube, Pinterest, and Snapchat. Use this graph to complete Exercises 59 – 60.



59. Write the set of platforms that more than 50% of marketers worked with in 2018 using roster method.

60. What is the cardinality of the set of platforms that increased in percentage from 2018 to 2019?

The top 10 best global brands in 2020 are shown in the graph below. The brand ranking by Interbrand is conducted every year using three key components: the brand's financial performance, the brand's purchasing decisions, and the brand's competitive strength. Use this graph to complete Exercises 61 – 62.



The values shown in the graph are in millions. For example, Coca-Cola shows $56,894 which means $56,894 million or $56,894,000,000.

61. Write the set of companies in the top five best global brands in 2020 using roster method.

62. What is the cardinality of the set of companies in the top 10 global brands whose brand value is under $100,000,000,000?

The research department at Money.co.uk conducted an analysis to determine the most misspelled brands in the world. The following chart shows the average number of times the given brands are misspelled in online searches per month as of October 29, 2001. Use this graph to complete Exercises 63 – 64.

63. Write the set of brands with more than 95,000 misspellings using the roster method.

64. What is the cardinality of the set of common misspellings for all shown brands? Count each misspelled brand name listed in the chart as a unique element.

Concept Review

65. What is a set?

66. Explain three ways to denote a set.

67. What is the empty set?

68. What is the difference between finite and infinite sets?

Section 1.1 Exercise Solutions

1. Set
2. Not a set
3. Set
4. Not a set
5. Set
6. Not a set
7. Set
8. $\{January, June, July\}$
9. $\{R. Harper, M. Jordan,\ T. Kukoc,$ $L. Longley, D. Rodman\}$
10. $\{\ \ \}$ or $\emptyset$
11. $\{1, 2, 3, \ldots, 9\}$
12. $\{42, 43, 44, \ldots, 52\}$
13. $\{2, 4, 6, \ldots, 20\}$
14. $\{x \mid x\ is\ a\ season\ of\ a\ year\}$
15. $\{x \mid x\ is\ a\ president\ on\ Mount$ $Rushmore\}$
16. $\{x \mid x\ is\ a\ vowel\ in\ the\ English$ $alphabet\}$
17. $\{x \mid x \in \mathbb{N}\ and\ x \leq 100\}$
18. $\{x \mid x \in \mathbb{N}\ and\ 280 < \ x \leq 300\}$
19. $\{3x \mid x \in \mathbb{N}\ and\ x \geq 1\}$
20. $\{4x \mid x \in \mathbb{N}\ and\ x \geq 1\}$
21. False
22. True
23. False
24. True
25. True
26. True
27. False
28. Empty
29. Not empty
30. Not empty
31. Empty
32. Empty
33. Empty
34. Empty
35. Not empty
36. Finite
37. Infinite
38. Finite
39. Infinite
40. Infinite
41. Finite
42. Infinite
43. Finite
44. Infinite
45. Infinite
46. $n(A) = 58$
47. $n(G) = 8$
48. $n(B) = 14$
49. $n(P) = 10$
50. $n(A) = 0$
51. $n(W) = 1$
52. $n(Y) = 1$ (since repeated items are counted only once)
53. False
54. False
55. True
56. False
57. True
58. False
59. $\{LinkedIn, Twitter, Instagram, Facebook\}$
60. 4
61. $\{Apple, Amazon, Microsoft, Google$ $Samsung\}$
62. 6

63. $\{Hyundai, Lamborghini, Ferrari, Hennessy\}$
64. 18
65. A set is a collection of objects whose contents can be clearly determined.
66. With a word description, the elements of a set are described in words. With roster method, elements are listed in braces and are separated by commas. With set-builder notation, a variable is listed in braces, along with a vertical bar separator and a statement indicating properties of the set.
67. The empty set is a set that does not contain any elements.
68. A finite set is one in which the elements of the set are fixed and countable. A set that is not finite is infinite.

Section 1.2 | Subsets

Objectives

- Define equal and equivalent sets
- Define subsets and proper subsets
- Count the number of subsets and proper subsets of a set
- Use Venn diagrams to display set relationships

Suppose a pizza can be made with a choice of five toppings. A pizza with none, some, or all of the available toppings can be ordered. How many different possible pizzas can be created as a result? It turns out that set theory can be used to answer this question and others like it! In this section, different types of sets will be examined. Additionally, visual representations will be used to explore relationships between sets. And, the number of pizzas in the scenario mentioned above will be counted.

Equal and Equivalent Sets

DEFINITION: Two sets are **equal** if they contain the same elements, regardless of order or possible repetition of elements. If sets A and B are equal, this relationship is denoted by $A = B$.

If sets A and B are equal, it follows that they will have the same cardinality.

DEFINITION: Two sets are **equivalent** if they contain the same *number* of distinct elements. If sets A and B are equivalent, this relationship is denoted by $A \simeq B$.

If sets A and B are equivalent, it follows that they will have the same cardinality and can be placed in a one-to-one correspondence with one another.

For example, since the sets $\{a, b, c, \ldots, z\}$ and $\{1, 2, 3, \ldots, 26\}$ are equivalent, they can be placed in a one-to-one correspondence. Note that both sets have a cardinality of 26.

$$\{a,\ b,\ c, \ldots,\ z\}$$

$$\{1,\ 2,\ 3, \ldots,\ 26\}$$

Similarly, the sets $\{x \mid x\ is\ a\ planet\ in\ the\ solar\ system\}$ and $\{1, 1, 2, 3, 4, 5, 5, 6, 7, 7, 8\}$ are equivalent and can be placed in a one-to-one correspondence. Note that both sets have a cardinality of 8. Recall that repeated elements are only counted once.

$$\{1,\quad 1,\quad 2,\quad 3,\quad 4,\quad 5,\quad 5,\quad 6,\quad 7,\quad 7,\quad 8\}$$

$$\{Mercury, Venus, Earth, Mars, Jupiter, Saturn, Uranus, Neptune\}$$

➢ **EXAMPLE 1.2.1:** Determine if the following sets are equal, equivalent, both, or neither. Explain.

a. $A = \{x \mid x \in \mathbb{N}\ and\ 47 < x < 73\}$ and $B = \{x \mid x \in \mathbb{N}\ and\ 48\ \leq x \leq 72\}$
b. $A = \{apple\}$ and $B = \{x \mid x\ is\ a\ prime\ number\ and\ x\ is\ divisible\ by\ 2\}$
c. $A = \{x \mid x\ is\ a\ species\ of\ \ a\ butterfly\}$ and $B = \{x \mid x\ is\ a\ month\ in\ a\ year\}$

SOLUTION:

a. Both. These sets are equal, because they contain the same elements (all natural numbers between 48 and 72, inclusive). They are also equivalent, because they share a cardinality of 25 and can be placed in a one-to-one correspondence with one another. Symbolically, we can write $A = B$ and $A \simeq B$.

b. These sets are equivalent, because they share a cardinality of 1 and can be placed in a one-to-one correspondence with one another. They are not equal, because they contain different elements. Note the only prime number that is divisible by 2 is 2 itself, as all other prime numbers are odd. Symbolically, we can write $A \simeq B$.

c. Neither. These sets do not contain the same elements nor do they have the same cardinality.

❖ **THINK ABOUT IT:** If sets A and B are equal, does that imply that they are also equivalent? If sets A and B are equivalent, does that imply that they are also equal? Why or why not?

ANSWER: If sets are equal, then they contain the same exact elements. This means that they also must have the same number of distinct elements, automatically making them equivalent. If sets are equivalent, then they have the same *number* of distinct elements. However, they don't have to contain the same exact elements, meaning that they are not necessarily equal. Thus, if sets are equal, they are automatically equivalent. On the other hand, if sets are equivalent, they are not necessarily equal.

Subsets and Proper Subsets

DEFINITION: If each element of set A is also an element of set B, then A is a **subset** of B. If set A is a subset of set B, this relationship is denoted by $A \subseteq B$.

If set A contains *at least one* element that is not contained in set B, then A is not a subset of B. This can be denoted by $A \nsubseteq B$.

➢ **EXAMPLE 1.2.2:** Determine whether each given set is a subset of the other.

a. $A = \{1, 3, 5, 7, 9, 11\}$ and $B = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$

b. $A = \{x \mid x \textit{ is an Emmy winner}\}$ and
$B = \{\textit{Michael J. Fox, Eddie Murphy, Archie Panjabi, RuPaul, Betty White}\}$

c. $A = \{x \mid x\ is\ a\ day\ of\ the\ week\ beginning\ with\ the\ letter\ S\}$ and $B = \{Saturday,\ Sunday\}$

d. $A = \{\ \ \}$ and $B = \{x \mid x \in \mathbb{N}\ and\ x > 100\}$

SOLUTION:

a. Because every element of set A is also an element of set B, it is true that $A \subseteq B$. However, there is *at least one* element of set B that is not listed in set A, so $B \nsubseteq A$.

b. Because every person in set B has won an Emmy, it is true that $B \subseteq A$. However, there is *at least one* member of set A that is not listed in set B (e.g., Kate Winslet), so $A \nsubseteq B$.

c. These sets are equal! Because every element of set A is also an element of set B, it is true that $A \subseteq B$. Further, because every element of set B is also an element of set A, it is true that $B \subseteq A$.

d. Because every element of set A is also an element of set B, it is true that $A \subseteq B$. However, there is *at least one* element of set B that is not in set A, so $B \nsubseteq A$. By definition, the empty set is a subset of every set!

The Empty Set as a Subset

Given any set A, it follows that $\{\ \ \} \subseteq A$.

❖ **THINK ABOUT IT:** If set A is a subset of set B, then *must* each element of set B also be an element of set A? Is there a situation in which this *can* be true?

ANSWER: No. If set A is a subset of set B, then the only requirement is that each element of A must also be an element of B. However, in the case that sets A and B are equal, each element of B will also be an element of A.

Equal Sets as Subsets

If given sets A and B are equal, then $A \subseteq B$ and $B \subseteq A$.

DEFINITION: Set A is considered a **proper subset** of set B if every element of A is also an element of B and sets A and B are not equal. If set A is a proper subset of set B, this relationship is denoted by $A \subset B$. If set A is not a proper subset of set B, this relationship is denoted by $A \not\subset B$.

- **EXAMPLE 1.2.3:** Determine whether each given set is a proper subset of the other.

  a. $A = \{1, 3, 5, 7, 9, 11\}$ and $B = \{1,2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$

  b. $A = \{x \mid x \text{ is an Olympic gold medalist}\}$ and
  $B = \{\textit{Simone Biles, Michelle Kwan, Katie Ledecky, Serena Williams}\}$

  c. $A = \{x \mid x = 3n + 1 \text{ where } n \in \mathbb{N}\}$ and $B = \{4, 7, 10, \dots\}$

  SOLUTION:

  a. Since set A is a subset of set B and sets A and B are not equal, it is true that $A \subset B$. On the other hand, since set B contains *at least one* element that is not in set A, it follows that $B \not\subset A$.

  b. Since set B is a subset of set A and sets A and B are not equal, it is true that $B \subset A$. On the other hand, since set A contains *at least one* element that is not in set B (e.g., Allyson Felix), it follows that $A \not\subset B$.

  c. Since sets A and B are equal, one cannot be a proper subset of the other.

❖ **THINK ABOUT IT:** Does $A \subset B$ imply $A \subseteq B$? Does $A \subseteq B$ imply $A \subset B$? Why or why not?

ANSWER: If $A \subset B$, then every element of set A is also an element of set B and sets A and B are not equal. By definition, it follows that A is a subset of B. On the other hand, if $A \subseteq B$, then every element of set A is also an element of set B and sets A and B *can* be equal. Thus, it follows that A is not necessarily a proper subset of B.

Counting Subsets and Proper Subsets

If A is a finite set, it is possible to count the number of subsets and proper subsets of A. It turns out, there is a connection between the number of elements of a set and the number of subsets of a set. See Figure 1.2.1.

FIGURE 1.2.1

Counting the Number of Subsets of a Set				
Set	**Number of Elements, n**	**List of Subsets**	**Number of Subsets, 2^n**	**Number of Proper Subsets**
$\{\ \}$	0	$\{\ \}$	$2^0 = 1$	0
$\{a\}$	1	$\{\ \}, \{a\}$	$2^1 = 2$	1
$\{a, b\}$	2	$\{\ \}, \{a\}, \{b\}, \{a, b\}$	$2^2 = 4$	3
$\{a, b, c\}$	3	$\{\ \}, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}$	$2^3 = 8$	7

According to the examples in Figure 1.2.1, it appears that the number of subsets of a set can be calculated by raising 2 to the power of the number of elements of the set. It turns out that this relationship is true for any set, not just for those listed in Figure 1.2.1.

Counting the Number of Subsets of a Set

If a set contains n elements, then the set will have 2^n subsets.

- **THINK ABOUT IT:** What do you notice about the relationship between the number of subsets and proper subsets for any finite set? Why does this relationship exist?

  ANSWER: According to the examples in Figure 1.2.1, it seems that there is one less proper subset than the number of subsets for a given set. This is because when counting subsets of a particular set, the one that is equal to the original set itself is not a proper subset.

There is a connection between the number of elements of a set and the number of proper subsets of a set. Recall that if set A is a proper subset of set B, then A is a subset of B and sets A and B are not equal. Thus, when counting the number of proper subsets of any set, the subset that is equal to the set itself is not proper and must be excluded.

Counting the Number of Proper Subsets of a Set

If a set contains n elements, then the set will have $2^n - 1$ proper subsets.

For example, suppose you would like to order a pizza and have a choice among five different toppings that you enjoy (green pepper, mushroom, olive, pepperoni, and sausage). You can opt to get all, some, or none of the toppings. How many different pizza orders are possible?

This is an example of counting the number of subsets of the set $\{g, m, o, p, s\}$ using abbreviations for the toppings.

The 32 subsets can be listed:

$$\{\quad\}, \{g\}, \{m\}, \{o\}, \{p\}, \{s\},$$

$$\{g, m\}, \{g, o\}, \{g, p\}, \{g, s\}, \{m, o\}, \{m, p\}, \{m, s\}, \{o, p\}, \{o, s\}, \{p, s\},$$

$$\{g, m, o\}, \{g, m, p\}, \{g, m, s\}, \{g, o, p\}, \{g, o, s\}, \{g, p, s\}, \{m, o, p\}, \{m, o, s\}, \{m, p, s\}, \{o, p, s\},$$

$$\{g, m, o, p\}, \{g, m, o, s\}, \{g, m, p, s\}, \{g, o, p, s\}, \{m, o, p, s\}, \{g, m, o, p, s\}$$

There is an easier way! To calculate the number of subsets, the formula 2^n can be used, where n is the number of elements of the set: $2^5 = 32$. Thus, there are 32 different possibilities for this pizza order.

➢ **EXAMPLE 1.2.4:** Count the number of subsets and proper subsets of each set.

a. $\{The\ Beatles, Boyz\ II\ Men, New\ Kids\ on\ the\ Block, NSYNC\}$
b. $\{x \mid x \in \mathbb{N}\ and\ 6 < x \leq 21\}$

SOLUTION:

a. This set has 4 elements, so there are $2^4 = 16$ subsets. Further, there are $2^4 - 1 = 15$ proper subsets. The set containing all four bands is the subset that is not a proper subset.

b. In roster method, this set is $\{7, 8, 9, \dots, 21\}$. This set has 15 elements, so there are $2^{15} = 32{,}768$ subsets. Further, there are $2^{15} - 1 = 32{,}767$ proper subsets. The set containing all original elements is the subset that is not a proper subset.

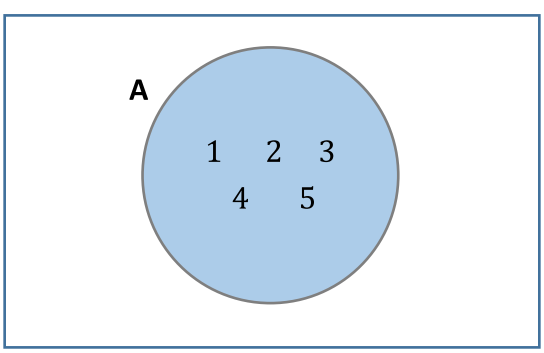
Venn Diagrams

DEFINITION: A **Venn diagram** can be used to visually represent the relationships among sets.

Sets within Venn diagrams are usually represented using circles or ovals and are enclosed in a rectangle. Elements of a set are placed within the circle or oval that is used to represent that set. See Figure 1.2.2 for a Venn diagram representing set $A = \{1, 2, 3, 4, 5\}$.

FIGURE 1.2.2

Venn diagram



Disjoint Sets

DEFINITION: Two sets that do not share a common element are called **disjoint** sets.

For example, set $A = \{1, 1, 2, 3, 5, 8, 13, 21\}$ and set $B = \{4, 7, 12, 19, 28, 52, 67\}$ are disjoint, because they do not have a common element.

- **EXAMPLE 1.2.5:** Determine whether the given sets are disjoint.

  a. $A = \{x \mid x = 2n \text{ } where \text{ } n \in \mathbb{N}\}$ and $B = \{x \mid x = 2n - 1 \text{ } where \text{ } n \in \mathbb{N}\}$
  b. $A = \{x \mid x \text{ } is \text{ } a \text{ } Grammy \text{ } winner\}$ and $B = \{x \mid x \text{ } is \text{ } an \text{ } Oscar \text{ } winner\}$

  SOLUTION:

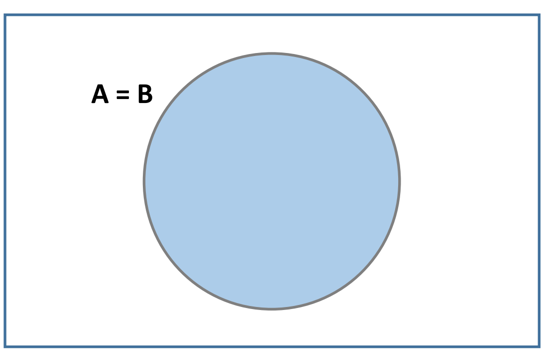
  a. Set $A = \{2 \cdot 1,\ 2 \cdot 2,\ 2 \cdot 3, \dots\} = \{2, 4, 6, \dots\}$, which consists of all even numbers. Set $B = \{2 \cdot 1 - 1,\ 2 \cdot 2 - 1,\ 2 \cdot 3 - 1, \dots\} = \{1, 3, 5, \dots\}$, which consists of all odd numbers. Since A and B do not have a common element, they are disjoint.

  b. The elements of set A consist of all people who have won a Grammy, and the elements of set B consist of all people who have won an Oscar. Since A and B share *at least one* element (e.g., Jennifer Hudson), they are not disjoint.

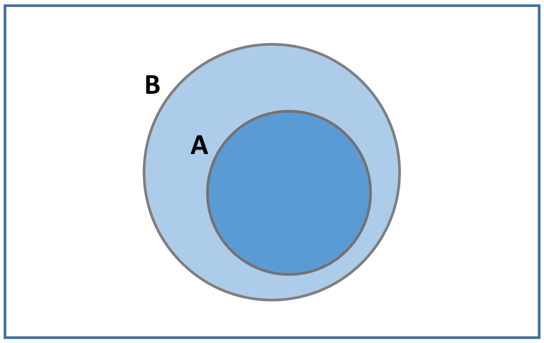
Set Relationships

Venn diagrams may be used to demonstrate the relationship among sets. For example, Venn diagrams can be used to represent equal sets, subsets, sets with common elements, and disjoint sets.

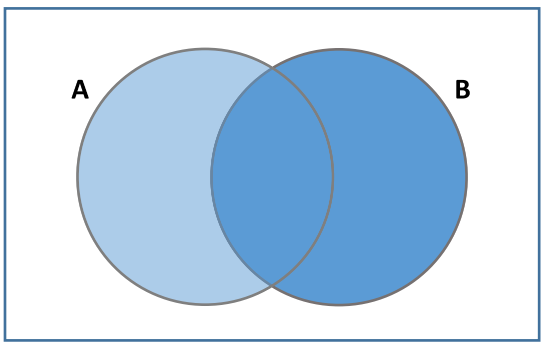
Equal Sets: If $A = B$, then these sets are represented using the same circle in the diagram.



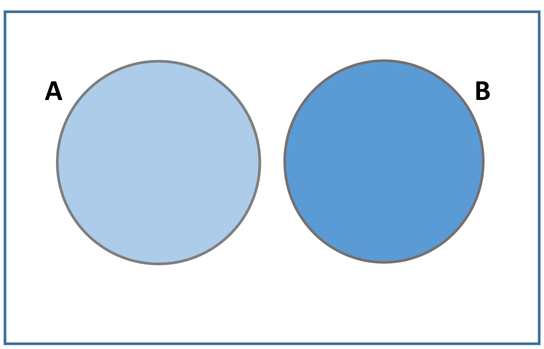
Proper Subsets: If $A \subset B$, then the circle representing set A is completely enclosed in the circle representing set B.



Overlapping Sets: If sets A and B have at least one common element, then the circles representing these sets will overlap. Common elements are placed in the middle section where the circles overlap.



Disjoint Sets: If sets A and B are disjoint, then the circles representing these sets will not intersect.

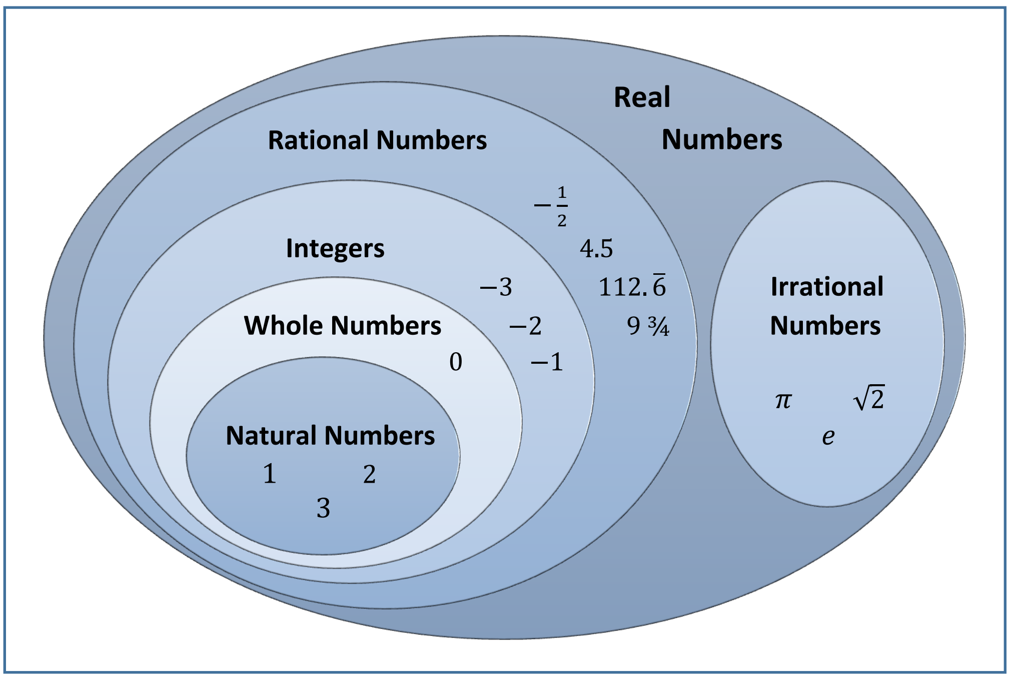


Visual Representation of Common Number Sets

There are many sets of common numbers, such as natural numbers, whole numbers, integers, rational numbers, irrational numbers, and real numbers. Figure 1.1.2 in Section 1.1 provides information about these sets. See Figure 1.2.3 for a visual representation of these sets.

FIGURE 1.2.3

Common Number Sets



Using Figure 1.2.3, we can now begin to analyze the relationships among the common number sets.

Recall these common number sets:

Natural Numbers: $\mathbb{N} = \{1,\ 2,\ 3, \dots\}$
Whole Numbers: $\mathbb{W} = \{0, 1,\ 2,\ 3, \dots\}$
Integers: $\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1,\ 2,\ 3, \dots\}$

Since $\mathbb{N} \subset \mathbb{W}$, the oval representing set $\mathbb{N}$ is completely enclosed in the oval representing set $\mathbb{W}$ in the diagram.

Similarly, since both $\mathbb{N} \subset \mathbb{Z}$ and $\mathbb{W} \subset \mathbb{Z}$, the ovals representing sets $\mathbb{N}$ and $\mathbb{W}$ are completely enclosed in the oval representing $\mathbb{Z}$ in the diagram.

Note that the sets of natural numbers, whole numbers, and integers are proper subsets of the set of rational numbers. Thus, the ovals representing sets $\mathbb{N}$, $\mathbb{W}$, and $\mathbb{Z}$ are completely enclosed in the oval representing the set of rational numbers.

Furthermore, rational numbers are those that can be written as a quotient of integers. Irrational numbers are numbers that cannot be written as a quotient of integers. Thus, the sets of rational and irrational numbers are disjoint and are represented in the diagram by ovals that don't overlap.

Finally, all of the sets discussed above are proper subsets of the set of real numbers, which is often represented by $\mathbb{R}$. Thus, the ovals representing $\mathbb{N}$, $\mathbb{W}$, $\mathbb{Z}$, the rational numbers, and the irrational numbers are completely enclosed in the oval representing $\mathbb{R}$.

Quick Review

- Two sets are **equal** if they contain the same elements, regardless of order or possible repetition of elements.
- Two sets are **equivalent** if they contain the same number of distinct elements.
- If each element of set A is also an element of set B, then A is a **subset** of B. This is denoted by $A \subseteq B$.
- If set A contains at least one element that is not contained in set B, then A is not a subset of B. This is denoted by $A \nsubseteq B$.
- Set A is a **proper subset** of set B if every element of A is also an element of B and sets A and B are not equal. This is denoted by $A \subset B$.
- If a set contains n elements, then it will have 2^n subsets and $2^n - 1$ proper subsets.
- A **Venn diagram** can be used to visually represent the relationships among sets.
- Two sets that do not share a common element are called **disjoint** sets.

Section 1.2 | Exercises

In Exercises 1 – 9, determine if the sets in each pair are equal, equivalent, both, or neither.

1. $C = \{triangle, square, pentagon, hexagon\}$
   $D = \{red, orange, yellow, green, blue, purple\}$

2. $C = \{1, 2, 3, 4, 5, 6, 7, 8, 9,10\}$
   $D = \{a, b, c, d, e, f, g, h, i, j\}$

3. $A = \{2, 4, 6, 8, 10\}$
   $B = \{1, 3, 5, 7, 9\}$

4. $A = \{x \mid x \in \mathbb{N} \; and \; 10 \; \leq x < 20\}$
   $B = \{x \mid x \in \mathbb{N} \; and \; 9 < x \leq 19\}$

5. $A = \{x \mid x \in \mathbb{N} \; and \; 5 < x \leq 10\}$
   $B = \{x \mid x \; is \; a \; multiple \; of \; 10 \; and \; 20 < x \leq 70\}$

6. $A = \{ \quad \}$
   $B = \{x \mid x \; is \; prime \; and \; even\}$

7. $A = \{y \mid y \in \mathbb{N} \; and \; 5 \leq y \leq 10\}$
   $B = \{5, 6, 7, 8, 9, 10\}$

8. $A = \{y \mid y \in \mathbb{N} \; and \; 5 \leq y \leq 10\}$
   $B = \{y \mid y \in \mathbb{Z} \; and \; -10 \leq y \leq -5\}$

9. $A = \{y \mid y \in \mathbb{N} \; and \; 5 \leq y \leq 10\}$
   $B = \{y \mid y \in \mathbb{N} \; and \; -10 \leq y \leq -5\}$

In Exercises 10 – 17, write ⊆ or ⊈ in each blank to create a true statement.

10. $\{0, 1, 2, 3, 4, 5\}$ ________ $\{1, 2, 3, \dots , 10\}$

11. $\{25, 26, 27, 28\}$ ________ $\{1, 3, 5, \dots , 99\}$

12. $\{x \mid x \text{ is even and } x < 100\}$ ________ $\{x \mid x \in \mathbb{N} \text{ and } x \leq 99\}$

13. $\{x \mid x \in \mathbb{N} \text{ and } x \leq 50\}$ ________ $\{x \mid x \text{ is divisible by } 5 \text{ and } x \leq 50\}$

14. $\{1, 3, 5, \ldots\}$ ________ $\{x \mid x = 2n - 1 \text{ where } n \in \mathbb{N}\}$

15. $\{1, 3, 5, 7, \ldots\}$ ________ $\{x \mid x = 3n \text{ where } n \in \mathbb{N}\}$

16. $\emptyset$ ________ $\{y \mid y \text{ is a city in Arizona}\}$

17. $\{Ron, Hermione, Harry\}$ ________ $\{x \mid x \text{ is a character in the Harry Potter series}\}$

In Exercises 18 – 28, determine if ⊆, ⊂, both, or neither can be placed in each blank to create a true statement.

18. $\{January, June, July\}$ ________ $\{x \mid x \text{ is a month beginning with the letter } J\}$

19. $\{x \mid x \text{ is a planet in the solar system}\}$ ________ $\{Mercury, Venus, Earth\}$

20. $\{fall, winter, spring, summer\}$ ________ $\{y \mid y \text{ is a season of a year}\}$

21. $\{x \mid x \text{ is a bone in the human hand}\}$ ________ $\{x \mid x \text{ is a bone in the human body}\}$

22. $\{x \mid x \text{ is an animal}\}$ ________ $\{x \mid x \text{ is a human}\}$

23. $\{y \mid y \text{ is a character in Guardians of the Galaxy}\}$ ________
$\{y \mid y \text{ is a character in Avengers}\}$

24. $\{\ \}$ ________ $\{x \mid x \text{ is a resident of Glen Ellyn, IL}\}$

25. $\{4, 5, 6, \ldots, 21\}$ ________ $\{1, 2, 3, \ldots, 100\}$

26. $\{26, 27, 28, \ldots, 74\}$ ________ $\{y \mid y \in \mathbb{N} \text{ and } 25 < y < 75\}$

27. $\{x \mid x = 2n \text{ where } n \in \mathbb{N}\}$ ________ $\{x \mid x = 2n - 1 \text{ where } n \in \mathbb{N}\}$

28. $\{3, 9, 27, \ldots\}$ ________ $\{x \mid x = 3^n \text{ where } n \in \mathbb{N}\}$

In Exercises 29 – 38, list the subsets of each given set.

29. $\{0\}$

30. $\{x \mid x \in \mathbb{N}$ and $x < 2\}$

31. $\{a, c\}$

32. $\{y \mid y \in \mathbb{N}$ and $36 < y \leq 38\}$

33. $\{\pi, e\}$

34. $\{vanilla, chocolate, strawberry\}$

35. $\{Han, Leia, Luke\}$

36. $\{x \mid x \in \mathbb{N}$ and $1 \leq x \leq 3\}$

37. $\{anise\ seed, basil, cumin, dill\}$

38. $\{\quad\}$

In Exercises 39 – 46, determine the number of subsets and proper subsets of each given set.

39. $\{x, y, z\}$

40. $\{5, 10, 15, \ldots, 50\}$

41. $\{Blanche, Dorothy, Rose, Sophia\}$

42. $\{rose, orchid, daisy, lily, carnation\}$

43. $\{x \mid x \in \mathbb{N}\ and\ 25 < x \leq 50\ \}$

44. $\{x \mid x \in \mathbb{N}\ and\ x < 8\}$

45. $\{y \mid y \in \mathbb{N}$ and $-10 \leq y \leq 10\}$

46. $\{y \mid y \in \mathbb{Z}$ and $-10 \leq y \leq 10\}$

In Exercises 47 – 58, determine if each statement is true or false.

47. $red \subseteq \{red, white, blue\}$

48. $Sam \subseteq \{Javier, Luke, Kai, Sam\}$

49. $\{red\} \in \{red, white, blue\}$

50. $\{yellow\} \in \{x \mid x\ is\ a\ color\ of\ the\ rainbow\}$

51. $\emptyset \subseteq \{\emptyset\}$

52. $\emptyset \subseteq \{\ \}$

53. $\emptyset \in \{\emptyset\}$

54. $\emptyset \in \emptyset$

55. $0 \notin \emptyset$

56. $0 \notin \{1, 3, 5, 7, 9\}$

57. $\{1, 2, 2, 3, 3, 3\} \nsubseteq \{1, 2, 3\}$

58. $\{x \mid x\ is\ a\ type\ of\ bird\} \nsubseteq \{\mathrm{owl}, canary, robin, blue\ jay\}$

Applications

59. You can order a hot dog with some, all, or none of these toppings: ketchup, mustard, onions, relish, cheese, tomato wedges, and peppers. How many different hot dogs can be ordered?

60. Gino's East offers 13 different veggie toppings. If a pizza can be ordered with some, all, or none of these veggie toppings, how many different pizzas can be ordered?

61. You have a penny, a nickel, a dime, a quarter, and a half-dollar. How many three-coin subsets are possible, and what is each worth?

62. You plan to visit Spain and the U.S. News and World Report recommends visiting Barcelona, Seville, Madrid, Granada, Ibiza, and Valencia. You only have time to visit four cities on your trip. How many four-city vacations to Spain are possible using this list?

63. If a set has 30 elements, how many subsets are there? If you could write down one subset per second, how many hours would it take for you to list them all? How many days would it take?

64. Pascal's triangle is a pattern of numbers in which each number is the sum of the two numbers immediately above it to the right and to the left. A portion of Pascal's triangle appears below. Additional lines can be added to the triangle by continuing the pattern. The first line is known as "row 0", the second line is known as "row 1", and so on.

    Among many other things, Pascal's triangle can be used to count subsets. For example, row 2 contains the numbers 1, 2, and 1. This tells us that if a set contains two elements, there is 1 subset without any elements, 2 subsets with one element, and 1 subset with two elements. Similarly, row 3 contains the numbers 1, 3, 3, and 1. This tells us that if a set contains three elements, there is 1 subset without any elements, 3 subsets with one element, 3 subsets with two elements, and 1 subset with three elements.

1	row 0
1 1	row 1
1 2 1	row 2
1 3 3 1	row 3
1 4 6 4 1	row 4
1 5 10 10 5 1	row 5

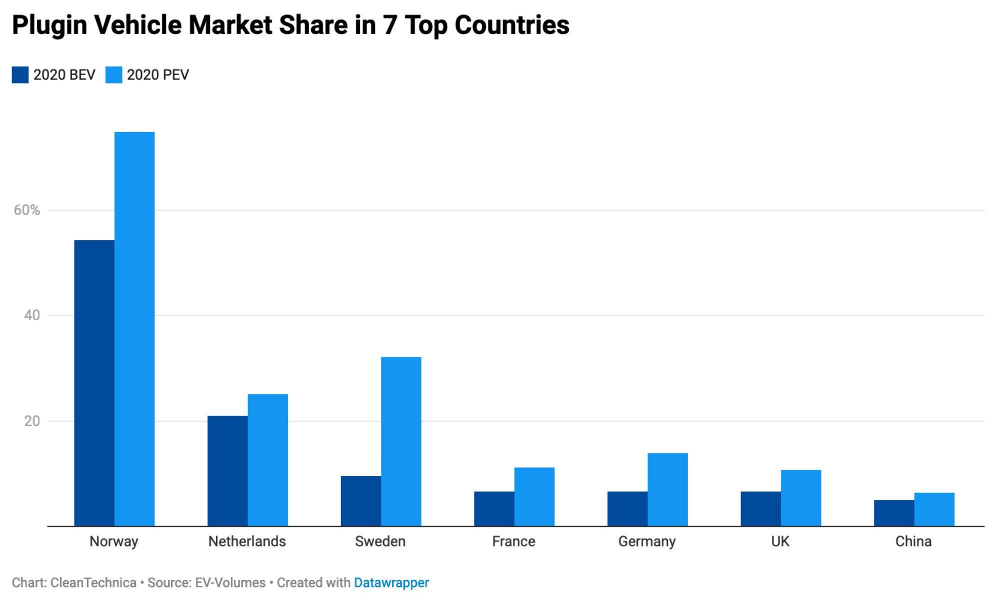
a. Interpret row 4 of Pascal's triangle.

b. If a set contains four elements, how many three-element subsets will the set contain?

c. List the values that would appear in row 6 of Pascal's triangle.

d. There are six cities you would like to visit on a road trip, but only have time to visit four. How many different selections of four cities can you make from the original six on your list? Hint: Use row 6 of Pascal's triangle.

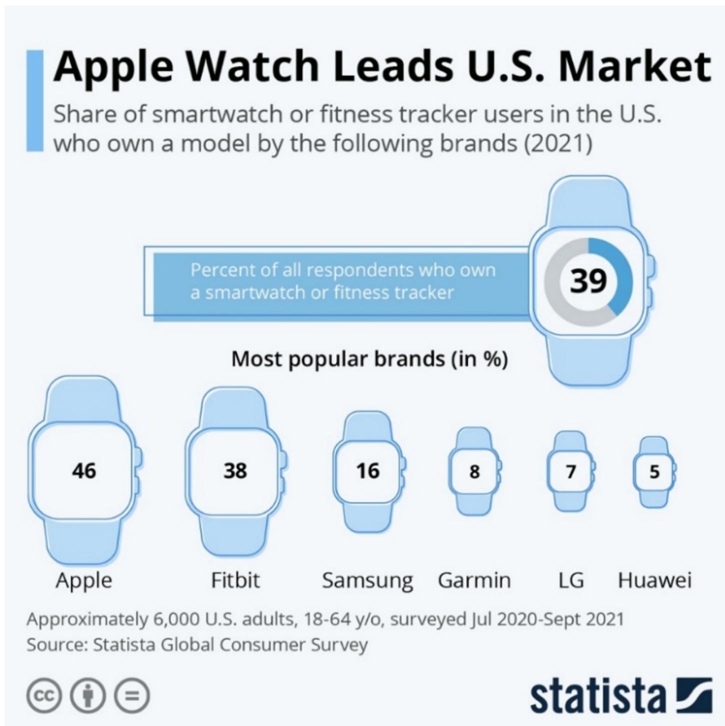
Using data from EV-Volumes, the market share percentages in 2020 for battery-electric vehicles (BEV) and plugin-electric vehicles (PEV) were calculated. The market share percentages for seven countries are displayed in the following graph by CleanTechnica. Use this graph to complete Exercises 54 – 66.



65. How many countries have at least 20% market share in battery-electric vehicles (BEV)? How many subsets are possible using only countries with at least 20% BEV market share?

66. How many countries have at least 20% market share in plugin-electric vehicles (PEV)? How many subsets are possible using only countries with at least 20% PEV market share?

The following infographic displays the most popular brands of smartwatch or fitness trackers in the U.S. in 2021. This data was collected and compiled by Statista. Let $S = \{Apple, Fitbit, Samsung, Garmin, LG, Huawei\}$. Use this infographic to complete Exercises 67 – 70.



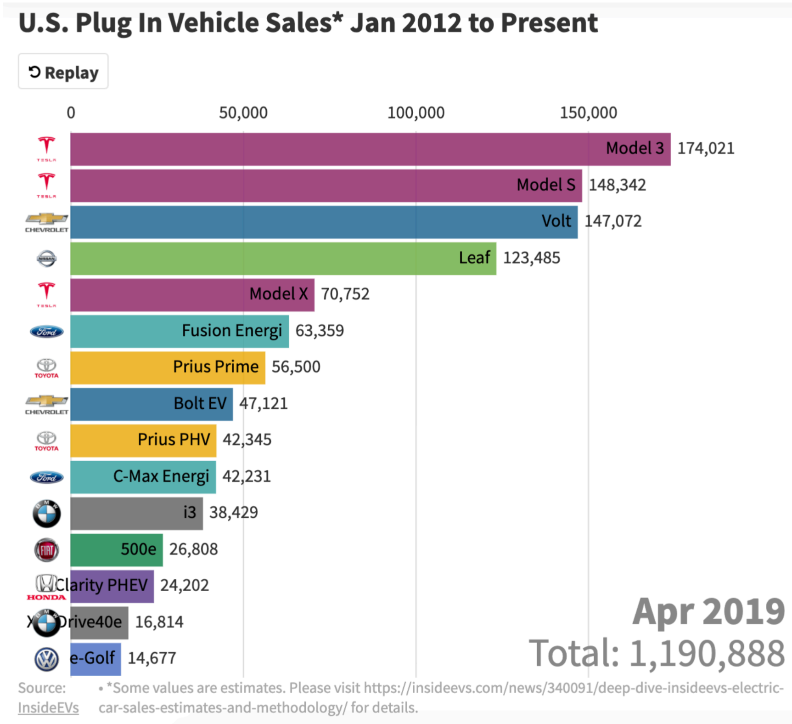
67. Is the following statement true or false? $\{Apple, LG\} \subseteq S$

68. Is the following statement true or false? $Samsung \notin S$

69. Given that the percentages given in the infographic add to more than 100 (they add to 120), some adults must own more than one fitness tracker or smartwatch. Suppose an adult owned two of the most popular branded fitness trackers or smartwatches. Which combinations of two brands could that adult possibly own? (That is, what are all the two-element subsets of S?)

70. How many of the approximately $6{,}000$ adults surveyed own a fitness tracker or smartwatch?

The following graph from InsideEVs displays U.S. plugin vehicle sales from January 2012 to April 2019. Use this graph to complete Exercises 71 – 72.



71. How many different electric vehicle models have at least 100,000 sales from January 2012 to April 2019? How many subsets are possible using only vehicles with at least 100,000 vehicles sold?

72. How many different electric vehicle models have at least 50,000 sales from January 2012 to April 2019? How many subsets are possible using only vehicles with at least 50,000 vehicles sold?

Concept Review

73. Give an example of two sets that are equivalent, but not equal. If this is not possible, explain why.

74. Give an example of two sets that are equal, but not equivalent. If this is not possible, explain why.

75. True or false? The set $\{0, 1, 2, \dots, 1000\}$ has $2^{1,000}$ subsets.

76. Is it possible to create a one-to-one correspondence for finite sets A and B if $A \subset B$? Why or why not?

77. Give an example of two sets A and B such that $A \subset B$ and $A \nsubseteq B$. If this is not possible, explain why.

78. Give an example of two sets A and B such that $A \subseteq B$ and $A \not\subset B$. If this is not possible, explain why.

79. A topic in statistics called bootstrapping involves set theory. With bootstrapping, statisticians use the original set of data to produce equivalent sets consisting of the same elements from the original set. Here, repeated elements are allowed and are counted as distinct elements. For example, $\{1, 1, 1, 1\}$, $\{2, 2, 2, 2\}$, $\{2, 4, 2, 4\}$, and $\{1, 3, 4, 1\}$ are bootstrap samples from the original set $\{1, 2, 3, 4\}$.

    a. Which of the following are bootstrap samples of $\{a, b, c, d\}$? (You may select more than one.)
        i. $\{a, a, a, a\}$
        ii. $\{a, b, b, c\}$
        iii. $\{a, b, c, f\}$
        iv. $\{a, b\}$
        v. $\{g\}$

    b. List two additional bootstrap samples of the set $\{a, b, c, d\}$. (There are many possible correct answers.)

    c. Is the set $\{a, b, c, d\}$ a bootstrap sample of $\{a, b, c, d\}$? Why or why not?

Section 1.2 | Exercise Solutions

1. Neither
2. Equivalent
3. Equivalent
4. Both
5. Equivalent
6. Neither
7. Both
8. Equivalent
9. Neither
10. $\not\subseteq$
11. $\not\subseteq$
12. $\subseteq$
13. $\not\subseteq$
14. $\subseteq$
15. $\not\subseteq$
16. $\subseteq$
17. $\subseteq$
18. $\subseteq$
19. Neither
20. $\subseteq$
21. Both
22. Neither
23. Neither
24. Both
25. Both
26. $\subseteq$
27. Neither
28. $\subseteq$
29. $\{\quad\}, \{0\}$
30. $\{\quad\}, \{1\}$
31. $\{\quad\}, \{a\}, \{c\}, \{a, c\}$
32. $\{\quad\}, \{37\}, \{38\}, \{37, 38\}$
33. $\{\quad\}, \{\pi\}, \{e\}, \{\pi, e\}$
34. $\{\quad\}, \{vanilla\}, \{chocolate\}, \{strawberry\}, \{vanilla, chocolate\}, \{vanilla, strawberry\}, \{chocolate, strawberry\}, \{vanilla, chocolate, strawberry\}$
35. $\{\quad\}, \{Han\}, \{Leia\}, \{Luke\}, \{Han, Leia\}, \{Han, Luke\}, \{Leia, Luke\}, \{Han, Leia, Luke\}$
36. $\{\quad\}, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}$
37. $\{\quad\}, \{anise\}, \{basil\}, \{cumin\}, \{dill\}, \{anise, basil\}, \{anise, cumin\}, \{anise, dill\}, \{basil, cumin\}, \{basil, dill\}, \{cumin, dill\}, \{anise, basil, cumin\}, \{anise, basil, dill\}, \{anise, cumin, dill\}, \{basil, cumin, dill\}, \{anise, basil, cumin, dill\}$
38. $\{\quad\}$
39. Subsets: $2^3 = 8$
    Proper subsets: $2^3 - 1 = 7$
40. Subsets: $2^{10} = 1{,}024$
    Proper subsets: $2^{10} - 1 = 1{,}023$
41. Subsets: $2^4 = 16$
    Proper subsets: $2^4 - 1 = 15$
42. Subsets: $2^5 = 32$
    Proper subsets: $2^5 - 1 = 31$
43. Subsets: $2^{25} = 33{,}554{,}432$
    Proper subsets: $2^{25} - 1 = 33{,}554{,}431$
44. Subsets: $2^7 = 128$
    Proper subsets: $2^7 - 1 = 127$
45. Subsets: $2^{10} = 1{,}024$
    Proper subsets: $2^{10} - 1 = 1{,}023$
46. Subsets: $2^{21} = 2{,}097{,}152$
    Proper subsets: $2^{21} - 1 = 2{,}097{,}151$

47. False
48. False
49. False
50. False
51. True
52. True
53. True
54. False
55. True
56. True
57. False
58. True
59. $2^7 = 128$
60. $2^{13} = 8{,}192$
61. There are 10 three-coin subsets. They are worth $16¢, 31¢, 36¢, 40¢,$ $56¢, 61¢, 65¢, 76¢, 80¢$, and $85¢$.
62. There are 15 vacations to four cities from the recommendation list.
63. There are $2^{30} = 1{,}073{,}741{,}824$ subsets. It would take approximately 298,261.6 hours or 12,427.6 days to list them all.
64. a. If a set contains four elements, there is 1 subset without any elements, 4 subsets with one element, 6 subsets with two elements, 4 subsets with three elements, and 1 subset with four elements.
    b. 4
    c. 1, 6, 15, 20, 15, 6, 1
    d. There are 15 four-city subsets
65. There are two countries with $2^2 = 4$ subsets.
66. There are three countries with $2^3 = 8$ subsets.
67. True
68. False
69. $\{Apple, FitBit\}, \{Apple, Samsung\}, \{Apple, Garmin\}, \{Apple, LG\}, \{Apple, Huawei\}, \{FitBit, Samsung\}, \{FitBit, Garmin\}, \{FitBit, LG\}, \{FitBit, Huawei\}, \{Samsung, Garmin\}, \{Samsung, LG\}, \{Samsung, Huawei\}, \{Garmin, LG\}, \{Garmin, Huawei\}, \{LG, Huawei\}$
70. $6{,}000 * 0.39 = 2{,}340$
71. There are four vehicles with $2^4 = 16$ subsets.
72. There are seven vehicles with $2^7 = 128$ subsets.
73. There are many correct answers. For example, the sets $\{a, b, c\}$ and $\{1, 2, 3\}$ are equivalent, but not equal.
74. This is not possible. If two sets are equal, they will automatically contain the same number of distinct elements. So, they will also be equivalent.
75. False. There are $2^{1{,}001}$ subsets.
76. It is not possible. If $A \subset B$, then every element of set A is also an element of set B, and sets A and B are not equal. This means that $n(A) < n(B)$. So, a one-to-one correspondence cannot be created.
77. It is not possible. If $A \subset B$, then every element of set A is also an element of set B. By definition, it follows that A is a subset of B.
78. There are many correct answers. It is possible for A to be a subset of B but not a proper subset of B. For example, consider sets $A = \{1, 2, 3\}$ and $B = \{3, 2, 1\}$. $A \subseteq B$ but A is not a proper subset of B.
79. a. i and ii
    b. $\{a, a, b, c\}$ and $\{d, d, d, d\}$
    c. Yes. It is equivalent to the original set.

Section 1.3 | Venn Diagrams

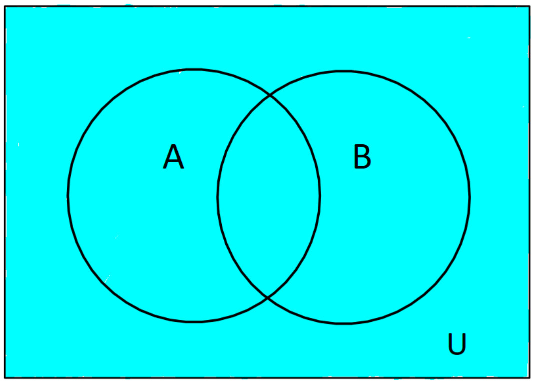
Objectives

- Define universal set, union, intersection, complement, and set difference for two sets and three sets
- Perform operations with sets using the above definitions
- Create Venn diagrams with two and three sets
- Use set notation to name sets given visual depictions of diagrams
- Use Venn diagrams to display set relationships

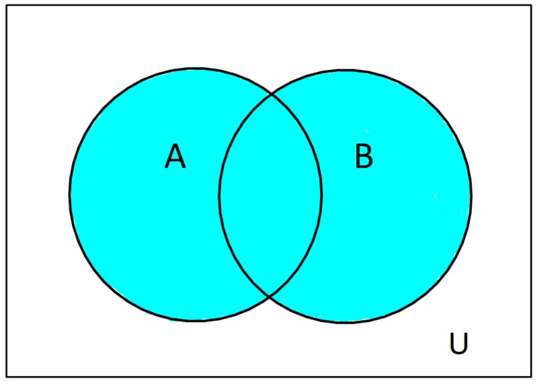
Recall from Section 1.2 that Venn diagrams can be used to visually represent the relationships among sets. This section studies the relationships between two sets and among three sets. Sets within Venn diagrams are usually represented using circles or ovals and are enclosed in a rectangle. This discussion starts with two sets then continues onto three sets.

Definitions for Two Sets

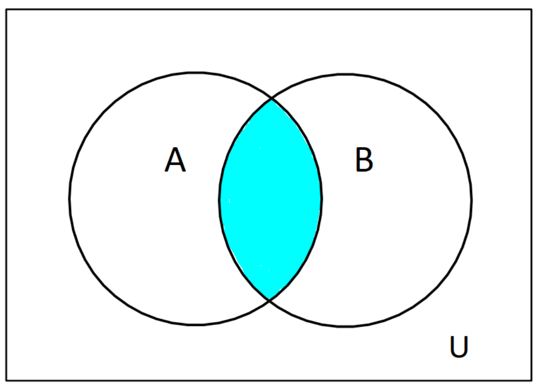
DEFINITION: The **universal set** contains all of the elements under consideration in a given situation. The universal set can be symbolized as U. Given two sets A and B, the universal set can be visualized as follows. The shaded region represents the universal set U.



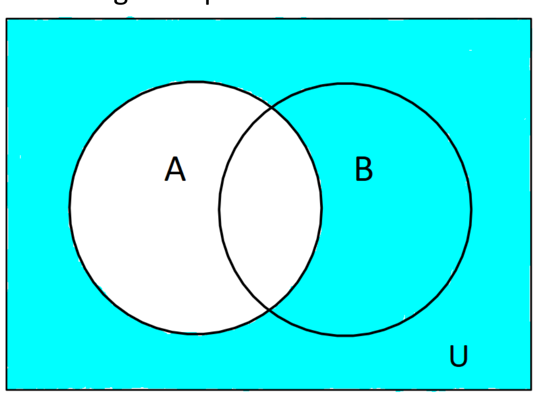
DEFINITION: Given two sets A and B, the **union** of sets A and B contains the elements in set A *OR* set B *OR* in both set A and set B. The union of sets A and B is symbolized as $A \cup B$. The union of two sets A and B can be visualized as follows. The shaded region represents $A \cup B$.



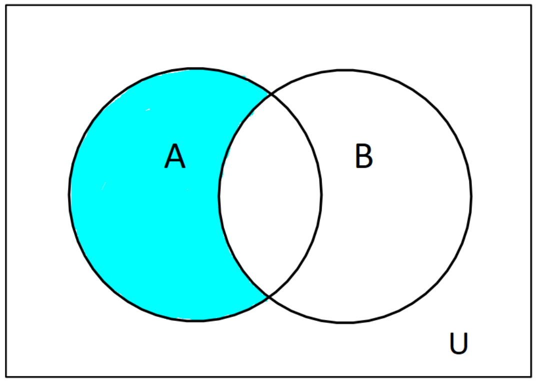
DEFINITION: Given two sets A and B, the **intersection** of sets A and B contains the elements in both set A *AND* set B. The intersection of sets A and B is symbolized as $A \cap B$. The intersection of two sets A and B can be visualized as follows. The shaded region represents $A \cap B$.



DEFINITION: The **complement** of set A contains the elements that are in the universal set but not in set A. The complement of set A is symbolized as A' or $\bar{A}$ or A^C. For this textbook, we will use the symbolization A' to represent the complement of A. The complement of set A can be visualized as follows. The shaded region represents A'.



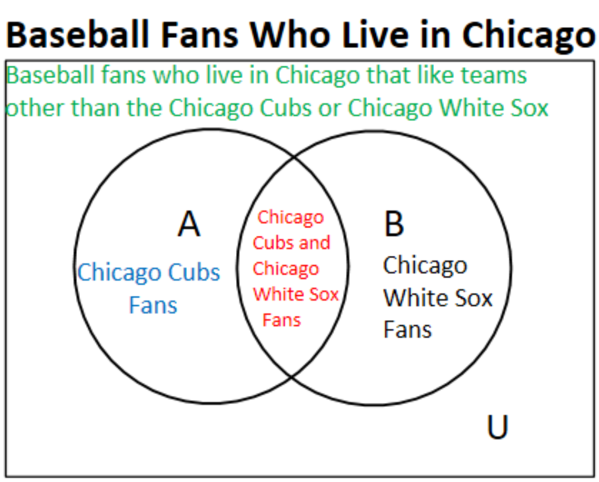
DEFINITION: Given sets A and B, the **set difference** of sets A and B contains the elements in set A but not in set B. The set difference of sets A and B is symbolized as $A - B$. The set difference of sets $A - B$ can be visualized as follows. The shaded region is $A - B$.



- **EXAMPLE 1.3.1:** Let the universal set U be baseball fans who live in Chicago. Let set A be Chicago Cubs fans. Let set B be Chicago White Sox fans. Create a Venn diagram and answer the following:

  a. Who would $A \cup B$ represent?
  b. Who would $A \cap B$ represent?
  c. Who would $A - B$ represent?
  d. Who would $B - A$ represent?
  e. Who would $U - (A \cup B)$ represent?
  f. Who would A' represent?
  g. Who would B' represent?

  SOLUTION:



a. $A \cup B$ represents those baseball fans who live in Chicago that like the Chicago Cubs or Chicago White Sox or both the Chicago Cubs and Chicago White Sox.

b. $A \cap B$ represents only those baseball fans who live in Chicago that like the Chicago Cubs and the Chicago White Sox.

c. $A - B$ represents those baseball fans who live in Chicago that like only the Chicago Cubs.

d. $B - A$ represents those baseball fans who live in Chicago that like only the Chicago White Sox.

e. $U - (A \cup B)$ represents those baseball fans who live in Chicago that like teams other than the Chicago Cubs or Chicago White Sox. For example, these are baseball fans who live in Chicago that like teams such as the Boston Red Sox, the St. Louis Cardinals, or the New York Yankees.

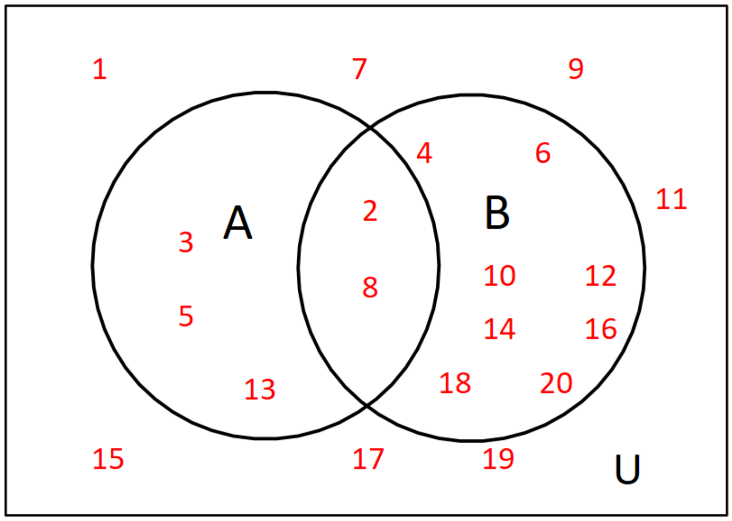
f. A' represents those baseball fans who live in Chicago that like teams other than the Chicago Cubs or Chicago White Sox and those baseball fans that like only the Chicago White Sox.

g. B' represents those baseball fans who live in Chicago that like teams other than the Chicago Cubs or Chicago White Sox and those baseball fans that like only the Chicago Cubs.

➢ **EXAMPLE 1.3.2:** Let $U = \{x | x \in \mathbb{N} \ and \ x \leq 20\}$. If $A = \{2, 3, 5, 8, 13\}$ and $B = \{2, 4, 6, 8, 10, 12, 14, 16, 18, 20\}$. Find the following:

a. $A \cup B$
b. $A \cap B$
c. $A - B$
d. $B - A$
e. $U - B$
f. A'

SOLUTION: It would be highly beneficial to create a Venn diagram of the example. Below is a visual representation of the example.



a. The union of A and B are all the elements in set A and in set B combined. So, the solution is $A \cup B = \{2, 3, 4, 5, 6, 8, 10, 12, 13, 14, 16, 18, 20\}$.

b. The intersection of sets A and B are the elements that are in common in both sets A and B. The solution is $A \cap B = \{2,8\}$.

c. To find the set difference $A - B$, take out the elements from set B that are in set A. $A = \{2, 3, 5, 8, 13\}$ $B = \{2, 4, 6, 8, 10, 12, 14, 16, 28, 2\}$. Take out the element 2 and 8 from set A. The solution is $A - B = \{3, 5, 13\}$

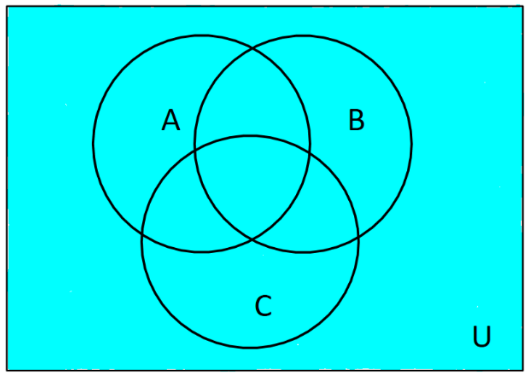
d. Similarly, to find the set difference $B - A$, take out the elements from set A that are in set B. The solution is $B - A = \{4, 6, 10, 12, 14, 16, 18, 20\}$.

e. Similar to parts c and d, to find the set difference $U - B$, take out the elements that are in set B that are in the universal set U. The solution is $U - B = \{1, 3, 5, 7, 9, 11, 13, 15, 17, 19\}$.

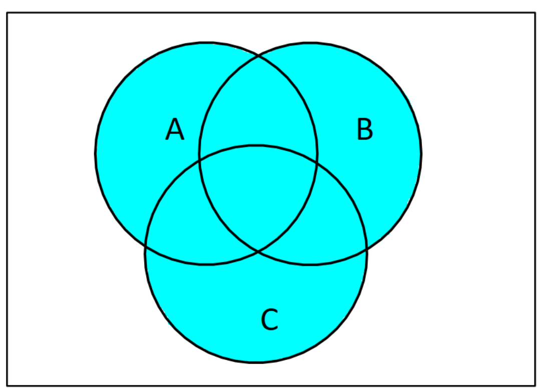
f. To find the complement of A, find the elements that are in the universal set U that are not in set A. The solution is $A' = \{1, 4, 6, 7, 9, 10, 11, 12, 14, 15, 16, 17, 18, 19, 20\}$.

Definitions for Three Sets

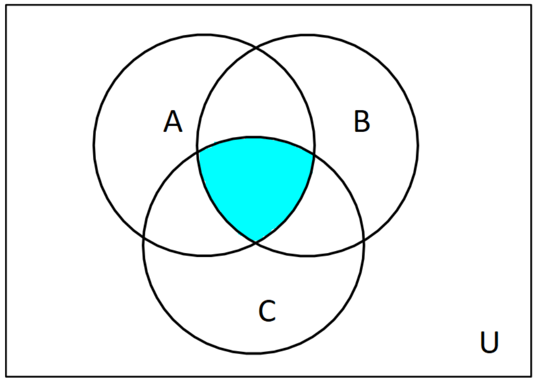
DEFINITION: The **universal set** contains all the elements under consideration in a given situation. The universal set can be symbolized as U. Given three sets A, B, and C, the universal set can be visualized as follows. The shaded region represents the universal set U.



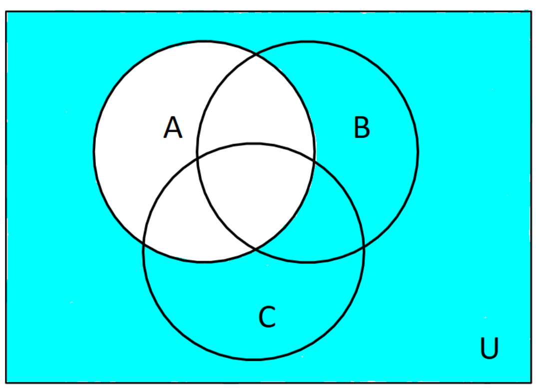
DEFINITION: Given three sets A, B, and C, the **union** of sets A, B, and C contains the elements in set A *OR* set B *OR* set C *OR* in any of their intersections. The union of sets A, B, and C is symbolized as $A \cup B \cup C$. The union of three sets A, B, and C can be visualized as follows. The shaded region represents $A \cup B \cup C$.



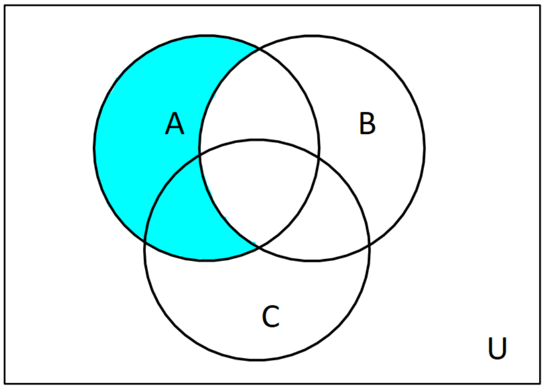
DEFINITION: Given three sets A, B, and C, the **intersection** of sets A, B, and C contains the elements in set A *AND* set B *AND* set C. The intersection of sets A, B, and C is symbolized as $A \cap B \cap C$. The intersection of three sets A, B, and C can be visualized as follows. The shaded region represents $A \cap B \cap C$.



DEFINITION: Given sets A, B, and C, the **complement** of set A contains the elements that are in the universal set but not in set A. The complement of set A is symbolized as A'. The complement of set A can be visualized as follows. The shaded region represents A'.



DEFINITION: Given sets A, B, and C, the **set difference** of sets A and B contains the elements in set A but not in set B. The set difference of sets A and B is symbolized as $A - B$. The set difference of set $A - B$ can be visualized as follows. The shaded region represents $A - B$.



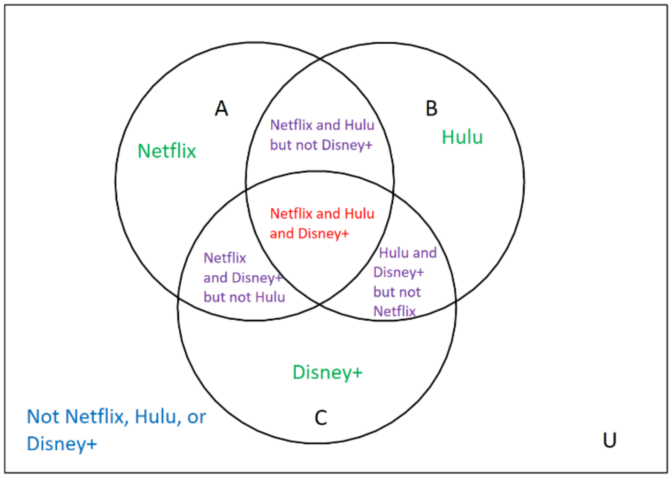
Note: When working with set operations such as $A \cup B \cap C$, work from left to right unless grouping symbols such as parentheses are given.

- **EXAMPLE 1.3.3:** Let the universal set U be students who use streaming television services. Let set A be students that use Netflix. Let set B be students that use Hulu. Let set C be students that use Disney+. Create a Venn diagram and answer the following:

  a. Who would $A \cup B \cap C$ represent?
  b. Who would $A \cap B \cap C$ represent?
  c. Who would $A \cap B - C$ represent?
  d. Who would $A \cup C - B$ represent?
  e. Who would $U - (A \cup B \cup C)$ represent?

SOLUTION:

Students Who Use Streaming Television Services



a. $A \cup B \cap C$ represents those students who use Netflix and Disney+ but not Hulu, students who use Netflix and Hulu and Disney+, and students who use Hulu and Disney+ but not Netflix. To interpret $A \cup B \cap C$, first find $A \cup B$, then find the intersection with C.

b. $A \cap B \cap C$ represents those students who use Netflix and Hulu and Disney+.

c. $A \cap B - C$ represents those students who use Nextflix and Hulu but not Disney+.

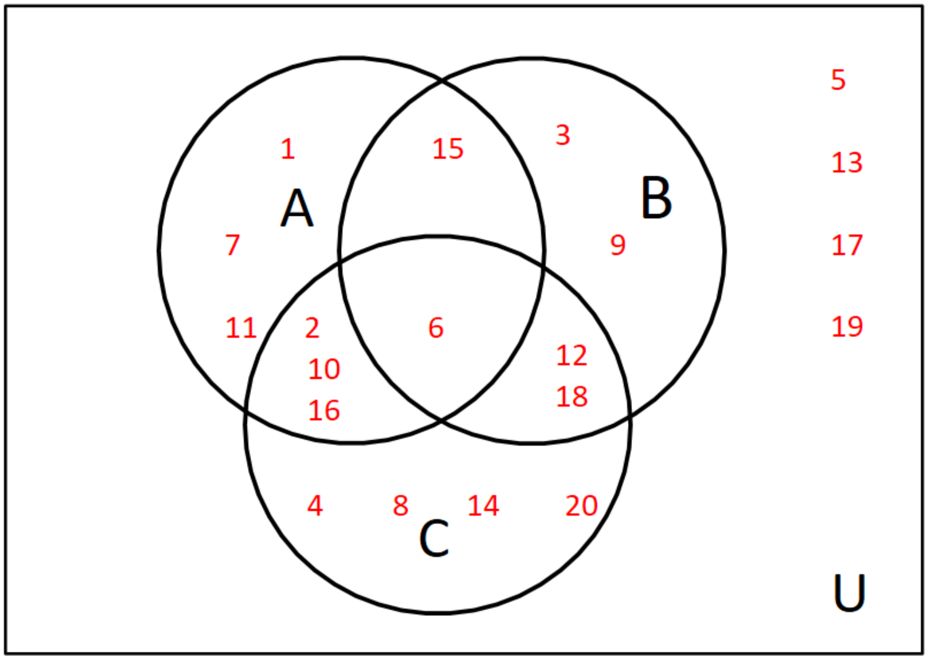
d. $A \cup C - B$ represents those students who use only Netflix, only Disney+, or Netflix and Disney+ but not Hulu.

e. $U - (A \cup B \cup C)$ represents those students who do not use Netflix, Hulu or Disney+. For example, this could be those students who use Amazon Prime Video, HBO Max, or no streaming service.

➢ **EXAMPLE 1.3.4:** Let $U = \{x | x \in \mathbb{N} \; and \; x \leq 20\}$. If $A = \{1, 2, 6, 7, 10, 11, 15, 16\}$, $B = \{x | x \; is \; a \; multiple \; of \; 3\}$, and $C = \{x | x \; is \; multiple \; of \; 2\}$. Find the following:

a. $A' \cup B' \cap C$
b. $(A \cap B) - C$
c. $B - (A' \cap C)$
d. $U - (A \cup B \cup C) - (A' \cap B' \cap C')$

SOLUTION: Again, as before, it would be highly beneficial to create a Venn diagram of the example. Below is a visual representation of the example.



a. Start off by finding A' and B'. Then, find the union of A' and B'.
$A' = \{3, 4, 5, 8, 9, 12, 13, 14, 17, 18, 19, 20\}$
$B' = \{1, 2, 4, 5, 7, 8, 10, 11, 13, 14, 16, 17, 19, 20\}$
$A' \cup B' = \{1, 2, 3, 4, 5, 7, 8, 9, 10, 11, 12, 13, 14, 16, 17, 18, 19, 20\}$
Now, intersect $A' \cup B'$ with C to get
$A' \cup B' \cap C = \{2, 4, 8, 10, 12, 14, 16, 18, 20\}$

b. First find the intersection of A and B, then find the set difference of $A \cap B$ and C. $A \cap B = \{6, 15\}$. For the set difference, take out any values in $A \cap B$ that are in C. In this case, take out the element 6. The solution is $(A \cap B) - C = \{15\}$.

c. First find A' then intersect A' with C. $A' = \{3, 4, 5, 8, 9, 12, 13, 14, 17, 18, 19, 20\}$
$A' \cap C = \{4, 8, 12, 14, 18, 20\}$ To find $B - (A' \cap C)$ take out any elements that are in $B = \{3, 6, 9, 12, 15, 18\}$ that are also in $A' \cap C$. The solution is
$B - (A' \cap C) = \{3, 6, 9, 15\}$.

d. First, find the union of A, B, and C. Then, find the intersection of A', B', and C'.
$A \cup B \cup C = \{1, 2, 3, 4, 6, 7, 8, 9, 10, 11, 12, 14, 15, 16, 18, 20\}$.
$A' \cap B' \cap C' = \{5, 13, 17, 19\}$. Then, from U take out the elements from $A \cup B \cup C$ and $A' \cap B' \cap C'$. The solution is $U - (A \cup B \cup C) - (A' \cap B' \cap C') = \emptyset$

➢ **YOU TRY IT 1.3.A:**

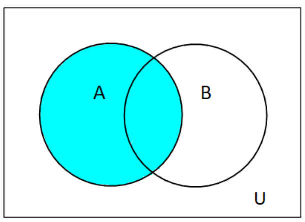
a. $A - (B \cup C)$
b. $U - (B' \cup C')$
c. $A' \cap B' \cap C'$

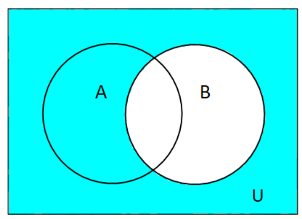
➢ **EXAMPLE 1.3.5:** Create Venn diagrams for the following sets.

a. $A \cup B'$
b. $A' \cap B \cap C'$

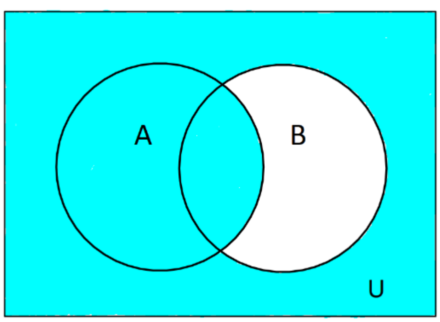
SOLUTION:

a. To find $A \cup B'$, first shade set A, then shade set B'. Then combine the two shaded regions. Below on the left is set A shaded. Below on the right is set B' shaded.

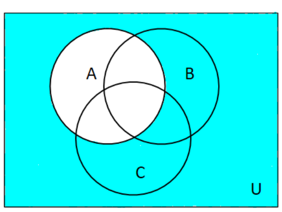


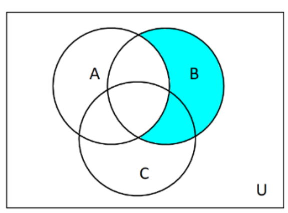


The shaded regions below is the solution to $A \cup B'$.

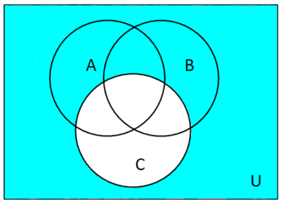


b. To find $A' \cap B \cap C'$, first find the intersection of $A' \cap B$, then find the intersection of $A' \cap B$ and C'. Below on the left is set A' shaded. Below on the right is set $A' \cap B$ shaded.

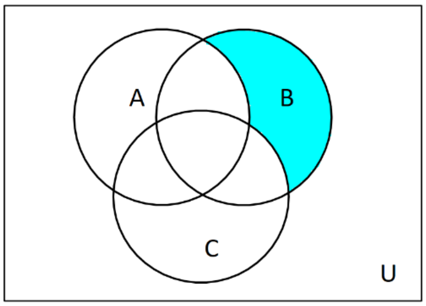




The set C' shaded is visualized as the Venn diagram below. Now find the intersection of $A' \cap B \cap C'$.



The shaded region below is the solution to $A' \cap B \cap C'$.

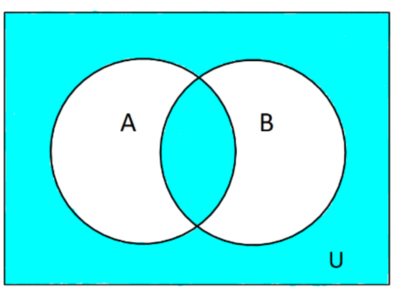


YOU TRY IT 1.3.B

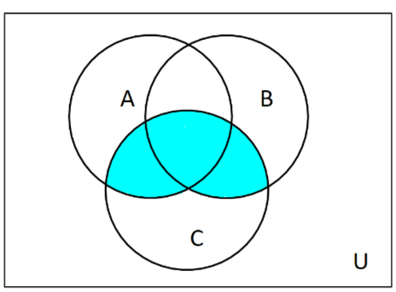
a. $A' \cap B$
b. $A \cup B' \cup C$

- **EXAMPLE 1.3.6:** Given the shaded regions, find two set representations for each set. Note that there may be more than two representations for each set.

a.



b.



SOLUTION:

a. Notice that there are two separate regions in the diagram. One region is the outside of the sets A and B. That can be represented as $U - (A \cup B)$. The other region is the intersection of A and B. That region would be represented as $A \cap B$. A solution to our diagram is $U - (A \cup B) \cup (A \cap B)$.

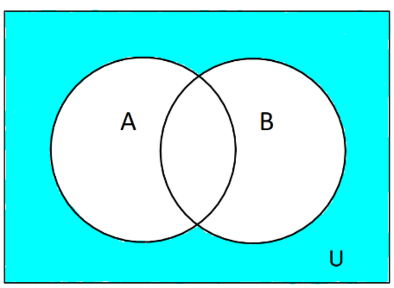
Another solution can be found by noticing that region outside of the sets A and B can be represented as the intersection of A' and B'. Again, the second region would be A intersected with B. A second solution to our diagram is $(A' \cap B') \cup (A \cap B)$.

b. Notice the regions that are shaded. One interpretation of the diagram is $A \cup B$ is intersected with C. That would give a solution of $A \cup B \cap C$.

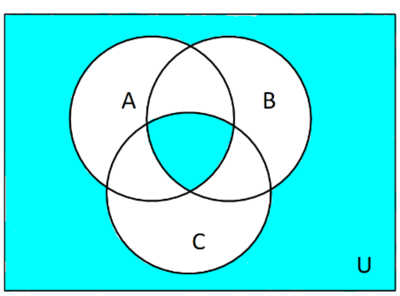
Another interpretation is that A is intersected with C and B is intersected with C. Combining the two regions would give a solution of $(A \cap C) \cup (B \cap C)$.

YOU TRY IT 1.3.C:

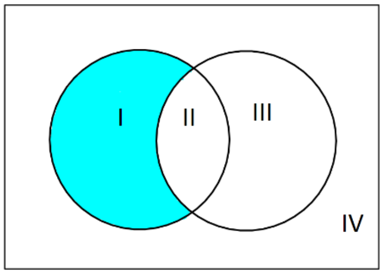
a.

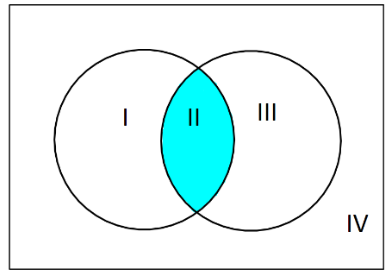


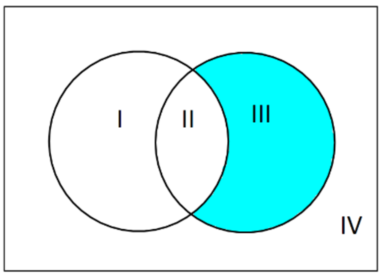
b.

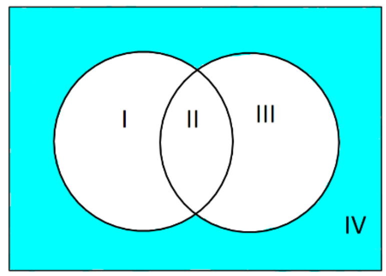


FINAL THOUGHTS: Given two sets that are not disjoint, there are four regions that comprise of the universal set. Often, we label these regions with Roman numerals. Below are the shaded visual representations of each region.

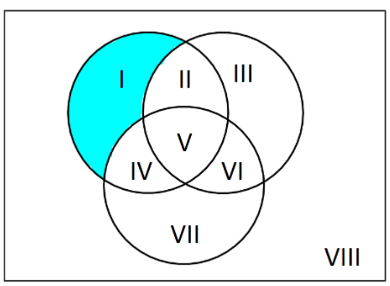


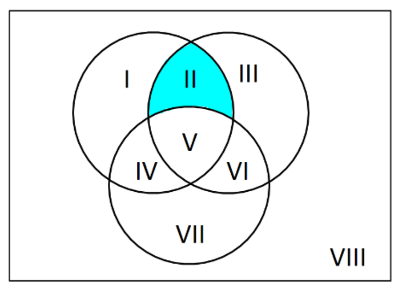


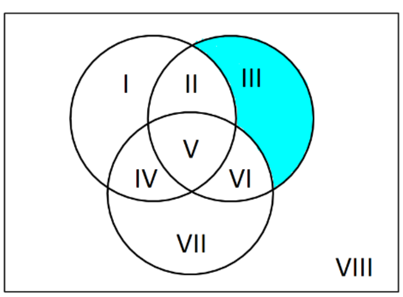


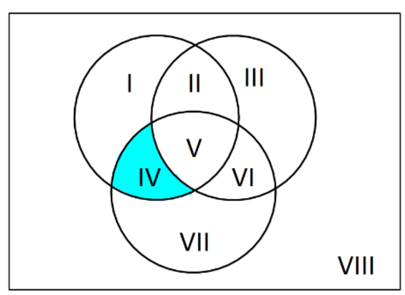


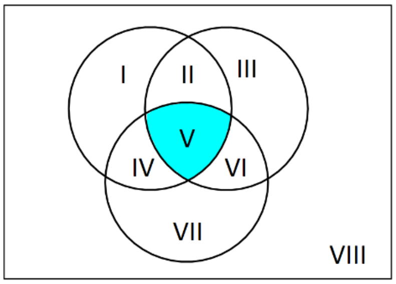
Similarly, given three sets that are not disjoint, there are eight regions that comprise of the universal set. Below and on the next page are the shaded visual representations of each region.

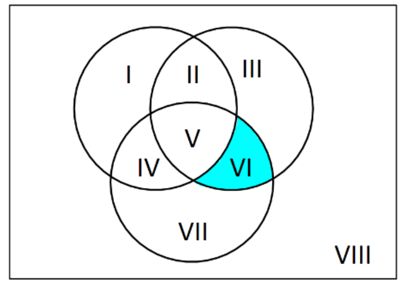


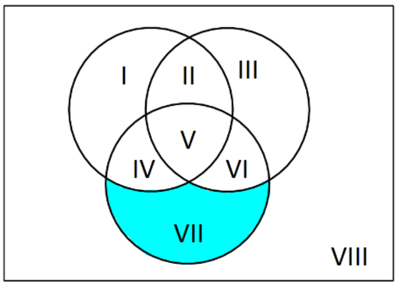


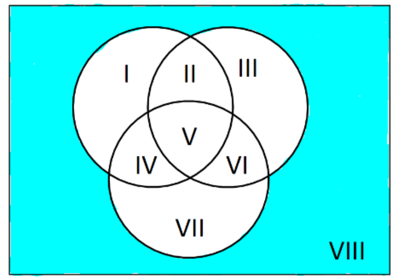










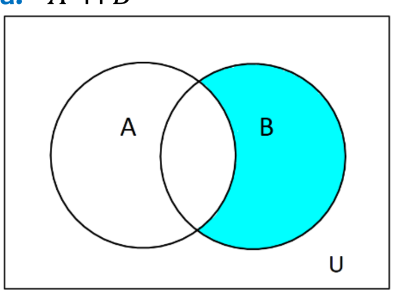


YOU TRY IT 1.3.A SOLUTION:

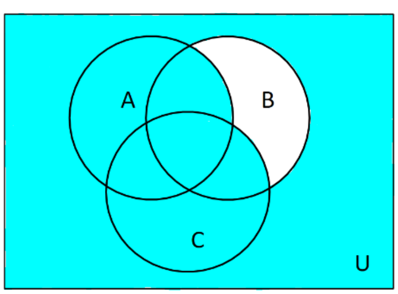
a. $A - (B \cup C) = \{1, 7, 11\}$
b. $U - (B' \cup C') = \{6, 12, 18\}$
c. $A' \cap B' \cap C' = \{5, 13, 17, 19\}$

YOU TRY IT 1.3.B SOLUTION:

a. $A' \cap B$



b. $A \cup B' \cup C$



YOU TRY IT 1.3.C SOLUTION:

a. $U - (A \cup B)$ and $A' \cap B'$

b. $U - (\mathrm{A} \cup \mathrm{B} \cup \mathrm{C}) \cup (\mathrm{A} \cap \mathrm{B} \cap \mathrm{C})$ and $(A' \cap B' \cap C') \cup (A \cap B \cap C)$

Section 1.3 Exercises

For exercises 1 – 4, create a Venn diagram and answer the questions regarding two sets.

1. Let the universal set consist of movie goers. Let set A represent movie goers who like action movies. Let set B represent movie goers who like dramas.

   a. Who would $A \cup B$ represent?
   b. Who would $A \cap B$ represent?
   c. Who would A' represent?
   d. Who would $U - A$ represent?
   e. Who would $U - (A \cap B)$ represent?

2. Let the universal set consist of married couples. Let set A represent couples who celebrate Sweetest Day. Let set B represent couples who celebrate Valentine's Day.

   a. Who would $A \cup B$ represent?
   b. Who would $A \cap B$ represent?
   c. Who would B' represent?
   d. Who would $U - B$ represent?
   e. Who would $U - (A \cup B)$ represent?

3. Let the universal set consist of people who enjoy playing sports. Let set A represent people who enjoy playing volleyball. Let set B represent people who enjoy playing soccer.

   a. Who would $(A \cup B)'$ represent?
   b. Who would $(A \cap B)'$ represent?
   c. Who would $A \cup B'$ represent?
   d. Who would $U - A'$ represent?
   e. Who would $U - (A \cup B)'$ represent?

4. Let the universal set consist of people who enjoy travel. Let set A represent people who have been to New York City. Let set B represent people who have been to Washington D.C.

    a. Who would $A \cup B$ represent?
    b. Who would $(A \cap B)'$ represent?
    c. Who would $A' \cap B'$ represent?
    d. Who would $U - B$ represent?
    e. Who would $U - (A \cap B)$ represent?

For exercises 5 – 9, create a Venn diagram and answer the questions regarding three sets.

5. Let the universal set consist of people at a fitness gym. Let set A represent people who run. Let set B represent people who practice yoga. Let set C represent people who lift weights.

    a. Who would $A \cup B \cap C$ represent?
    b. Who would $A \cap B \cap C'$ represent?
    c. Who would $A \cap B - C$ represent?
    d. Who would $A \cup C - B'$ represent?
    e. Who would $U - (A \cup B \cup C)$ represent?

6. Let the universal set consist of students attending a community college. Let set A represent students who use Snapchat. Let set B represent students who use Instagram. Let set C represent students who use TikTok.

    a. Who would $A \cap B \cap C$ represent?
    b. Who would $A' \cup B \cup C$ represent?
    c. Who would $A \cup B - C$ represent?
    d. Who would $A \cap C - B'$ represent?
    e. Who would $U - (A \cap B \cap C)$ represent?

7. Let the universal set consist of pet owners. Let set A represent those who have cats. Let set B represent those who have dogs. Let set C represent those who have reptiles.

    a. Who would $A \cup B \cup C$ represent?
    b. Who would $(A \cup B)' \cup C$ represent?
    c. Who would $(A \cap C)' - B$ represent?
    d. Who would $B' \cup (C - A')$ represent?
    e. Who would $U - (A \cap B \cap C)'$ represent?

8. Let the universal set consist of music enthusiasts. Let set A represent those who enjoy rock. Let set B represent those who enjoy hip hop. Let set C represent those who enjoy country.

    a. Who would $(A \cup B \cup C)'$ represent?
    b. Who would $(A \cap B) \cup (A \cap C)$ represent?
    c. Who would $C - (A' \cap C)'$ represent?
    d. Who would $(C' \cup B) \cap (A \cap B')$ represent?
    e. Who would $U - (A \cup B)'$ represent?

9. Let the universal set consist of people who enjoy travel. Let set A represent people who have been to Chicago. Let set B represent people who have been to Los Angeles. Let set C represent people who have been to Miami.

    a. Who would $(A \cup B) \cap C$ represent?
    b. Who would $(A \cap C) - B$ represent?
    c. Who would $U - (A \cap B \cap C)$ represent?
    d. Who would $(A \cap B) \cup (A \cap C)$ represent?
    e. Who would $(A \cup B)' \cap C'$ represent?

For exercises 10 – 15, answer the questions given the descriptions of each set.

10. Let $U = \{x | x \in \mathbb{N} \; and \; x \leq 30\}$, $A = \{2, 5, 11, 17, 23, 28\}$, and $B = \{2, 5, 9, 12, 15, 19, 22, 25, 29\}$.
    a. B'
    b. $A \cup B$
    c. $A \cap B'$
    d. $B - A'$
    e. $(A - B)'$
    f. $U - (A' \cup B)$

11. Let $U = \{x | x \in \mathbb{N} \; and \; x \leq 23\}$, $A = \{1, 7, 8, 9, 13, 15, 19, 23\}$, $B = \{2, 4, 7, 13, 14, 18, 22, 23\}$, and $C = \{3, 6, 9, 13, 14, 16, 19, 21, 22, 23\}$.
    a. $A' \cup B' \cap C$
    b. $(A \cap B) - C$
    c. $B - (A' \cap C)$
    d. $U - (A \cup B \cup C) - (A' \cap B' \cap C')$

12. Let $U = \{1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144\}$, $A = \{1, 3, 5, 13, 21, 55, 89\}$, $B = \{2, 8, 34, 144\}$, and $C = \{5, 21, 55, 144\}$.
    a. $A \cup (B \cap C)$
    b. $(A \cup B) \cap (B \cup C)$
    c. $C' \cup (B - A)'$
    d. $(A - B') \cap (A' \cap C')$

13. Let $U = \{1, 2, 3, \dots, 20\}$, $A = \{1, 2, 5, 7, 13, 15, 18, 20\}$, $B = \{1, 3, 7, 9, 11, 13\}$, and $C = \{4, 8, 12, 16, 19, 20\}$.
    a. $(A \cup C) \cap B$
    b. $(A \cap B) - (B \cap C)$
    c. $(C' - B) \cap A'$
    d. $(A \cup C')' \cap (B' \cap C)$

14. Let $U = \{x | x \in \mathbb{N}, x\ is\ a\ multiple\ of\ 2, and\ x \leq 30\}$, $A = \{x | x\ is\ a\ multiple\ of\ 4\}$, and $B = \{x | x\ is\ a\ multiple\ of\ 6\}$.
    a. $A \cup B$
    b. $A \cap B$
    c. $A - B$
    d. $B' - A$
    e. $U - B$
    f. A'

15. Let $U = \{x | x \in \mathbb{N}\ and\ x \leq 30\}$, $A = \{x | x\ is\ odd\}$, $B = \{1, 2, 3, 5, 8, 13, 21\}$, and $C = \{x | x\ is\ a\ multiple\ of\ 5\}$
    a. $A \cup B \cap C$
    b. $A' \cap B$
    c. $(B \cup C)' \cap A$
    d. $U - (A \cup B)$

For exercises 16 – 21, create Venn diagrams for the following sets.

16. $U - (A \cap B)$

17. $(A \cup B)'$

18. $B' \cup (A' - B)$

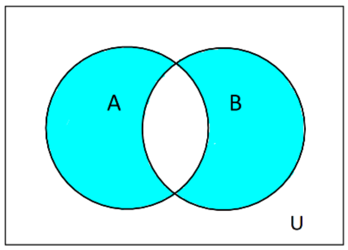
19. $(B \cup C) - A$

20. $(A \cup B)' \cap C'$

21. $(A \cap C') - B$

For exercises 22 – 27, given the shaded regions, find two set representations for each set. Note that there may be more than two representations for each set.

22.



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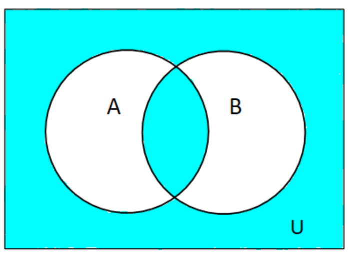

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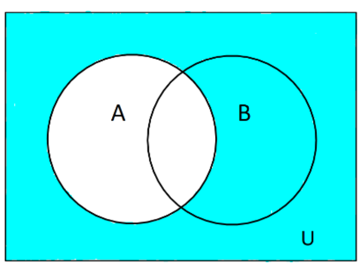

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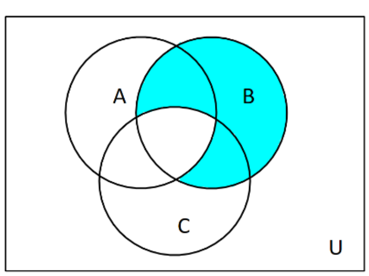

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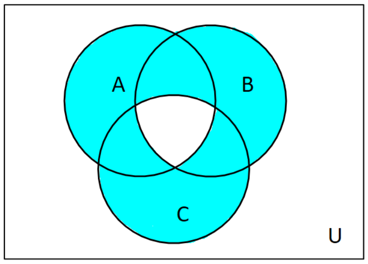

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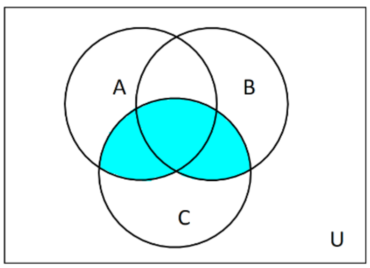
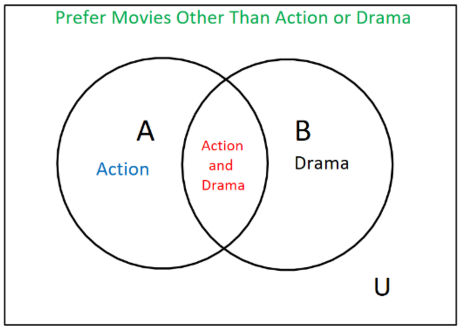
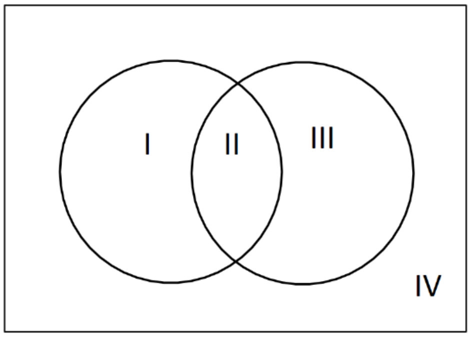


Section 1.3 | Exercise Solutions

1.





The Venn diagram on the right represents the two sets A and B. Use the Venn diagram on the left to label the four regions.

a. $A \cup B$ is represented by regions I, II, and III. Specifically, $A \cup B$ represents movie goers who like only actions, both actions and dramas, and only dramas.

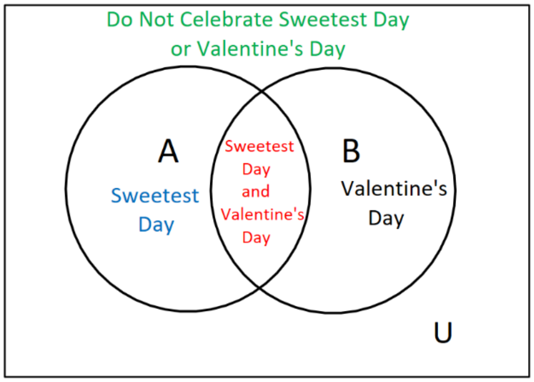
b. $A \cap B$ is represented by region II which is movie goers who like actions and dramas.

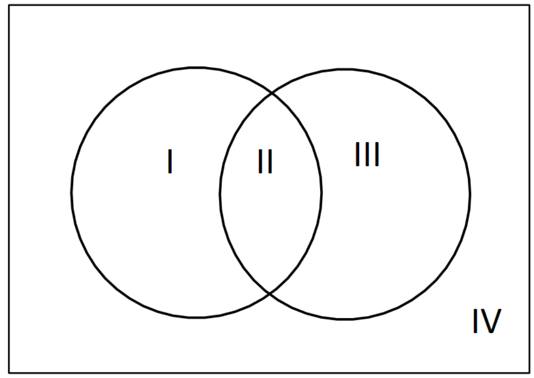
c. A' represents movie goers who prefer movies other than actions or dramas and movie goes who like dramas only. Regions III and IV represent A'.

d. $U - A$ represents movie goers who prefer movies other than actions or dramas and movie goes who like dramas only. Regions III and IV represent A'.
Note: $U - A = A'$.

e. $U - (A \cap B)$ represents movie goers who prefer only actions, only dramas, or prefer movies other than actions or dramas. Regions I, III, and IV represent $U - (A \cap B)$.

2.





a. $A \cup B$ is represented by regions I, II, and III. Specifically, $A \cup B$ represents married couples who celebrate only Sweetest Day, both Sweetest Day and Valentine’s Day, and only Valentine’s Day.

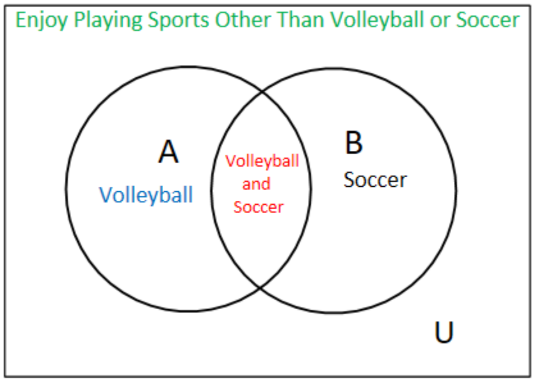
b. $A \cap B$ represents married couples who celebrate Sweetest Day and Valentine’s Day. $A \cap B$ is represented by region II.

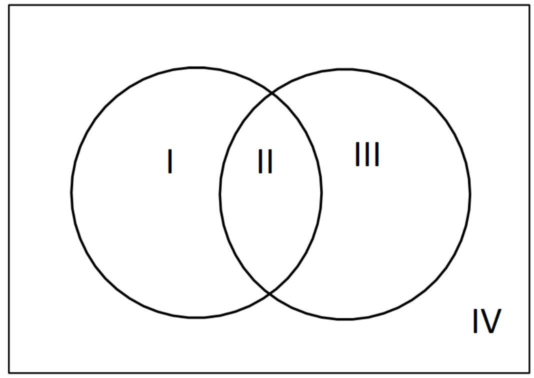
c. B' represents married couples who do not celebrate Sweetest Day or Valentine’s Day and married couples who celebrate only Sweetest Day. Regions I and IV represent B'.

d. $U - B$ represents married couples who do not celebrate Sweetest Day or Valentine’s Day and married couples who celebrate only Sweetest Day. Regions I and IV represent B'. Note: $U - B = B'$.

e. $U - (A \cup B)$ represents married couples who do not celebrate Sweetest Day or Valentine’s Day. This is represented by region IV.

3.





a. $(A \cup B)'$ is represented by region IV. Specifically, $(A \cup B)'$ represents those who enjoy playing sports other than volleyball or soccer.

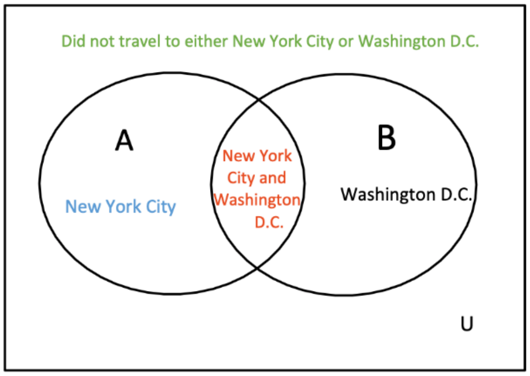
b. $(A \cap B)'$ is represented by regions I, III, and IV. Specifically, $(A \cap B)'$ represents those who like playing only soccer, only volleyball, and who enjoy playing sports other than volleyball or soccer.

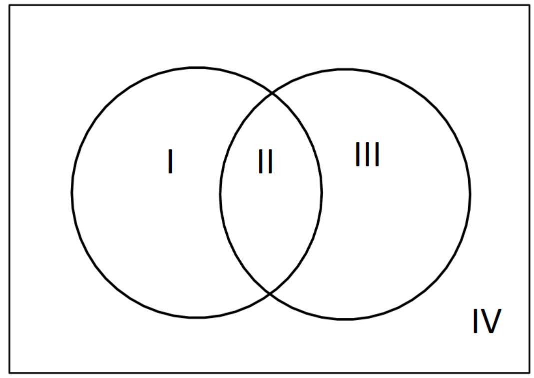
c. $A \cup B'$ is represented by regions I, II, and IV. Specifically, $A \cup B'$ represents those who like playing only soccer, both soccer and volleyball, or who enjoy playing sports other than volleyball or soccer.

d. $U - A'$ is represented by regions I and II. Specifically, $U - A'$ represents those who like playing only volleyball and both volleyball and soccer.
Note: $U - A' = A$.

e. $U - (A \cup B)'$ is represented by regions I, II, and III. Specifically, $U - (A \cup B)'$ represents those who like playing only soccer, only volleyball, and both soccer and volleyball. Note: $U - (A \cup B)' = A \cup B$.

4.





a. $A \cup B$ is represented by regions I, II and III. Specifically, $A \cup B$ represents people who have traveled to only New York City, New York City and Washington D.C., or to only Washington D.C.

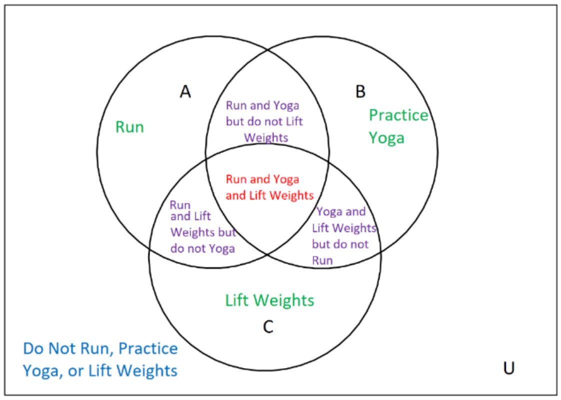
b. $(A \cap B)'$ is represented by regions I, III, and IV. Specifically, $(A \cap B)'$ represents people who have traveled to only New York City, only Washington D.C. or did not travel to either New York City or Washington D.C.

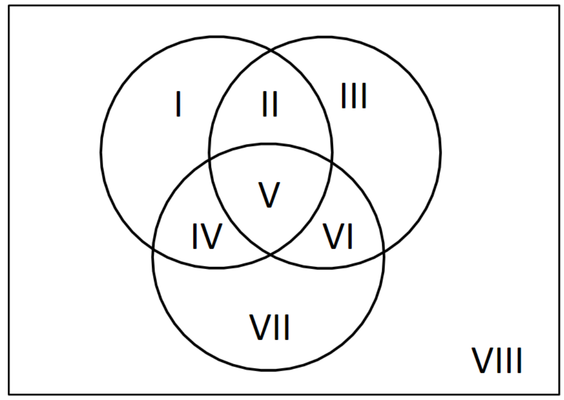
c. $A' \cap B'$ is represented by region IV. Specifically, $A' \cap B'$ represents people who did not travel to either New York City or Washington D.C.

d. $U - B$ is represented by regions I and IV. Specifically, $U - B$ represents people who have traveled to only New York City or people who did not travel to either New York City or Washington D.C.

e. $U - (A \cap B)$ is represented by regions I, III, or IV. Specifically, $U - (A \cup B)'$ represents those people who have traveled to only New York City, only Washington D.C. or who did not travel to either New York City or Washington D.C.

5.





a. $A \cup B \cap C$ is represented by regions IV, V, and VI.
Region IV: People who run and lift weights but do not practice yoga
Region V: People who run, practice yoga, and lift weights
Region VI: People who practice yoga and lift weights but do not run

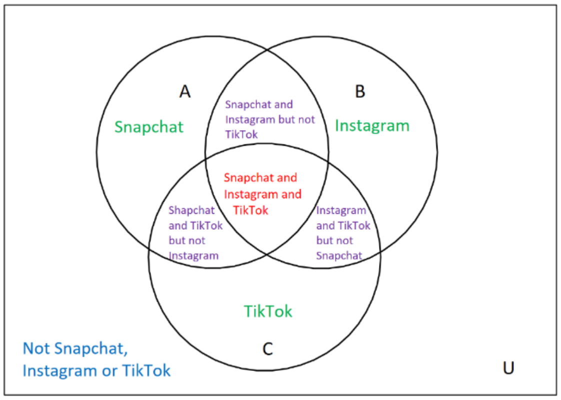
b. $A \cap B \cap C'$ is represented by region II.
Region II: People who run and practice yoga but do not lift weights

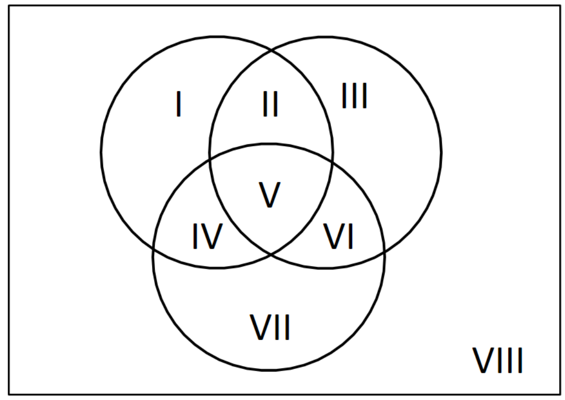
c. $A \cap B - C$ is represented by region II.
Region II: People who run and practice yoga but do not lift weights
Note: $A \cap B - \ C = A \cap B \cap C'$

d. $A \cup C - B'$ is represented by regions II, V, and VI.
Region II: People who run and practice yoga but do not lift weights
Region V: People who run, practice yoga, and lift weights
Region VI: People who practice yoga and lift weights but do not run

e. $U - (A \cup B \cup C)$ is represented by region VIII.
Region VIII: People who do not run, practice yoga, or lift weights

6.





a. $A \cap B \cap C$ is represented by region V.
Region V: Students who use Snapchat, Instagram, and TikTok

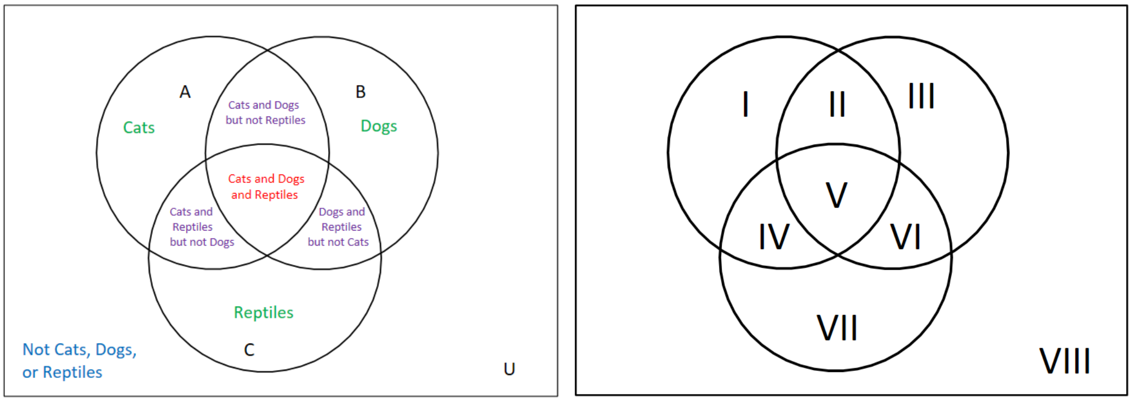
b. $A' \cup B \cup C$ is represented by regions II, III, IV, V, VI, VII, and VIII.
Region II: Students who use Snapchat and Instagram but not TikTok
Region III: Students who use only Instagram
Region IV: Students who use Snapchat and TikTok but not Instagram
Region V: Students who use Snapchat, TikTok, and Instagram
Region VI: Students who use Instagram and TikTok but not Snapchat
Region VII: Students who use only TikTok
Region VIII: Students who do not use Snapchat, Instagram, or TikTok

c. $A \cup B - C$ is represented by regions I, II, and III.
Region I: Students who use only Snapchat
Region II: Students who use Snapchat and Instagram but not TikTok
Region III: Students who use only Instagram

d. $A \cap C - B'$ is represented by region V.
Region V: Students who use Snapchat, Instagram, and TikTok

e. $U - (A \cap B \cap C)$ is represented by regions I, II, III, IV, VI, VII, VIII.
Region I: Students who use only Snapchat
Region II: Students who use Snapchat and Instagram but not TikTok
Region III: Students who use only Instagram
Region IV: Students who use Snapchat and TikTok but not Instagram
Region VI: Students who use Instagram and TikTok but not Snapchat
Region VII: Students who use only TikTok
Region VIII: Students who do not use Snapchat, Instagram, or TikTok

7.



a. $A \cup B \cup C$ is represented by regions I, II, III, IV, V, VI, and VII.
Region I: People who have cats but not dogs or reptiles
Region II: People who have cats and dogs but not reptiles
Region III: People who have dogs but not cats or reptiles
Region IV: People who have cats and reptiles but not dogs
Region V: People who have cats, dogs, and reptiles
Region VI: People who have dogs and reptiles but not cats
Region VII: People who have reptiles but not cats or dogs

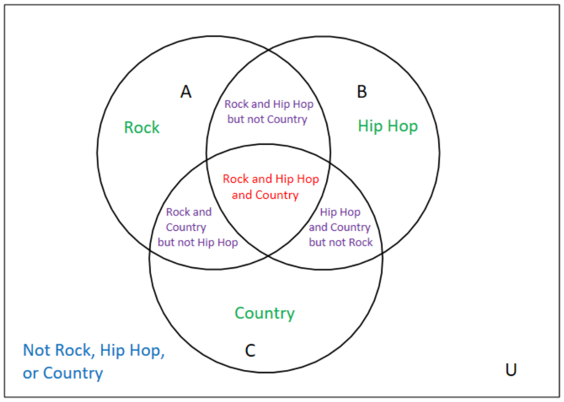
b. $(A \cup B)' \cup C$ is represented by regions IV, V, VI, VII, and VIII.
Region IV: People who have cats and reptiles but not dogs
Region V: People who have cats, dogs, and reptiles
Region VI: People who have dogs and reptiles but not cats
Region VII: People who have reptiles but not cats or dogs
Region VIII: People who do not have cats, dogs, or reptiles

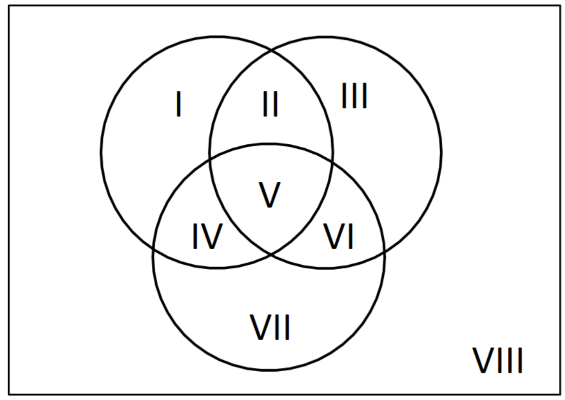
c. $(A \cap C)' - B$ is represented by regions I, VII, and VIII.
Region I: People who have cats but not dogs or reptiles
Region VII: People who have reptiles but not cats or dogs
Region VIII: People who do not have cats, dogs, or reptiles

d. $B' \cup (C - A')$ is represented by regions I, IV, V, VII, and VIII
Region I: People who have cats but not dogs or reptiles
Region IV: People who have cats and reptiles but not dogs
Region V: People who have cats, dogs, and reptiles
Region VII: People who have reptiles but not cats or dogs
Region VIII: People who do not have cats, dogs, or reptiles

e. $U-(A\cap B\cap C)'$ is represented by region V.
Region V: People who have cats, dogs, and reptiles
Note: $U-(A\cap B\cap C)'=A\cap B\cap C$.

8.





a. $(A\cup B\cup C)'$ is represented by region VIII.
Region VIII: People who do not enjoy rock, hip hop, or country

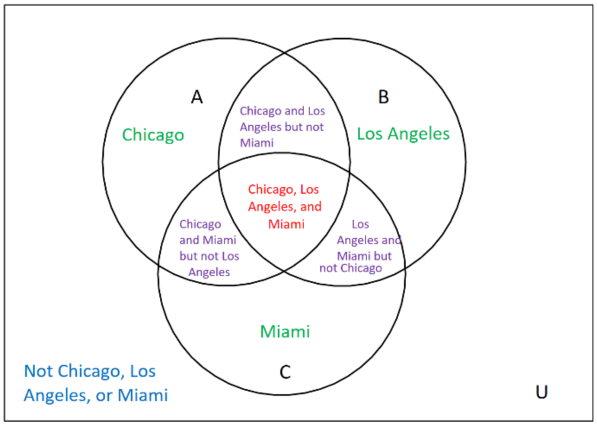
b. $(A\cap B)\cup(A\cap C)$ is represented by regions II, IV, and V.
Region II: People who enjoy rock and hip hop but not country
Region IV: People who enjoy rock and country but not hip hop
Region V: People who enjoy rock, hip hop, and country

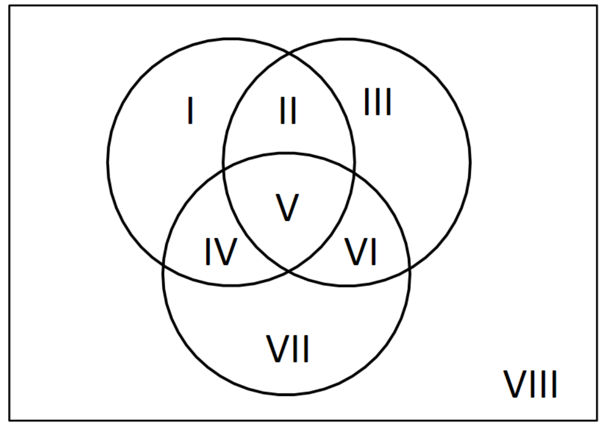
c. $C-(A'\cap C)'$ is represented by regions VI and VII.
Region VI: People who enjoy hip hop and country but not rock
Region VII: People who enjoy country but not rock or hip hop

d. $(C'\cup B)\cap(A\cap B')$ is represented by region I.
Region I: People who enjoy rock but not hip hop or country

e. $U-(A\cup B)'$ is represented by regions I, II, III, IV, V, and VI.
Region I: People who enjoy rock but not hip hop or country
Region II: People who enjoy rock and hip hop but not country
Region III: People who enjoy hip hop but not rock or country
Region IV: People who enjoy rock and country but not hip hop
Region V: People who enjoy rock, hip hop, and country
Region VI: People who enjoy hip hop and country but not rock
Note: $U-(A\cup B)'=A\cup B$.

9.





a. $A \cup B \cap C$ is represented by regions IV, V, and VI.
Region IV: People who have traveled to Chicago and Miami but not Los Angeles
Region V: People who have traveled to Chicago, Los Angeles, and Miami
Region VI: People who have traveled to Los Angeles and Miami but not Chicago

b. $(A \cap C) - B$ is represented by region IV.
Region IV: People who have traveled to Chicago and Miami but not Los Angeles

c. $U - (A \cap B \cap C)$ is represented by regions I, II, III, IV, VI, VII, and VIII.
Region I: People who have traveled to Chicago but not Los Angeles and not Miami
Region II: People who have traveled to Chicago and Los Angeles but not Miami
Region III: People who have traveled to Los Angeles but not Chicago and not Miami
Region IV: People who have traveled to Chicago and Miami but not Los Angeles
Region VI: People who have traveled to Los Angeles and Miami but not Chicago
Region VII: People who have traveled to Miami but not Chicago and not Los Angeles
Region VIII: People who have not traveled to Chicago and not traveled to Los Angeles and not traveled to Miami.
(Another way to represent Region VIII: People who have not traveled to any of the three cities Chicago, Los Angeles or Miami)

d. $(A \cap B) \cup (A \cap C)$ is represented by regions II, IV and V.
Region II: People who have traveled to Chicago and Los Angeles but not Miami
Region IV: People who have traveled to Chicago and Miami but not Los Angeles
Region V: People who have traveled to Chicago, Los Angeles, and Miami

e. $(A \cup B)' \cap C'$ is represented by region VIII.
Region VIII: People who have not traveled to Chicago and not traveled to Los Angeles and not traveled to Miami.
(Another way to represent Region VIII: People who have not traveled to any of the three cities Chicago, Los Angeles or Miami)

10.

a. $B' = \{1, 3, 4, 6, 7, 8, 10, 11, 13, 14, 16, 17, 18, 20, 21, 23, 24, 26, 27, 28, 30\}$
b. $A \cup B = \{2, 5, 9, 11, 12, 15, 17, 19, 22, 23, 25, 28, 29\}$
c. $A \cap B' = \{11, 17, 23, 28\}$
d. $B - A' = \{2, 5\}$
e. $(A - B)' =$
$\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 14, 15, 16, 18, 19, 20, 21, 22, 24, 25, 26, 27, 29, 30\}$
f. $U - (A' \cup B) = \{11, 17, 23, 28\}$

11.

a. $A' \cup B' \cap C = \{3, 6, 9, 14, 16, 19, 21, 22\}$
b. $(A \cap B) - C = \{7\}$
c. $B - (A' \cap C) = \{2, 4, 7, 13, 18, 23\}$
d. $U - (A \cup B \cup C) - (A' \cap B' \cap C') = \emptyset$

12.

a. $A \cup (B \cap C) = \{1, 3, 5, 13, 21, 55, 89, 144\}$
b. $(A \cup B) \cap (B \cup C) = \{2, 5, 8, 21, 34, 55, 144\}$
c. $C' \cup (B - A)' = \{1, 2, 3, 5, 8, 13, 21, 34, 55, 89\}$
d. $(A - B') \cap (A' \cap C') = \emptyset$

13.

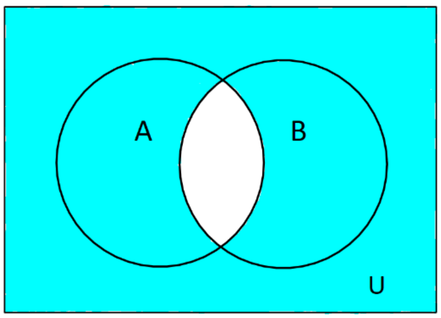
a. $(A \cup C) \cap B = \{1, 7, 13\}$
b. $(A \cap B) - (B \cap C) = \{1, 7, 13\}$
c. $(C' - B) \cap A' = \{6, 10, 14, 17\}$
d. $(A \cup C')' \cap (B' \cap C) = \{4, 8, 12, 16, 19\}$

14.

a. $A \cup B = \{4, 6, 8, 12, 16, 18, 20, 24, 28, 30\}$
b. $A \cap B = \{12, 24\}$
c. $A - B = \{4, 8, 16, 20, 28\}$
d. $B' - A = \{2, 10, 14, 22, 26\}$
e. $U - B = \{2, 4, 8, 10, 14, 16, 20, 22, 26, 28\}$
f. $A' = \{2, 6, 10, 14, 18, 22, 26, 30\}$

15. Let $U = \{x|x \in \mathbb{N}\ and\ x \leq 30\}$, $A = \{x|x\ is\ odd\}$, $B = \{1, 2, 3, 5, 8, 13, 21\}$, and $C = \{x|x\ is\ a\ multiple\ of\ 5\}$
    a. $A \cup B \cap C = \{5, 15, 25\}$
    b. $A' \cap B = \{2, 8\}$
    c. $(B \cup C)' \cap A = \{7, 9, 11, 17, 19, 23, 27, 29\}$
    d. $U - (A \cup B) = \{4, 6, 10, 12, 14, 16, 18, 20, 22, 24, 26, 28, 30\}$

16.



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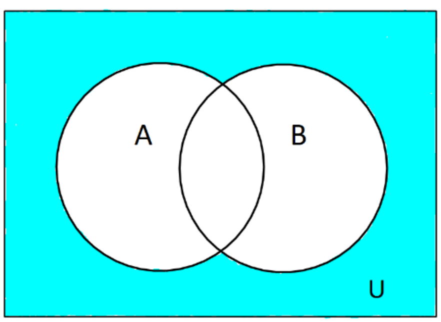

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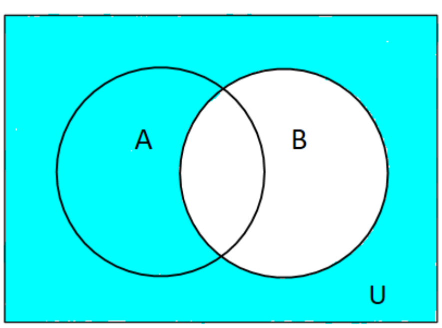

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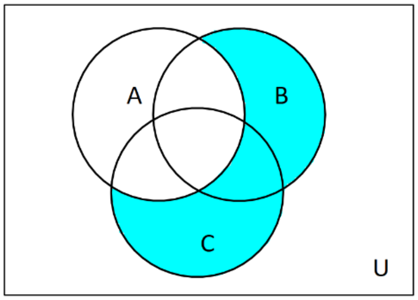

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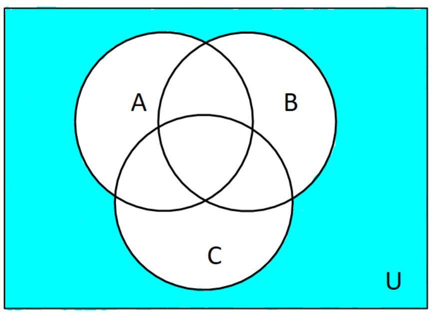

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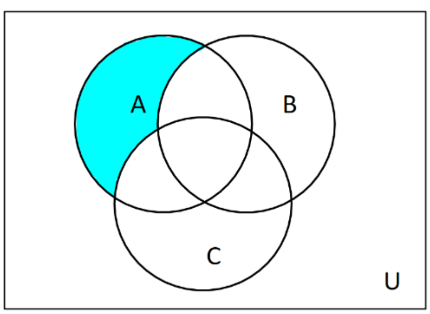


22. $(A \cup B) - (A \cap B)$,
$(U - A') \cup (U - B') - (A \cap B)$, and
$(A - B) \cup (B - A)$ are three possible solutions.

23. $(A \cup B)' \cup (A \cap B)$,
$(A \cap B) \cup (A' \cap B')$, and
$\big(U - (A \cup B)\big) \cup (A \cap B)$ are three possible solutions.

24. $U - A$,
A', and
$(B - A) \cup (A \cup B)'$ are three possible solutions.

25. $B - (A \cap B \cap C)$,
$(U - B') - (A \cap B \cap C)$, and
$(A' \cup C') \cap B$ are three possible solutions.

26. $(A \cup B \cup C) - (A \cap B \cap C)$,
$(A' \cup B' \cup C') - (A' \cap B' \cap C')$, and
$U - (A \cap B \cap C) - (A' \cap B' \cap C')$ are three possible solutions.

27. $(A \cap C) \cup (B \cap C)$,
$C - (A' \cap B')$, and
$C \cap (A \cup B)$ are three possible solutions.

Section 1.4 | DeMorgan's Laws and Cardinality of Unions and Intersections

Objectives

- State DeMorgan's Laws
- Use DeMorgan's Laws to find the negation of unions and intersections of sets
- State the cardinality of union and intersection formulas
- Find the cardinality of the union and intersection of sets

DeMorgan's Laws

When working with set operations, DeMorgan's Laws are key to simplify a more complex group of set operations. For example, for two sets A and B, what does $(A \cup B)'$ simplify to? That is, what expression also represents the set $(A \cup B)'$ but doesn't use parentheses?

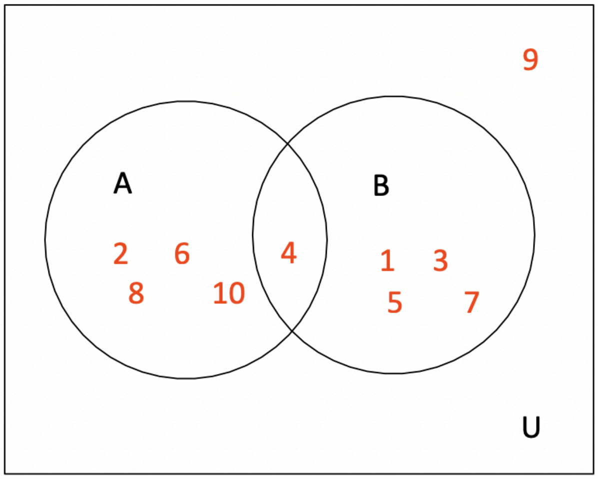
Consider two sets, A and B.

$$U = \{x \mid x \in \mathbb{N} \; and \; 1 \leq x \leq 10\}$$

$$A = \{2, 4, 6, 8, 10\}$$

$$B = \{1, 3, 4, 5, 7\}$$

These sets can be visually displayed as follows.



Remember the question: is there a way to simplify $(A \cup B)'$ so that the expression is written without parentheses yet describes the same set?

List the elements in the complement of $(A \cup B)$. Remember to begin inside the parentheses, so with $A \cup B$.

$$A \cup B = \{1, 2, 3, 4, 5, 6, 7, 8, 10\}$$

$$(A \cup B)' = \{9\}$$

Now explore a few possible equivalent sets, such as $A' \cup B'$ or $A' \cap B'$ or $A \cup B'$. Note here there are many possible sets to consider; these are just a few.

Start by finding the complement of A and the complement B.

$$A' = \{1, 3, 5, 7, 9\}$$

$$B' = \{2, 6, 8, 9, 10\}.$$

$$A' \cup B' = \{1, 2, 3, 5, 6, 7, 8, 9, 10\}$$

$$A' \cap B' = \{9\}$$

$$A \cup B' = \{2, 4, 6, 8, 9, 10\}$$

Notice that $(A \cup B)' = \{9\}$ and $A' \cap B' = \{9\}$.

What about $(A \cap B)'$? Is there a way to write an expression without parentheses describing the same set?

Start by finding $A \cap B$.

$$A \cap B = \{4\}$$

$$(A \cap B)' = \{1, 2, 3, 5, 6, 7, 8, 9, 10\}$$

Notice from above that $(A \cap B)' = A' \cup B' = \{1, 2, 3, 5, 6, 7, 8, 9, 10\}$.

A possible conjecture to make at this point is that for any sets A and B, $(A \cup B)' = A' \cap B'$ and $(A \cap B)' = A' \cup B'$.

Note that one example does not constitute a proof of a law; it is merely enough to support a conjecture. For a conjecture to become a proof, it is required to show that for every conceivable set A and B the observed relationship holds. This one example, however, should provide support for the viability of DeMorgan's Laws.

This brings us to DeMorgan's Laws.

DeMorgan's Laws

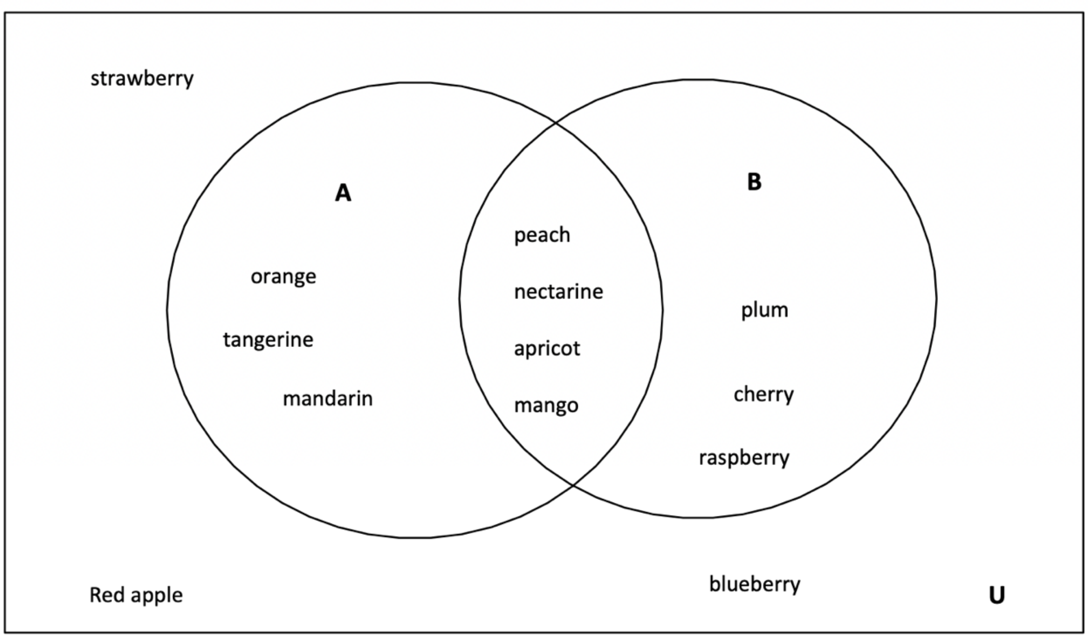
For any two sets A and B,

$$(A \cup B)' = A' \cap B'$$
$$(A \cap B)' = A' \cup B'$$

❖ **THINK ABOUT IT:** A common misconception is that $(A \cup B)' = A' \cup B'$ and $(A \cap B)' = A' \cap B'$. Provide an example showing this isn't true.

ANSWER: There are infinitely many possible examples showing that this isn't true. Consider the set of fruits that are yellow-orange and the set of fruits with a pit. The universal set is the set of all fruits.

For simplicity, list just a few. Let set $A = \{$yellow-orange fruits$\} = \{$orange, mandarin, tangerine, peach, nectarine, apricot, mango$\}$ and set $B = \{$fruits with a pit$\} = \{$peach, plum, cherry, nectarine, apricot, mango, raspberry$\}$. Set $U = \{$peach, plum, cherry, nectarine, apricot, mango, raspberry, orange, mandarin, tangerine, red apple, blueberry, strawberry$\}$.



Then $A \cup B =$ {peach, plum, cherry, nectarine, apricot, mango, raspberry, orange, mandarin, tangerine}. This set is the set of yellow-orange fruits that have a pit.

$(A \cup B)' =$ {red apple, blueberry, strawberry}. This is the set of fruits that are neither yellow-orange nor have a pit.

$A' =$ {plum, cherry, raspberry, red apple, blueberry, strawberry}

$B' =$ {orange, mandarin, tangerine, red apple, blueberry, strawberry}

$A' \cup B' =$ {plum, cherry, orange, mandarin, tangerine, raspberry, red apple, blueberry, strawberry}. This is the set of fruits that either aren't yellow-orange or don't have a pit.

Notice that $(A \cup B)' \neq A' \cup B'$. Similarly, $(A \cap B)' \neq A' \cap B'$.

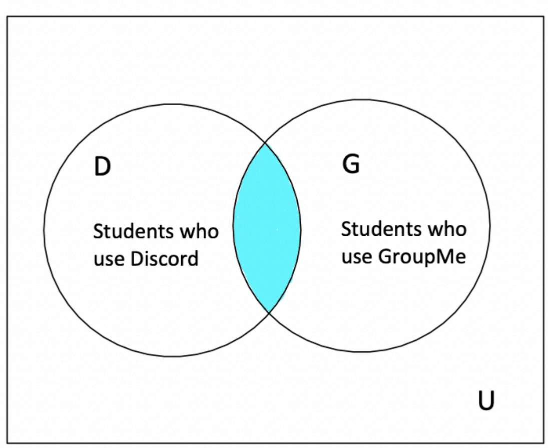
This means that the set of fruits which are neither yellow-orange nor have a pit is not the same as the set of fruits which aren't yellow-orange *or* don't have a pit. Many times, stating these set relationships in set notation is simpler than using words.

The complement sign in $(A \cup B)'$ and $(A \cap B)'$ cannot be distributed. It all comes down to order of set operations. Remember that any operations inside the parentheses must be completed first (just like in the order of operations with arithmetic). This means $(A \cup B)'$ first takes the union of sets A and B and then takes the complement. However, $A' \cup B'$ first finds the complement of A, then the complement of B, and then takes the union of the two. It is actually very rare for sets to be equal when set operations are applied in different orders.

The distributive property does not apply here. (Recall that the distributive property doesn't apply in other situations as well, such as when squaring a polynomial. That is, $(a+b)^2 \neq a^2 + b^2$ for most a and b.)

- **EXAMPLE 1.4.1:** A recent survey found that of 100 College of DuPage students, 41 use both Discord and GroupMe to communicate in their classes. How many students of the 100 don't use Discord or don't use GroupMe to communicate in their classes?

SOLUTION: Draw the situation using a Venn Diagram.



Let D be the set of students that use Discord to communicate in their courses and G be the set of students that use GroupMe to communicate in their courses. Then the set of students who use both Discord and GroupMe to communicate in their classes is

$$D \cap G$$

$D \cap G$ is shaded above.

The cardinality of the set $D \cap$ G is 41 (since 41 students use both Discord and GroupMe).

$$n(D \cap G) = 41$$

This means the cardinality of the set $(D \cap G)'$ has to be 59 (the other 59 students surveyed).

$$n((D \cap G)') = 100 - 41 = 59$$

By DeMorgan's Laws,

$$(D \cap G)' = D' \cup G'$$

so $D' \cup G'$ also has cardinality 59. That is,

$$n((D \cap G)') = n(D' \cup G') = 59$$

Notice that $D' \cup G'$ is the set of students that don't use Discord or don't use GroupMe to communicate in their classes.

- **EXAMPLE 1.4.2:** Let A and B be sets. Use DeMorgan's laws to simplify the expression $[(A \cup B)' \cap B']'$ so that it contains no parentheses yet describes the same set.

SOLUTION: As with the order of operations, start inside the innermost parentheses here.

$(A \cup B)' = A' \cap B'$

Now return to the beginning expression.

$$\begin{aligned} [(A \cup B)' \cap B]' &= [A' \cap B' \cap B']' \\ &= [A' \cap B']' \\ &= (A' \cap B')' \\ &= A \cup B \end{aligned}$$

Notice that A' is intersected with the same set (namely B') twice. It isn't necessary to write the same set twice.

The complement of the complement of a set is the original set, i.e. $A'' = A$.

Cardinality of the Union of Sets

It is also important to discuss the cardinality of the union and intersection of sets.

- **EXAMPLE 1.4.3:** Find the cardinality of $A \cap B$ given the following:

$$U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$$

$$A = \{2, 4, 6, 8, 10\}$$

$$B = \{1, 3, 4, 5, 7\}$$

SOLUTION: The cardinality of A is 5 and the cardinality of B is also 5. First find $A \cap B$ and then count the elements in $A \cap B$.

$$A \cap B = \{4\}$$

The cardinality of $A \cap B$ is 1.

- **EXAMPLE 1.4.4:** Find the cardinality of $A \cup B$ given sets A and B.

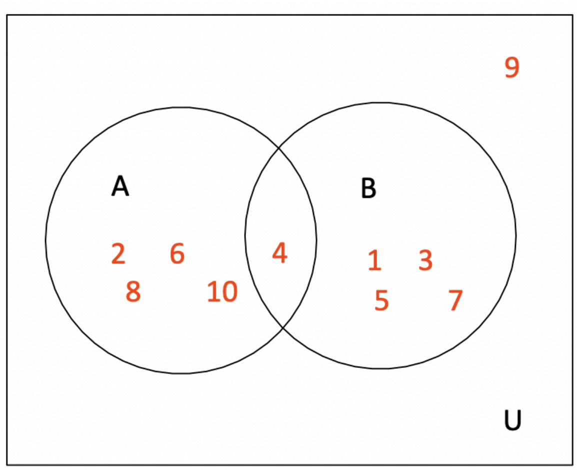
$$A = \{2, 4, 6, 8, 10\}$$

$$B = \{1, 3, 4, 5, 7\}$$

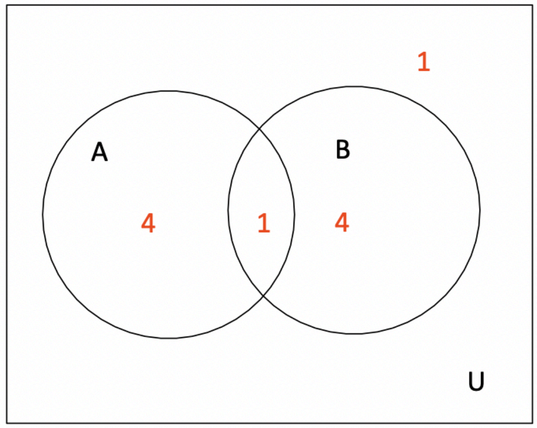
SOLUTION: First find $A \cup B = \{1, 2, 3, 4, 5, 6, 7, 8, 10\}$ and then count the number of elements in $A \cup B$.

$$n(A \cup B) = 9$$

A visual diagram for sets A and B would look like this.



A different visual diagram, to display the cardinality of each region, could also be made.



This diagram now is displaying cardinalities. For example, the 1 in the region of $A \cap B$ means that the cardinality of $A \cap B$ is 1 (that is, $n(A \cap B) = 1$), not that the region $A \cap B$ contains the element 1. The region $A \cap B$ doesn't contain the element 1, in fact; that region contains the element 4. (Yet the region has cardinality 1 since it contains one element.) This is an important and subtle distinction to make.

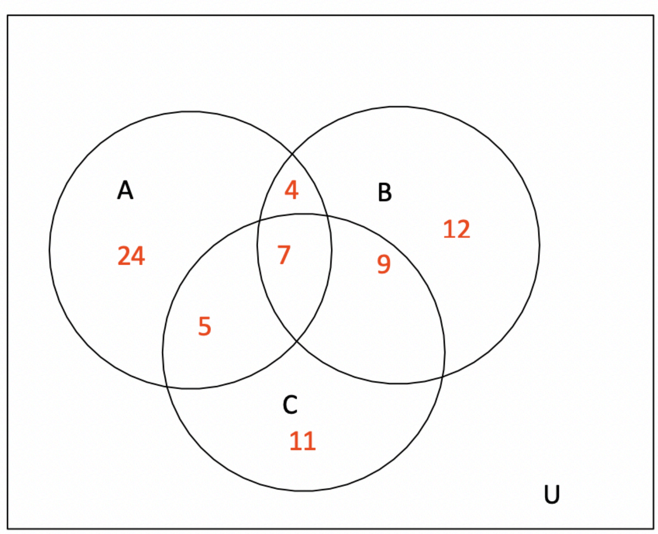
- **EXAMPLE 1.4.5:** Use the diagram displaying the cardinalities of the regions shown above to find the cardinality of $A \cup B$.

  SOLUTION: This time it isn't possible to formally find $A \cup B$ because only the cardinalities are listed, not the elements in A and B. Instead count the number of elements in A or B or both.

$$n(A \cup B) = 4 + 1 + 4 = 9$$

- **YOU TRY 1.4.A:** In the following Venn diagram, the cardinality of each region is listed. For example, $n(A \cap B \cap C) = 7$; it is not known if $A \cap B \cap C$ contains the element 7.

  Find $n(A \cup B - C)$.



➢ **EXAMPLE 1.4.6:** Find the cardinality of $H \cup G$ for $H = \{x \mid x \in \mathbb{N} \text{ and } 10 \le x \le 20\}$ and $G = \{16, 18, 20, 22, 24, 26\}$.

SOLUTION: First find $H \cup G$ and then count the number of elements in $H \cup G$. It might help to list H in roster notation first.

$$H = \{10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20\}$$

$$H \cup G = \{10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 22, 24, 26\}$$

$$n(H \cup G) = 14$$

There is a way to find the cardinality of the union of sets without having to write, count, or otherwise identify the number of elements in the union.

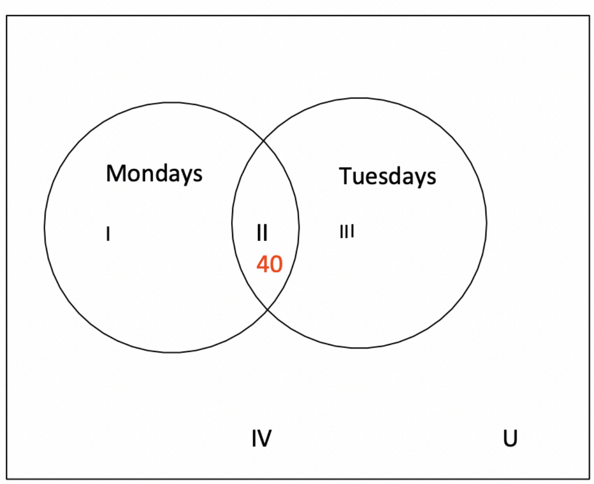
Cardinality of the Union of Sets

For any two sets A and B, the cardinality of $A \cup B$ is

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

➢ **EXAMPLE 1.4.7:** At a given company, 140 employees typically work on Mondays, 130 employees work on Tuesdays, and 40 employees work on both Mondays and Tuesdays. How many employees work on Monday or Tuesday (or both)?

SOLUTION: Let M be the set of employees who work on Mondays and T be the set of employees who work on Tuesdays. Then $n(M \cap T) = 40$ since 40 people work on Mondays and Tuesdays.



Further, it is known that the number of elements in region I plus the number of elements in region II adds to 140. Also, the number of elements in region II plus the number of elements in region III adds to 130. It is possible to find exactly the number of elements in region I; however, that won't be necessary for this problem.

$$n(M) = 140$$

$$n(T) = 130$$

$$n(M \cap T) = 40$$

By the formula, $n(M \cup T) = n(M) + n(T) - n(M \cup T) = 140 + 130 - 40 = 230$

That is, 230 employees work on either Monday or Tuesday or both.

It might be tempting to do $140 + 130$ but that would be counting the number of employees who work Mondays and Tuesdays (so the number of people in $M \cap T$) twice – once as an element of set M and once as an element of set T.

There will be more problems like this in section 1.5.

❖ **THINK ABOUT IT:** Is it possible to find $n(A \cup B)$ by finding the cardinality of A, then finding the cardinality of B, then adding the two together? That is, is $n(A \cup B) = n(A) + n(B)$?

ANSWER: Use the sets A and B as defined above.

$$A = \{2, 4, 6, 8, 10\}$$

$$B = \{1, 3, 4, 5, 7\}$$

$$n(A) = 5$$

$$n(B) = 5$$

$$n(A) + n(B) = 10$$

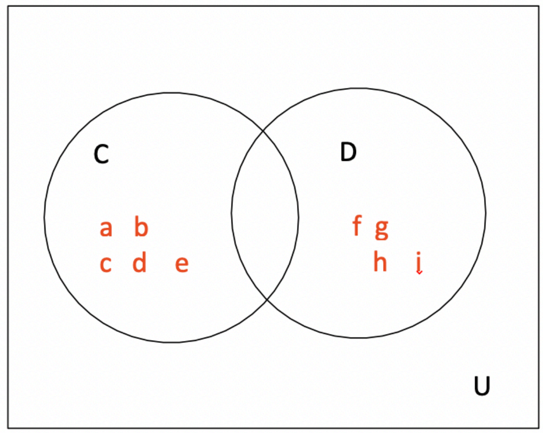
As found earlier, $n(A \cup B) = 9$.

While it is tempting to finding the cardinality of $A \cup B$ simply by adding the cardinality of A to the cardinality of B, that counts the element 4 twice – once as an element of A and once as an element of B. To find the cardinality of the union, then, be careful to count those elements in the intersection only once.

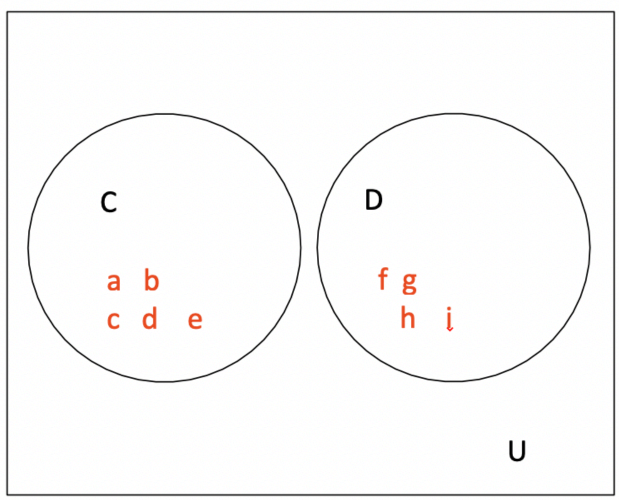
It is worth exploring if there are any special cases, or exceptions, to the cardinality of the union of two sets formula. For instance, are disjoint sets an exception?

❖ **THINK ABOUT IT:** What is the cardinality of the intersection of disjoint sets? What is the cardinality of the union of disjoint sets?

ANSWER: Define sets $C = \{a, b, c, d, e\}$ and $D = \{f, g, h, i\}$ to demonstrate. C and D are disjoint sets.



Also note that the Venn diagram could be drawn differently here. Since there are no values in the intersection, the Venn diagram could be represented as below, where C and D are shown as disjoint sets.



$$n(C) = 5$$

$$n(D) = 4$$

$$n(C) + n(D) = 9$$

$$C \cup D = \{a, b, c, d, e, f, g, h, i\}$$

$$n(C \cup D) = 9$$

$$C \cap D = \emptyset$$

$$n(C \cap D) = 0$$

That is, the cardinality of the union of disjoint sets is the same as the sum of the cardinalities of the individual sets. The cardinality of the intersection of disjoint sets is 0.

Again, it is worth noting that one example does not constitute a proof, but one example can instead strengthen the belief of a conjecture. A conjecture of note here is that the cardinality of the union of disjoint sets can be found by simply adding the cardinalities of the individual sets. This is because the cardinality of the intersection of disjoint sets is 0.

Cardinality of the Union of Disjoint Sets

For any two disjoint sets A and B, the cardinality of the union is

$$n(A \cup B) = n(A) + n(B)$$

Cardinality of the Intersection of Disjoint Sets

For any two disjoint sets A and B, the cardinality of the intersection is

$$n(A \cap B) = 0$$

These two facts together show that the cardinality of the union equation

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

does work when A and B are disjoint sets.

YOU TRY IT 1.4.A SOLUTION:

This time it isn't possible to formally find $A \cup B - C$ because only the cardinalities are listed, not the elements in A and B and C. Instead count the number of elements.

$$n(A \cup B - C) = 24 + 4 + 12 = 40$$

Section 1.4 | Exercises

For exercises 1-6, let $U = \{a, b, c, d, \dots, m, n, o\}$, $A = \{a, e, i, o\}$, $B = \{j, a, c, k, i, e\}$.

1. Find $n(A \cup B)$.

2. Find $n(A \cap B)$.

3. Do the results for $n(A \cup B)$ and $n(A \cap B)$ change if U is expanded to include the entire alphabet?

4. Find $n(A \cup B)'$.

5. Find $n(A \cap B)'$.

6. Do the results for $n(A \cup B)'$ and $n(A \cap B)'$ change if U is expanded to include the entire alphabet?

For exercises 7 – 12, let $U = \{a, c, e, g, i, k, l, n, o, q, r, s, t, u, w, y\}$, $A = \{a, e, i, o, u\}$, $B = \{r, s, t, l, n, e\}$.

7. Find $n(A \cup B)$.

8. Find $n(A \cap B)$.

9. Do the results for $n(A \cup B)$ and $n(A \cap B)$ change if U is expanded to include the entire alphabet?

10. Find $n(A \cup B)'$.

11. Find $n(A \cap B)'$.

12. Do the results for $n(A \cup B)'$ and $n(A \cap B)'$ change if U is expanded to include the entire alphabet?

For exercises 13 – 16, let $U = \{1, 2, 3, 4, \dots, 19, 20\}$, $C = \{2x \mid x \in \mathbb{N} \text{ and } 1 \leq x \leq 10\}$, and $D = \{x \mid x \in \mathbb{N} \text{ and } 10 \leq x \leq 20\}$. (Hint – the set C contains the elements 2, 4, 6, 8, ….)

13. Find $n(C \cup D)$.

14. Find $n(C \cap D)$.

15. Find $n(C \cup D)'$.

16. Find $n(C' \cap D')$.

For exercises 17 – 20, let $U = \{1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, \dots\}$, $A = \{3, 5, 11, 15, 19\}$, $B = \{5, 9, 15, 17, 21\}$.

17. Find $n(A \cup B)$.

18. Find $n(A \cap B)$.

19. Find $n(U \cup A)$.

20. Find $n(U \cap B)$.

For exercises 21 – 25, let $U = \{1, 2, 3, 4, \dots, 19, 20\}$, $E = \{2x \mid x \in \mathbb{N} \text{ and } 1 \leq x \leq 10\}$, $F = \{1, 5, 10, 15, 16, 20\}$, and $G = \{1, 3, 5, 7, 9, 11, 13, 15, 17, 19\}$.

21. Consider sets E and F.
    a. Find $n(E \cup F)$.
    b. Find $n(E \cap F)$.

22. Consider sets E and G.
    a. Find $n(E \cup G)$.
    b. Find $n(E \cap G)$.

23. Consider sets F and G.
    a. Find $n(F \cup G)$.
    b. Find $n(F \cap G)$.

24. What do you notice about $n(E \cap F)$ and $n(F \cap G)$? Why is this true?

25. What do you notice about $n(E \cup F)$ and $n(F \cup G)$? Why is this true?

For exercises 26 – 28, let $U = \{1, 2, 3, \ldots., 28, 29, 30\}$, $E = \{3x \mid x \in \mathbb{N}, 1 \leq x \leq 10\}$, $F = \{6x \mid x \in \mathbb{N}, 1 \leq x \leq 5\}$, $G = \{5x \mid x \in \mathbb{N}, 1 \leq x \leq 6\}$.

26. Consider sets E and F:
    a. Find $n(E \cup F)$.

    b. Find $n(E \cap F)$.

27. Consider sets F and G:
    a. Find $n(F \cup G)$.

    b. Find $n(F \cap G)$.

28. Consider sets U and G:
    a. Find $n(U \cap G)$.

    b. Find $n(U - G)$.

29. Use DeMorgan's Laws to simplify $(A \cap B)' \cup A$.

30. Use DeMorgan's Law to simplify $(A' \cap B)'$.

31. Use DeMorgan's Law to simplify $[(A' \cup C)']'$.

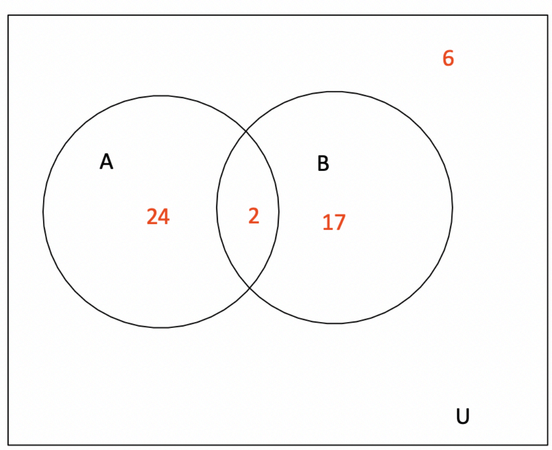
32. Use DeMorgan's Laws to simplify $[(C \cap D)' \cup D]'$.

33. Use DeMorgan's Laws to simplify $(A \cup B)' \cup (A \cap B)'$.

34. Use DeMorgan's Law to simplify $[(A' \cap B)' \cap C']'$.

35. Use DeMorgan's Law to simplify $[(A \cup B')' \cup C]'$.

For exercises 36 – 41, use the following diagram. The diagram shows the cardinalities of each region. For example, $n(A \cap B) = 2$. This means that $A \cap B$ contains two elements (not necessarily the number two). Find the following cardinalities.



36. $n(A \cup B)$

37. $n(U)$

38. $n(A - B)$

39. $n(B - A)$

40. $n(A' \cup B)$

41. $n(A \cap B)'$

For exercises 42 – 49, use the following diagram. The diagram shows the cardinality of each region. Find the following cardinalities.

42. $n(A)$

43. $n(B)$

44. $n(A \cup B)$

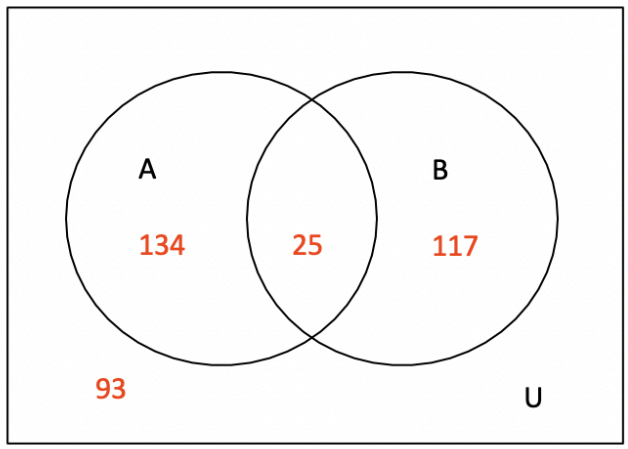
45. $n(A - B)$

46. $n(B - A)$

47. $n(U - B)$

48. $n(U - A)$

49. $n(U)$



For exercises 50-60, use the following diagram. The diagram shows the cardinalities of each region of the Venn diagram. For example, $n(A \cap B \cap C) = 1$. This means that $A \cap B \cap C$ contains one element (which is not necessarily the number 1). Find the following cardinalities.

50. $n(A \cup B)$

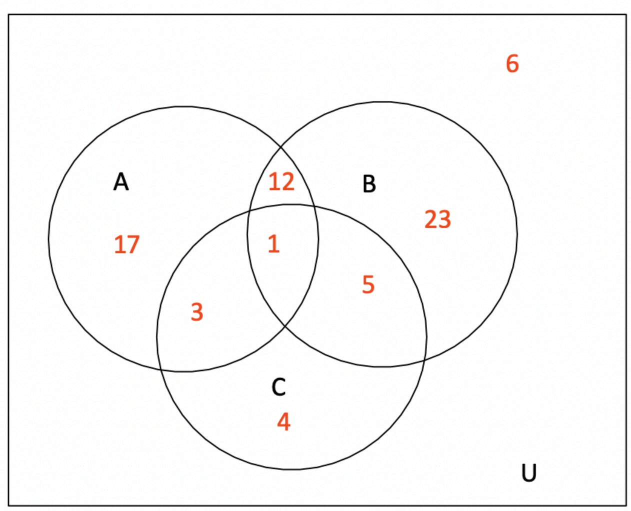
51. $n(B \cup C)$

52. $n(C \cup A)$

53. $n(A \cup B \cup C)$

54. $n(A \cap B)$

55. $n(A \cup C - B)$



56. $n(A \cap B)'$

57. $n(C \cup B)'$

58. $n(A' \cup C)$

59. $n(B' - C')$

60. $n(A' \cap B' \cap C')$

For exercises 61 – 67, use the following diagram. The diagram shows the cardinality of each region. Find the following cardinalities.

61. $n(A - B)$

62. $n(B - C)$

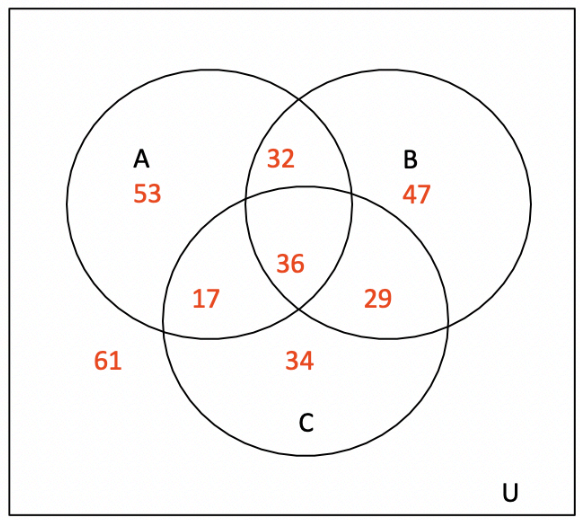
63. $n(U - A)$

64. $n(A \cap B \cup C)$

65. $n(A \cup B \cap C)$

66. $n(A - (B \cup C))$

67. $n\big((B \cup C) - A\big)$



Applications

68. At Noodles and Company (as of October 2021), there are 25 different noodles dishes. 22 of them contain dairy and 11 of them contain soy.
    a. How many dishes at Noodles and Company contain dairy or soy? If you can answer this question, do so. If you can't, state why not.

    b. In addition to the above numbers, 9 dishes contain dairy and soy. How many dishes at Noodles and Company contain dairy or soy? If you can answer this question, do so. If you can't, state why not.

    c. Remember 9 dishes contain dairy and soy. Suppose a person is avoiding all dairy or soy in their diet. How many dishes do they have to choose from? Hint – Let

D be the set of dishes containing dairy and S be the set of dishes containing soy. Then $n(D \cap S) = 9$ and maybe you can use DeMorgan's Laws.

69. Of the 330 freshmen, 270 are taking a math course and 260 are taking a science course.
    a. How many students are taking a math and science course? If you can answer this question, do so. If you can't, state why not.

    b. How many students are taking a math or science course? If you can answer this question, do so. If you can't state why not.

    c. Suppose 40 students are not taking either a math or science course. How many students are taking a math and science course? If you can answer this question, do so. If you can't state why not.

    d. Suppose it is also known that 40 students are not taking either a math or science course. Can you determine the number of students taking a math or science course? If you can answer this question, do so. If you cannot answer this question, state why not.

Concept Review

70. State DeMorgan's Laws.

71. State the cardinality rule for a union of two sets.

72. Let A and B be sets. What is known about the relationship between sets A and B so that $n(A \cup B) = n(A) + n(B)$?

73. Let C and D be sets. What is known about the relationship between sets C and D so that $n(C - D) = n(D - C)$?

74. Explain why $(A \cap B)' \neq A' \cap B'$ in general.

75. Explain why $(A \cup B)' \neq A' \cup B'$ in general.

Section 1.4 Exercise Solutions

1. 7
2. 3
3. They remain the same. This is because the large rectangular box that represents the universal set does not have an impact on the union or intersection of sets.
4. 8
5. 12
6. They do not remain the same. This is because the complement of a set is formed using the universal set. So, a change in the universal set will influence a change in the complement of a set.
7. $n(A \cup B) = 10$
8. $n(A \cap B) = 1$
9. No
10. $n(A \cup B)' = 6$
11. $n(A \cap B)' = 15$
12. They do not remain the same. $n(A \cup B)$ and $n(A \cap B)$ would remain the same however.
13. 15
14. 6
15. 5
16. 5
17. $n(A \cup B) = 8$
18. $n(A \cap B) = 2$
19. No cardinality, set is infinite.
20. $n(U \cap B) = 5$
21.
    a. 13
    b. 3
22.
    a. 20
    b. 0
23.
    a. 13
    b. 3
24. Those sets have the same cardinality. This is because F has the same number of odd integers that it does even.
25. They also have the same cardinality. This is because E and G are the same size and E and F and G and F have the same number of overlapping elements.
26.
    a. $n(E \cup F) = 10$
    b. $n(E \cap F) = 5$
27.
    a. $n(F \cup G) = 10$
    b. $n(F \cap G) = 1$
28.
    a. $n(U \cap G) = 6$
    b. $n(U - G) = 24$
29. U (the universal set)
30. $A \cup B'$
31. $A' \cup C$
32. $\emptyset$
33. $(A' \cap B') \cup (A' \cup B')$ and because $A' \cap B' \subseteq A' \cup B'$ this simplifies to $A' \cup B'$
34. $A' \cap B \cup C$
35. $A \cup B' \cap C'$
36. 43
37. 49
38. 24

39. 17
40. 25
41. 47
42. $n(A) = 159$
43. $n(B) = 142$
44. $n(A \cup B) = 276$
45. $n(A - B) = 134$
46. $n(B - A) = 117$
47. $n(U - B) = 227$
48. $n(U - A) = 210$
49. $n(U) = 369$
50. 61
51. 48
52. 42
53. 65
54. 13
55. 24
56. 58
57. 23
58. 42
59. 7
60. 6
61. $n(A - B) = 70$
62. $n(B - C) = 79$
63. $n(U - A) = 171$
64. $n(A \cap B \cup C) = 148$
65. $n(A \cup B \cap C) = 82$
66. $n\big(A - (B \cup C)\big) = 53$
67. $n\big((B \cup C) - A\big) = 110$
68.
    a. This question cannot be answered. The number of dishes that contain dairy and soy must be known.
    b. 24
    c. 1
69.
    a. Cannot answer this question because you need to know how many students are taking a math course or a science course.
    b. Cannot answer this question yet.
    c. 240
    d. 290
70. For two sets A and B, $(A \cap B)' = A' \cup B'$ and $(A \cup B)' = A' \cap B'$.
71. $n(A \cap B) = 0$ for disjoint sets A and B. $n(C \cup D) = n(C) + n(D) - n(C \cap D)$ for sets C and D.
72. Sets A and B must be disjoint.
73. Sets C and D must be equivalent.
74. $(A \cap B)'$ represents the elements in the universal set that are not in the intersection of A and B, including elements that are only in set A and elements that are only in set B. Whereas, $A' \cap B'$ represents the elements in the universal set that are completely outside of sets A and B, which excludes elements from those sets altogether.
75. $(A \cup B)'$ represents the elements in the universal set that are completely outside of sets A and B, which excludes elements from those sets altogether. Whereas, $A' \cup B'$ represents the elements in the universal set that are not common to both sets A and B, but includes elements that are only in set A and elements that are only in set B.

Section 1.5 | Using Venn Diagrams for Survey Application Problems

Objectives

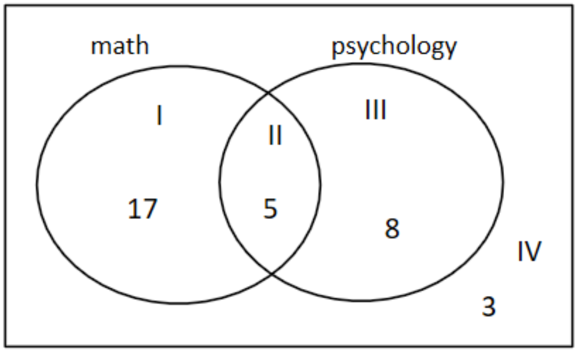
- Use Venn diagrams in problem solving
- Use Venn diagrams to solve survey problems with two sets
- Use Venn diagrams to solve survey problems with three sets

In Sections 1.2 and 1.3 of this book, Venn diagrams were introduced as a visual way to represent sets. This section will expand on the use of Venn diagrams, using them as a visual way to organize set information from application problems.

Two Set Venn Diagram Survey Application Problems

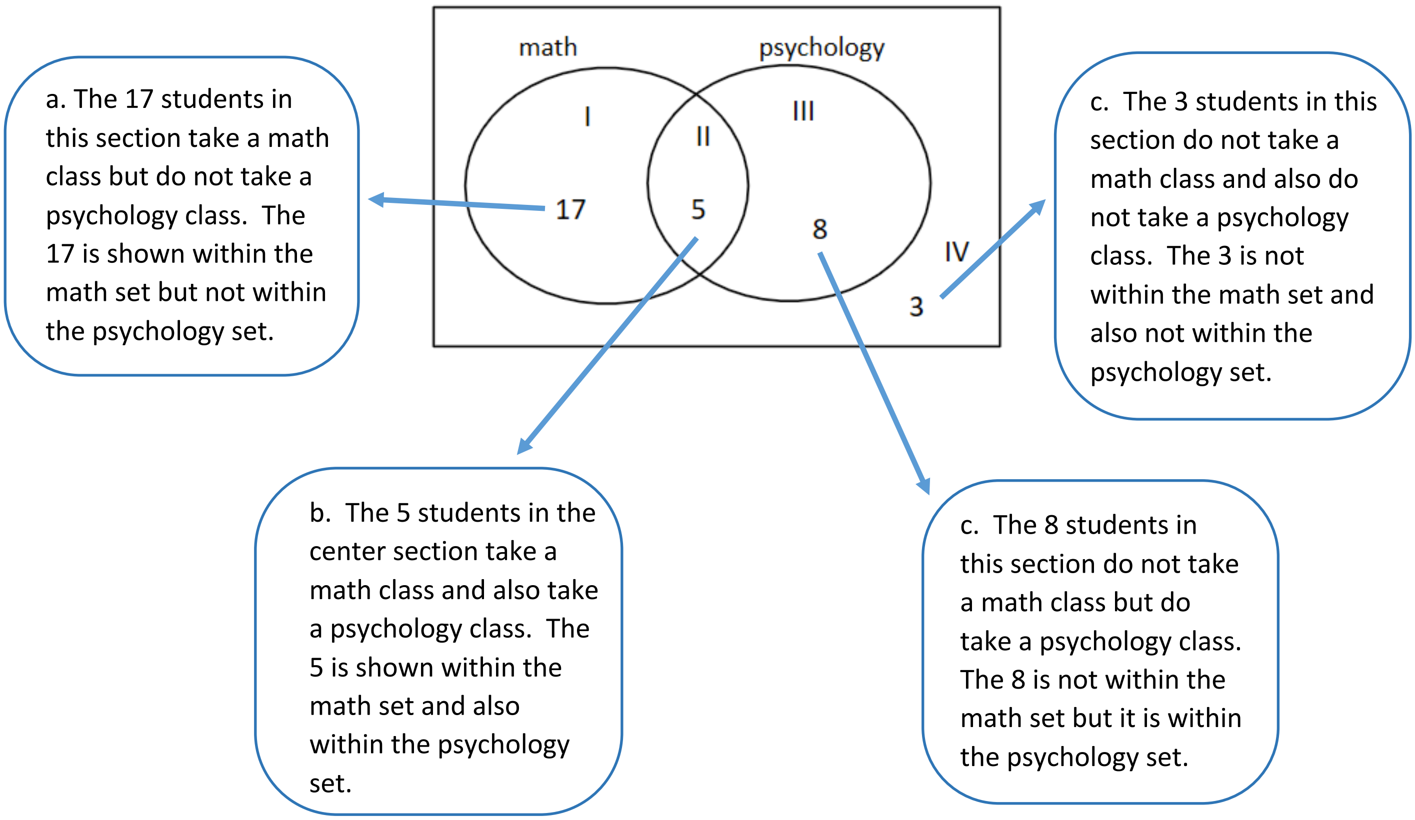
➢ **EXAMPLE 1.5.1:** Consider the Venn diagram below. This is a Venn diagram summary of the information from a survey of students. The survey asked students if they were taking a math class and/or a psychology class. Therefore, you see two sets; one for math and one for psychology.

Describe what each region represents.



a. What is known about the 17 people listed in region I of the Venn diagram?
b. What is known about the 5 people listed in region II of the Venn diagram?
c. What is known about the 8 people listed in region III of the Venn diagram?
d. What is known about the 3 people listed in region IV of the Venn diagram?
e. How many total people were surveyed?

SOLUTION:



a. There are 17 students taking math but not taking psychology.

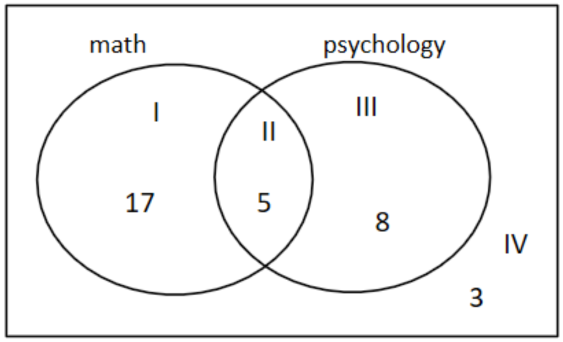
b. There are 5 students taking both math AND psychology.

c. There are 8 students taking psychology but not taking math.

d. There are 3 people that do not take math and also do not take psychology.

e. To find the number of people surveyed, remember that the definition of a universal set contains all of the elements under consideration for a particular situation. In this example, the rectangular universal set would represent all of the students surveyed. Find the sum of all of the students in the universal set.
The sum shows 17 + 5 + 8 + 3 = 33 students were surveyed.

The Venn diagram can provide more information.
For example, how many students are taking a math class? How many students are not taking a math class? How many students are taking a psychology class? How many students are not taking a psychology class?



Is a student taking math?

If **yes**, then that student is counted within any region within the math set.
There are 17 + 5 = 22 students taking a math class.

If **no,** then that student is counted in one of the regions that lie outside the math set.
There are 8 + 3 = 11 students not taking a math class.

Is a student taking psychology?

If **yes**, then that student is counted within any region within the psychology set.
There are 5 + 8 = 13 students taking a psychology class.

If **no,** then that student is counted in one of the regions that lie outside the psychology set.
There are 17 + 3 = 20 students not taking a psychology class.

It is time to approach these application questions from a different perspective. Sometimes multiple statements are given. A Venn diagram can be used to display the information and provide a detailed view of the statements.

Note: Consider Venn diagram application problems like a puzzle. Information statements are given and it is important to determine which order to use these statements to complete the Venn diagram. The information statements are not always provided in the order they can be used. Next is an example of an application problem where a Venn diagram is helpful.

➢ **EXAMPLE 1.5.2**: A survey asked students about the items in their school lunch. Use the information below to complete a Venn diagram and answer the questions.

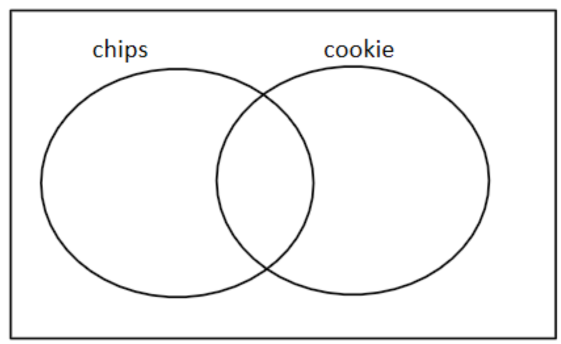
17 students had chips in their lunch
14 students had a cookie in their lunch
11 students had both chips and a cookie in their lunch
25 students were surveyed about what they packed in their lunch

a. How many students had chips but not a cookie for their lunch?
b. How many students did not have a cookie in their lunch?

SOLUTION:

It is often beneficial to complete a Venn diagram to display the survey information.

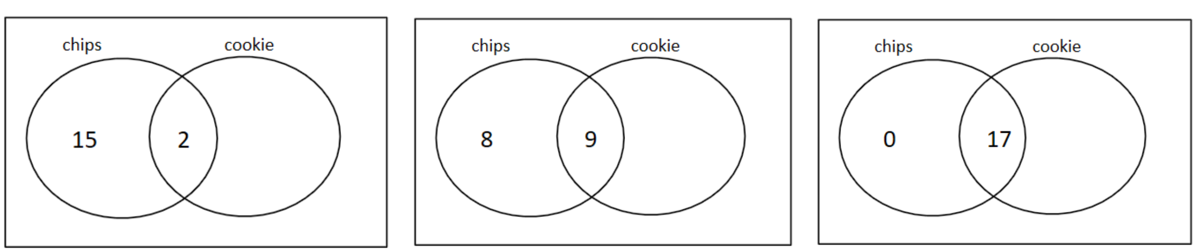
First, determine the number of sets needed in the Venn diagram. In this case, the statements provide information about whether a student had chips in their lunch and also information about whether a student had a cookie in their lunch. Therefore, two sets are needed; one for the chips set and one for the cookie set.



Step 1: Find a statement and try to use it.
17 students had chips in their lunch

Right now, this statement cannot be used because there is not enough information. This statement gives the total of all regions within the chips set as 17. How are the 17 students separated into the two regions that make up the chips set?

Is it separated as 15 and 2? Is it separated as 8 and 9? Is it separated as 0 and 17?

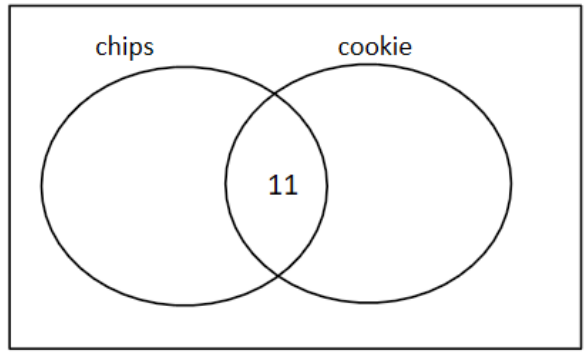


There is not enough information to use this statement right now.
Find a statement that is as specific as possible. Since the problem has two sets, find a statement that gives specific YES/NO information about both sets.

The third statement in the list has information about both chips and cookies.

11 students had both chips and a cookie in their lunch

The number 11 should be in the region within both the chips set and the cookie set. In short, eleven students must go in the region where the chips set and cookie set intersect.

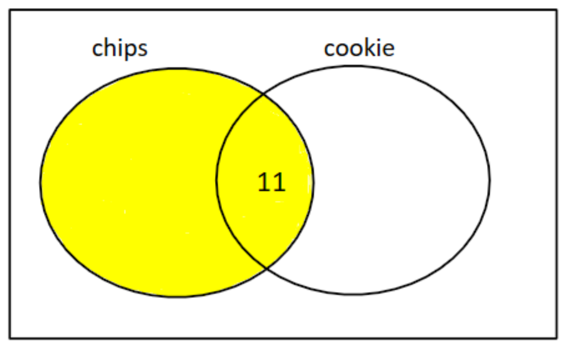


Check off the statement that was used and continue.

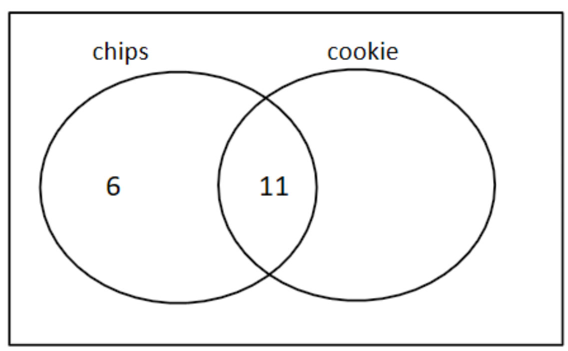
17 students had chips in their lunch
14 students had a cookie in their lunch
✓**~~11 students had both chips and a cookie in their lunch~~**
25 students were surveyed about what they packed in their lunch

Step 2: Go back to the statement:

17 students had chips in their lunch.
The chip set is highlighted.



The sum of all of the regions in the chips set needs to total 17. Now that 11 students are already listed in the center region, it can be determined that
11 + ? = 17 or that 6 students must be counted in the left region within the chip set.

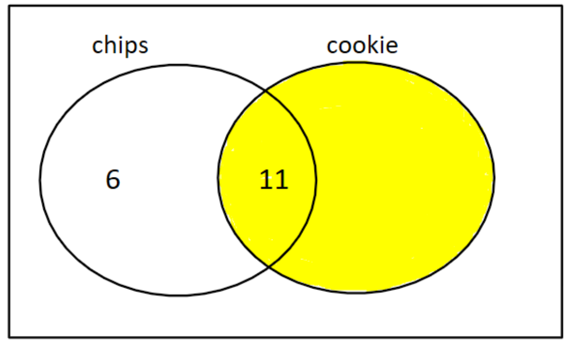


Check off that the statement was used and continue to another statement.

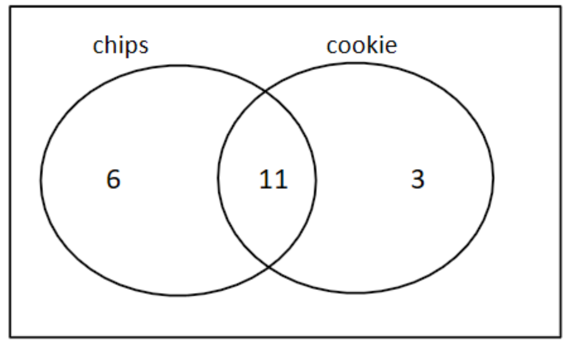
✓**~~17 students had chips in their lunch~~**
14 students had a cookie in their lunch
✓**~~11 students had both chips and a cookie in their lunch~~**
25 students were surveyed about what they packed in their lunch

Step 3: In a similar method, use the second statement listed.

14 students had a cookie in their lunch
The cookie set is highlighted.



The sum of all of the regions in the cookie set needs to total 14. With 11 students are already listed in the center region, it can be determined that 11 + ? = 14 or that 3 students must be counted in the right region within the cookie set.

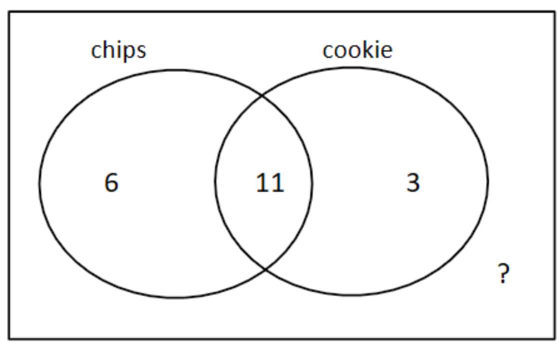


Check off the statement that was used and continue to another statement.

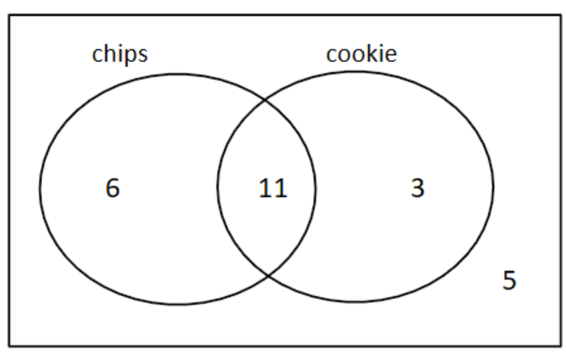
- ✓ **~~17 students had chips in their lunch~~**
- ✓ **~~14 students had a cookie in their lunch~~**
- ✓ **~~11 students had both chips and a cookie in their lunch~~**
- **25 students were surveyed about what they packed in their lunch**

Step 4: Only one statement remains.

25 students were surveyed about having chips or a cookie packed in their lunch
This statement refers to the sum of all regions within the universal set.
6 + 11 + 3 + ? = 25



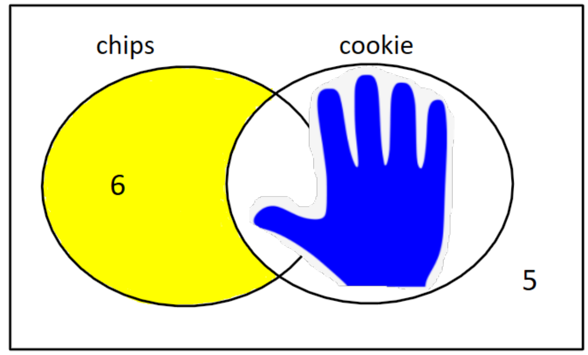
Solving, it can be determined that the region outside the chips set and cookie sets is 5.



Now that all of the regions of the Venn diagram are complete, use the information in the Venn diagram to answer questions about the students in the survey.

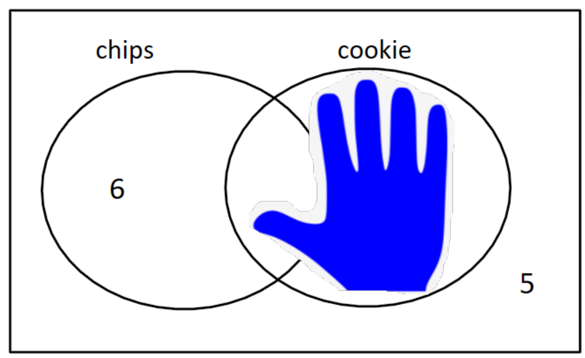
a. To find the number of students who had chips but not a cookie, find the region that lies within the chips set (highlighted) but is not within the cookie set. (Hint: cover/hide the values shown within the cookie set and find the sum of values remaining within the chips set.)

Answer = 6 students



b. To find the number of students who did not have a cookie, include all regions that are NOT inside the cookie set. (Hint: cover/hide the values shown within the cookie set and find the sum of all of the other regions). This gives 6 + 5 = 11.

Answer = 11 students



➢ **EXAMPLE 1.5.3**: A survey asked people about their pizza choices. Use the information below to complete a Venn diagram and answer the questions.

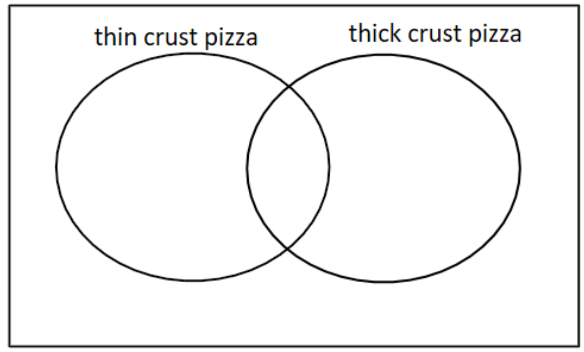
104 people like thin crust pizza.
87 people like thick crust pizza.
63 people do not like thin crust pizza.
14 people do not thin crust pizza or thick crust pizza.

a. How many people were surveyed?
b. How many people like both thin crust pizza and thick crust pizza?

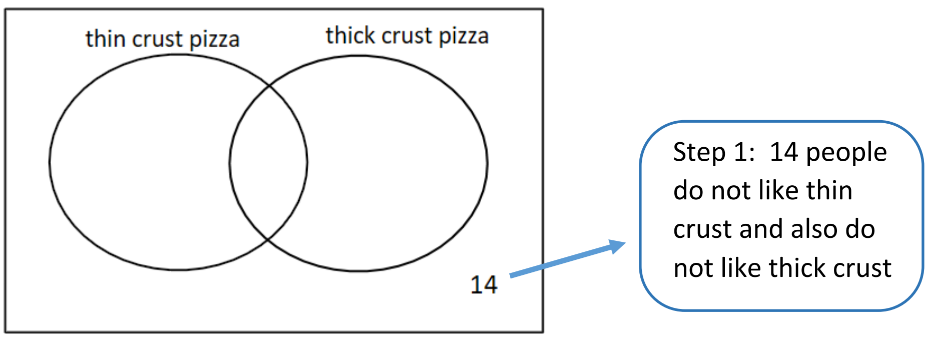
SOLUTION:

It is often beneficial to complete a Venn diagram to display the survey information.

First, determine the number of sets needed in the Venn diagram. In this case, the statements provide information about thin crust pizza and thick crust pizza. Therefore, two sets are needed.



Step 1: Begin with the statement with the most information. The only statement with specific information about both thin crust pizza and thick crust pizza is 14 people do not like thin crust pizza or thick crust pizza.

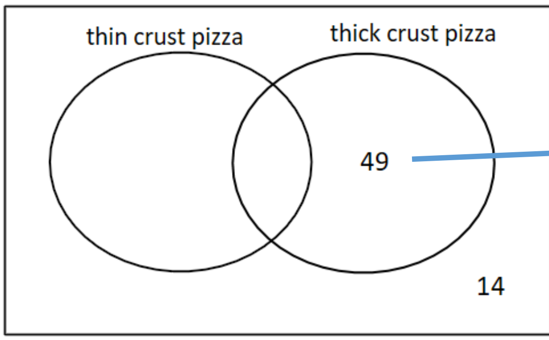


Check off the statement that was used and continue to another statement.

104 people like thin crust pizza.
87 people like thick crust pizza.
63 people do not like thin crust pizza.
✓ **~~14 people do not like thin crust pizza or thick crust pizza.~~**

Step 2: At this time, the first two statements in the list cannot be used because there is not enough information to determine how to distribute the 104 people into the two regions of the thin crust set or how to distribute the 87 people into the two regions of the thick crust set. (See Step 1 of Example 1.5.2). Continue with another statement.

63 people do not like thin crust pizza.



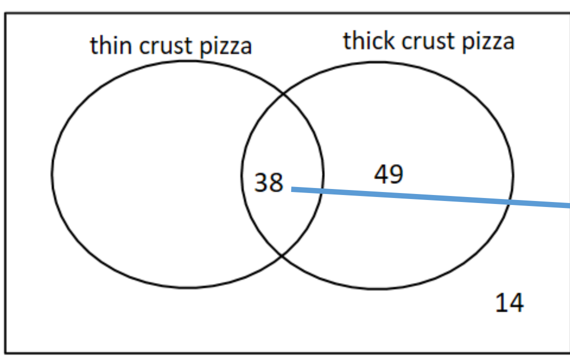
Step 2: 63 people do NOT like thin crust pizza. The total of all regions outside the thin crust set is 63. Therefore, 14 + ? = 63 or that 49 people are in the remaining region.

Check off the statement that was used and continue to another statement.

104 people like thin crust pizza.
87 people like thick crust pizza.
✓ **~~63 people do not like thin crust pizza.~~**
✓ **~~14 people do not like thin crust pizza or thick crust pizza.~~**

Step 3: Find a statement to help complete the thick crust pizza set.

87 people like thick crust pizza.



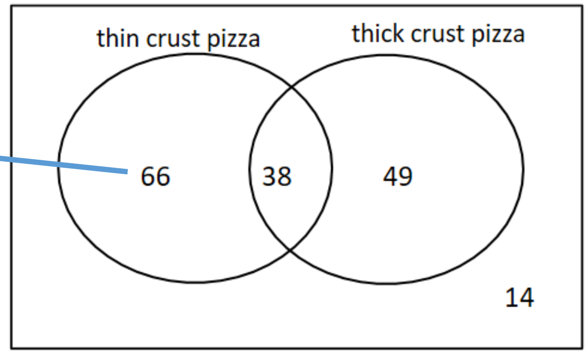
Step 3: 87 people like thick crust pizza. The total of all regions within the thick crust set is 87. Therefore, 49 + ? = 87 or 38 people are in the remaining region of the thick crust set.

Check off the statement that was used and continue to another statement.

104 people like thin crust pizza.
✓ **~~87 people like thick crust pizza.~~**
✓ **~~63 people do not like thin crust pizza.~~**
✓ **~~14 people do not like thin crust pizza or thick crust pizza.~~**

Step 4: The final statement provided: **104 people like thin crust pizza.**

Step 4: 104 people like thin crust pizza. Therefore, 38 + ? = 104 or 66 people are in the remaining region of the thin crust set.



Now use the completed Venn diagram to answer the questions.

a. To find the number of people surveyed, find the sum of all regions.
66 + 38 + 49 + 14 = 167 people were surveyed

b. The number of people who like both thin crust pizza and thick crust pizza is found in the intersection of those sets.
38 people like both thin crust and thick crust.

YOU TRY IT 1.5.A:

Two hundred people living in a high-rise building were asked whether they use the stairs or elevator in their building. Use the information below to complete a Venn diagram and answer the questions.

161 people use the elevator
112 people use the stairs
85 people use both the elevator and stairs
(Hint: Don't forget about the first statement in the paragraph that states 200 people were surveyed).

a. How many people use the stairs only?
b. How many people use neither the stairs nor the elevator?

Three Set Venn Diagram Survey Application Problems

Follow a similar strategy when working with survey application problems with three sets.

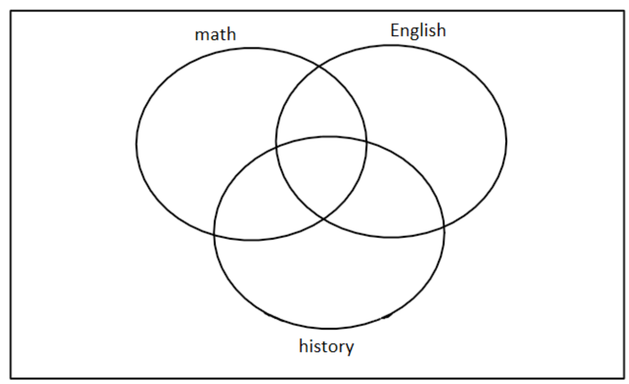
- **EXAMPLE 1.5.4:** Forty students were surveyed about their class schedule. Of those surveyed 18 students were taking math, 21 students were taking English, 15 students were taking history, 5 students were taking math and English, 4 students were taking math and history, 13 students were taking English and history, and 4 students were taking all three subjects.

  a. How many students did not take history?
  b. How many students took math and English but not history?
  c. How many students took English but not history?
  d. How many students took math or English?

SOLUTION:

Complete a Venn diagram to display the survey information.

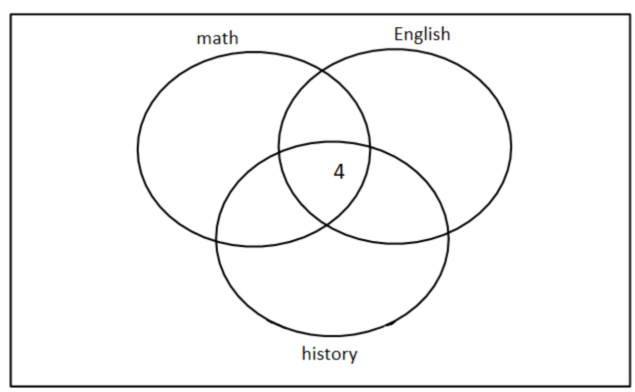
First, determine the number of sets needed in the Venn diagram. In this case, the statements provide information about three types of classes: math, English and history. Draw a three set Venn diagram and label the sets.



Step 1: Find a statement that can be used and begin filling in the Venn diagram. Hint: Start with statements that have information about all three sets.

4 students were taking math, English, and history

Since the students in this statement take all three classes, the 4 goes in the center where all three sets intersect.



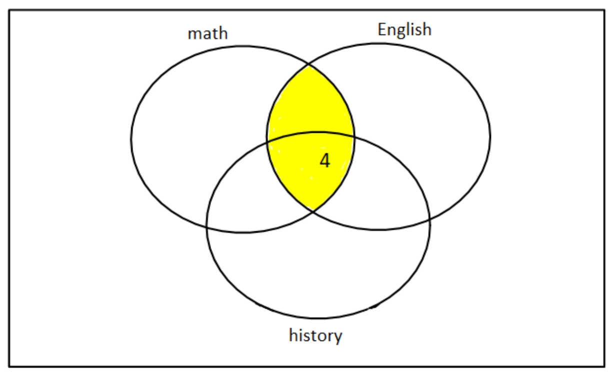
Check off the statement that was used and continue with another statement. Since there are no other statements with information about all three sets, move to the statements with information about two sets (highlighted below). Remember to use one statement at a time.

40 students were surveyed about their class schedule. The following summarize the results of the survey.

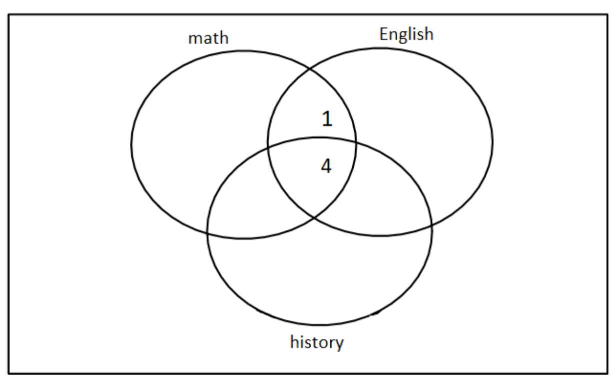
18 students were taking math
21 students were taking English
15 students were taking history
5 students were taking math and English
4 students were taking math and history
13 students were taking English and history
✓ ~~**4 students were taking math, English, and history**~~

Step 2: 5 students were taking math and English

The students in this statement must be within both the math set and the English set. Therefore, the sum of regions where the math set and English set intersect (highlighted) is 5.

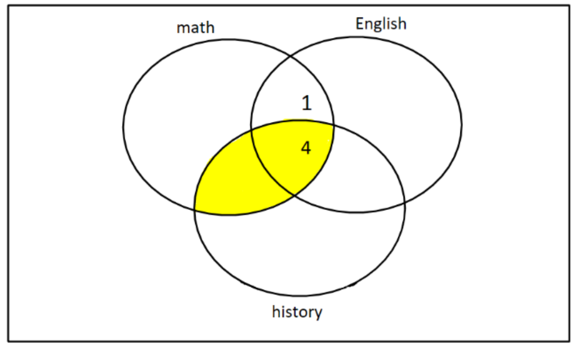


The sum of all of the regions where the math set and English set intersect is 5. Now that 4 students are already listed in the center region, it can be determined that 4 + ? = 5 or that 1 student must be counted in the remaining region of the intersection.

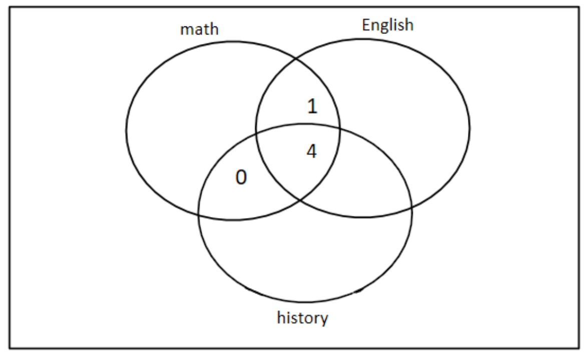


Step 3: 4 students were taking math and history

The students in this statement must be within both the math set and the history set. Therefore, the sum of regions where the math set and history set intersect (highlighted) is 4.

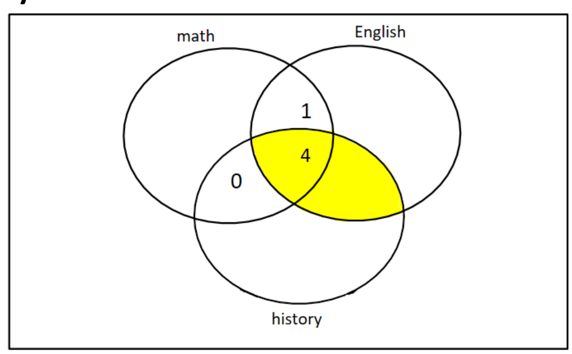


The sum of all of the regions where the math set and history set intersect is 4. Now that 4 students are already listed in the center region, it can be determined that 4 + ? = 4 or that 0 students must be counted in the remaining region of the intersection.

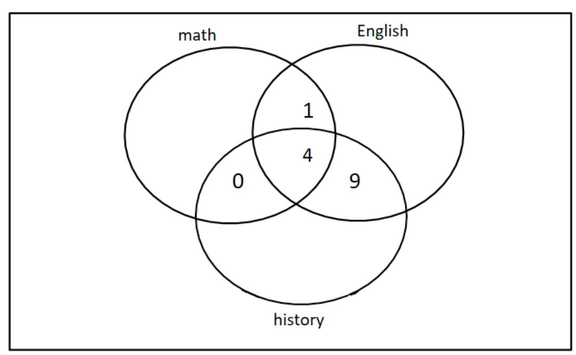


Step 4: 13 students were taking English and history

The students in this statement must be within both the English set and the history set. Therefore, the sum of regions where the English set and history set intersect (highlighted) is 13.



The sum of all of the regions where the English set and history set intersect is 13. Now that 4 students are already listed in the center region, it can be determined that 4 + ? = 13 or that 9 students must be counted in the remaining region of the intersection.



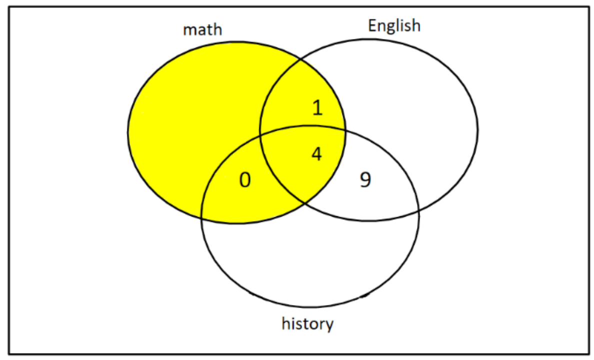
Check off the statements that were used in steps 2, 3 and 4 and continue on to the remaining statements that have information about one set (highlighted below). Remember to use one statement at a time.

40 students were surveyed about their class schedule. The following summarize the results of the survey.

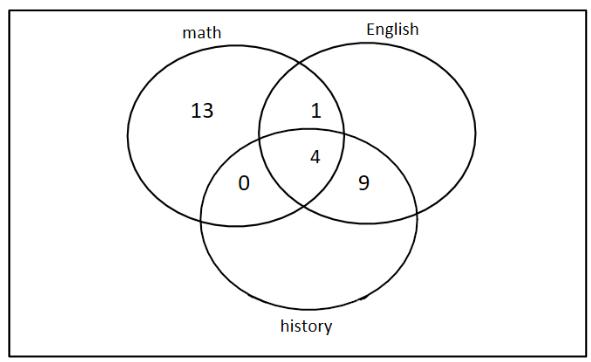
18 students were taking math
21 students were taking English
15 students were taking history
~~**5 students were taking math and English**~~
~~**4 students were taking math and history**~~
~~**13 students were taking English and history**~~
~~**4 students were taking math, English, and history**~~

Step 5: 18 students were taking math

The students in this statement must be within the math set (highlighted).

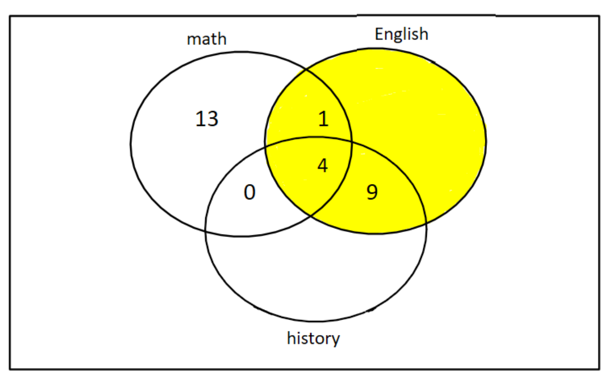


The sum of all of the regions within the math set is 18. Now that three of the four regions are already completed using previous statements, it can be determined that 0 + 4 + 1 + ? = 18 or that 13 students must be counted in the remaining region of the math set.

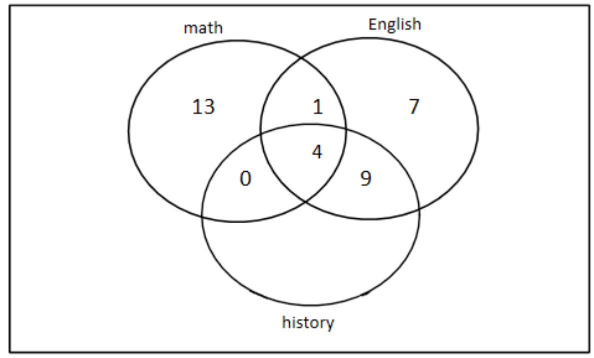


Step 6: 21 students were taking English

The students in this statement must be within the English set (highlighted).

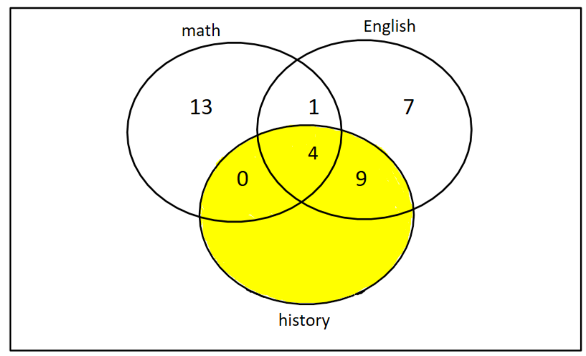


The sum of all of the regions within the English set is 21. Now that three of the four regions are already completed using previous statements, it can be determined that 1 + 4 + 9 + ? = 21 or that 7 students must be counted in the remaining region of the English set.

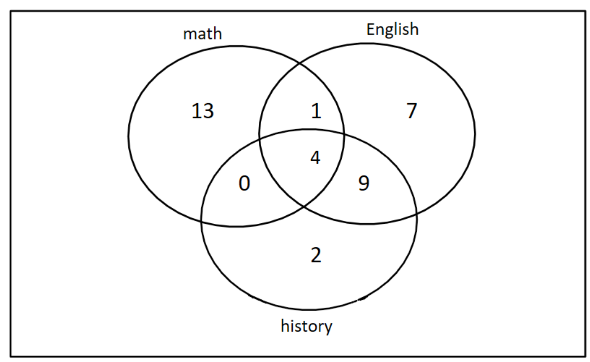


Step 7: **15 students were taking history**

The students in this statement must be within the history set (highlighted).



The sum of all of the regions within the history set is 15. Now that three of the four regions are already completed using previous statements, it can be determined that 0 + 4 + 9 + ? = 15 or that 2 students must be counted in the remaining region of the history set.

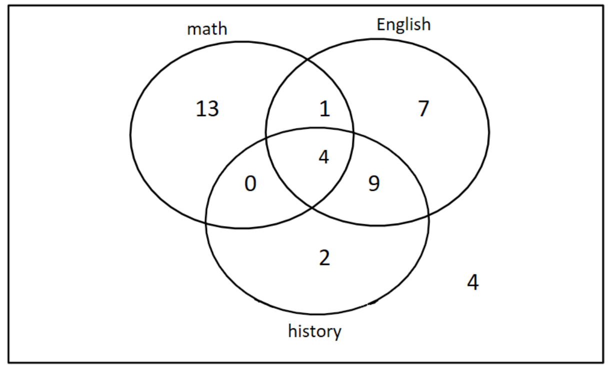


Step 8: One statement remains.

40 students were surveyed about their class schedule. The following summarize the results of the survey.

- ✓ ~~18 students were taking math~~
- ✓ ~~21 students were taking English~~
- ✓ ~~15 students were taking history~~
- ✓ ~~5 students were taking math and English~~
- ✓ ~~4 students were taking math and history~~
- ✓ ~~13 students were taking English and history~~
- ✓ ~~4 students were taking math, English, and history~~

The sum of all of the regions within the universal set is 40. Now that all regions except one have been completed from previous steps, it can be determined that 13 + 1 + 7 + 0 + 4 + 9 + 2 + ? = 40 or that 4 students must be counted in the remaining region of the history set.

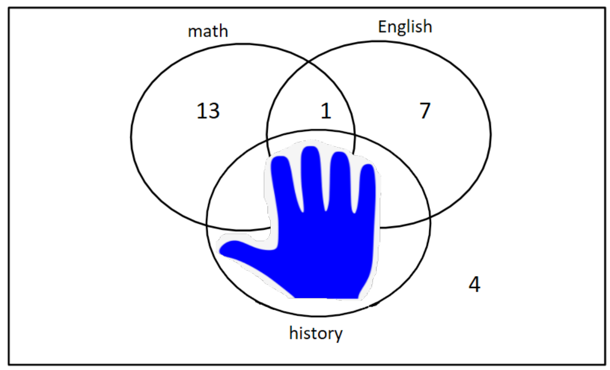


Now use the completed Venn diagram to answer the questions.

a. Find the region(s) that meet the criteria. In order to determine the number of students who did not take history, find the total of all of the regions that are not inside the history set.

Hint: cover/hide the regions in the history set and find the sum of all of the other regions.

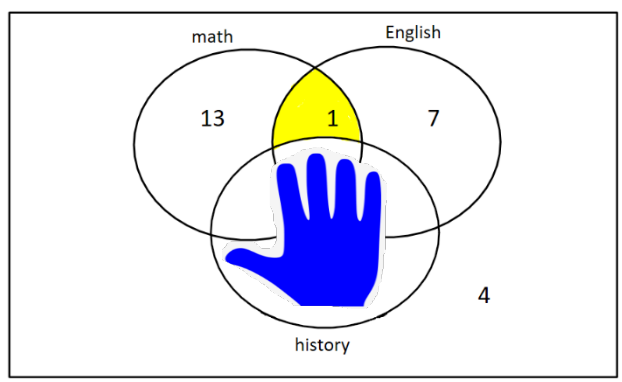
This gives a total of
13 + 1 + 7 + 4 = 25 students who were not taking history.



b. Find the region(s) that meet the criteria. In this case, find the region(s) that are within both the math and English set while also not inside the history set.

Hint: Highlight the intersection of math and English, then cover/hide the regions within the history set.

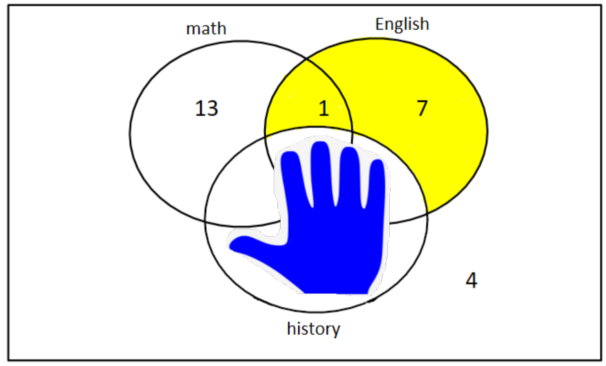
There is just one region that meets the criteria and there is 1 student in this region.



c. Find the region(s) that meet the criteria. In this case, find the region(s) that are within the English set while not inside the history set.

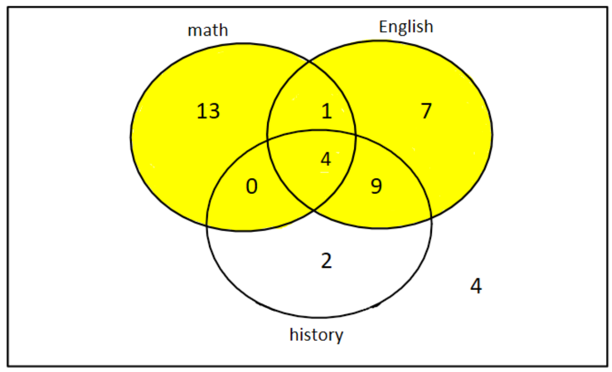
Hint: Highlight the English set, then cover/hide the regions in the history set.

There are two regions with a total of 1 + 7 = 8 students taking English but not history.



d. Find the region(s) that meet the criteria. In this case, find the region(s) that are within the math set OR the English set. Consider the OR similar to union in Section 1.3.

Hint: Highlight the math set and English set. Find the sum of all regions within either of these sets. This gives a total of
13 + 1 + 7 + 0 + 4 + 9 = 34 students were taking math or English.



- **EXAMPLE 1.5.5**: The three most commonly visited European cities are Paris, London and Istanbul. A survey asked college students about travel to these cities. Use the information below to complete a Venn diagram and answer the questions.

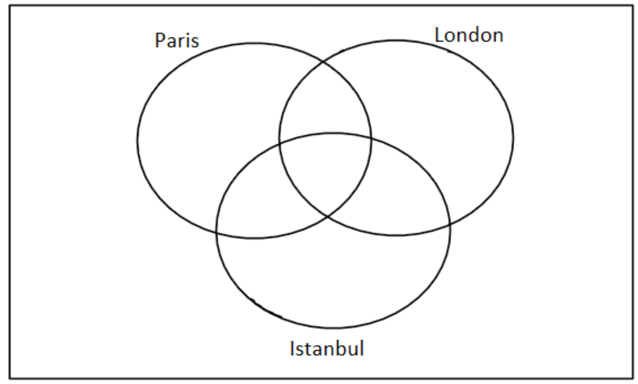
378 students have traveled to Paris
325 students have traveled to London
209 students have traveled to Istanbul
69 students have traveled to London and Istanbul
110 students have traveled to London and Paris
146 students have not traveled to any of these cities
89 students have traveled to Istanbul only (not Paris and not London)
43 students have traveled to London and Istanbul but have not traveled to Paris

a. How many students were surveyed?
b. How many students have traveled to Paris only?

SOLUTION:

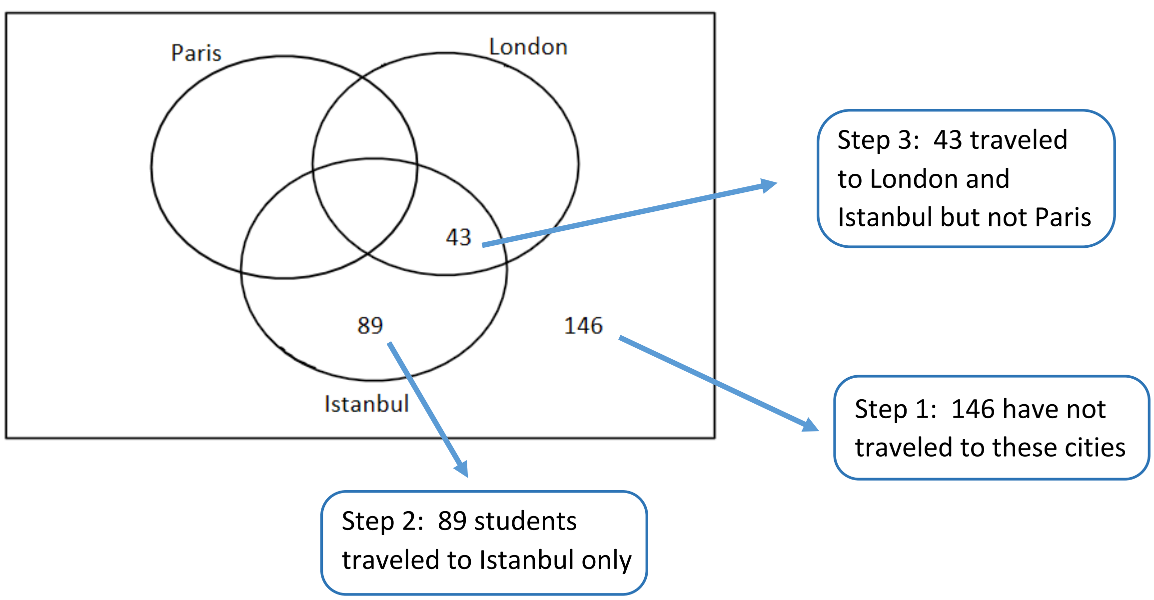
It is often beneficial to complete a Venn diagram to display the survey information.

First, determine the number of sets needed in the Venn diagram. In this case, the statements provide information about three cities: Paris, London and Istanbul. Draw a three set Venn diagram and label the sets.



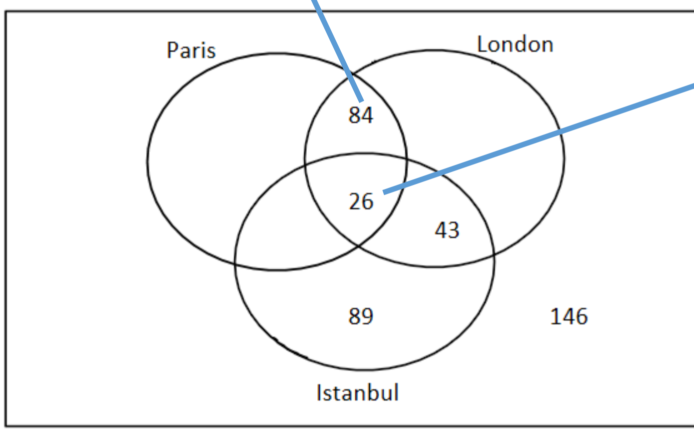
Step 1 – Step 3: Find statements that have information about all three sets. In this case, three statements could be used as a starting point.
Remember to use each statement individually.

Step 1: 146 students have not traveled to any of these cities
Step 2: 89 students have traveled to Istanbul only (not Paris and not London)
Step 3: 43 students have traveled to London and Istanbul but have not traveled to Paris



Step 4 – Step 5: Begin working with statements that have information on two sets.
Step 4: 69 students have traveled to London and Istanbul
Step 5: 110 students have traveled to London and Paris

Step 5: 110 traveled to London and Paris
Think: 26 + ? = 110
Or 110 – 26 already in the intersection of London and Paris = 84 left for the empty region

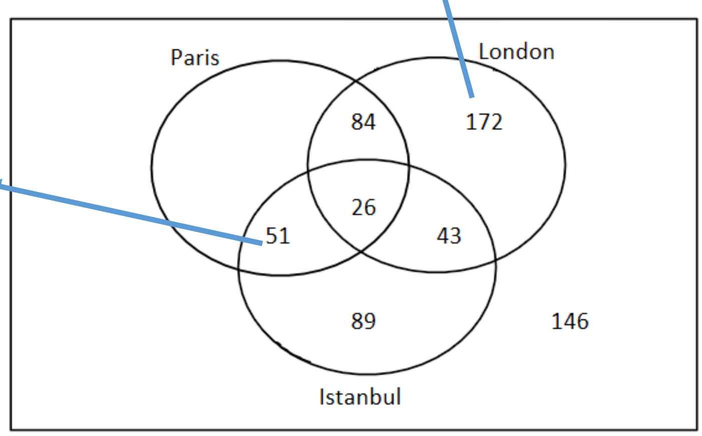


Step 4: 69 traveled to London and Istanbul.
Think: 43 + ? = 69
Or 69 – 43 already in the intersection of London and Istanbul = 26 left for the empty region

Step 6 – Step 7: Begin working with statements that have information on only one set.
Step 6: 325 students have traveled to London
Step 7: 209 students have traveled to Istanbul

Step 6: 325 traveled to London
Think: 84 + 26 + 43 + ? = 325
Or 172 left for the empty region in the London set

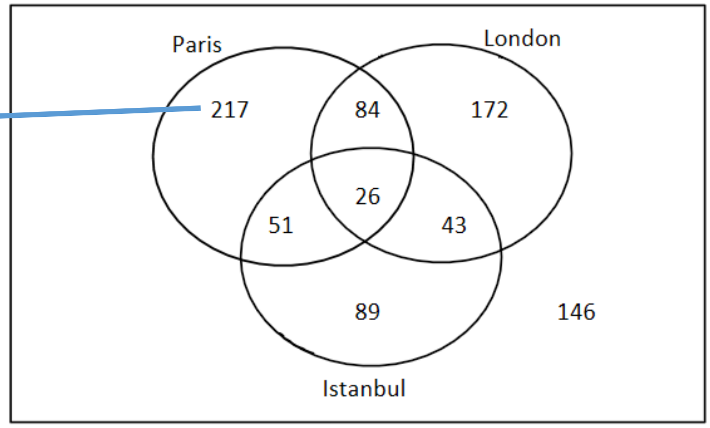
Step 7: 209 traveled to Istanbul
Think: 26 + 43 + 89 + ? = 209
Or 51 left for the empty region in the Istanbul set



Step 8: Now that three of the four regions of the Paris set are completed, use the last remaining statement to complete the Venn diagram.

378 students have traveled to Paris

Step 8: 378 traveled to Paris
Think: 84 + 26 + 51 + ? = 378
Or 217 left for the empty region in the Paris set



Use the Venn diagram information to answer the questions.

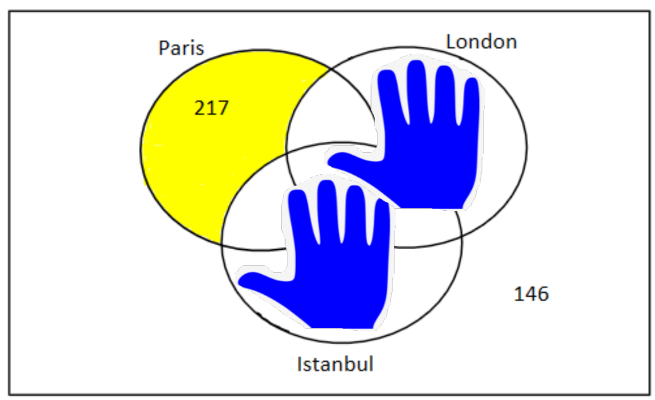
a. Add all of the values in the universal set.

217 + 84 + 172 + 51 + 26 + 43 + 89 + 146 = 828 total people surveyed.

b. Find the region(s) that meet the criteria. In this case, find the region(s) that are within the Paris set but not inside the London set and also not inside the Istanbul set.

Hint: Highlight the Paris set and cover/hide the regions of the London set and also cover/hide the regions of the Istanbul set.

This gives a single region with 217 students.



HELPFUL HINTS FOR SOLVING VENN DIAGRAM SURVEY APPLICATION PROBLEMS

1. Determine how many sets are discussed in your application and draw your blank Venn diagram with the sets labeled.
2. Use one statement at a time.
3. Work with statements that have information about as many sets as possible first.
   - In a three-set Venn diagram, start with statements that include information about all three sets. Then move to statements that include information about two sets. Finally, use statements that provide information for only one set.
   - In a two-set Venn diagram, start with statements that include information about both sets. Then move to statements that include information about only one set.

YOU TRY IT 1.5.B:

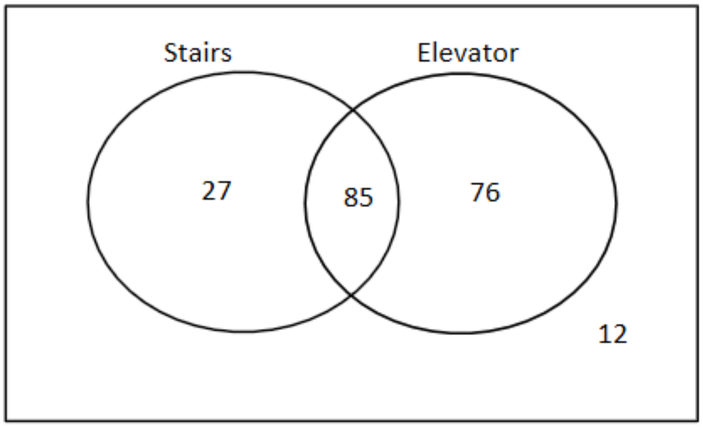
A survey asked people about their preferred toppings for hot dogs. Use the information below to complete a Venn diagram and answer the questions.

72 people like mustard
52 people like onion
95 people like mustard or relish
14 people like onion only (not mustard and not relish)
9 people like mustard and onion but not relish
32 people like mustard and relish
36 people like mustard and onion
103 people do not like onion

- **a.** Complete a Venn diagram using the survey information
- **b.** How many people like mustard only?
- **c.** How many people like mustard or relish?
- **d.** How many people do not like mustard?
- **e.** How many people were surveyed?

YOU TRY IT 1.5.A SOLUTION:

a. Complete a Venn diagram to display the survey information.

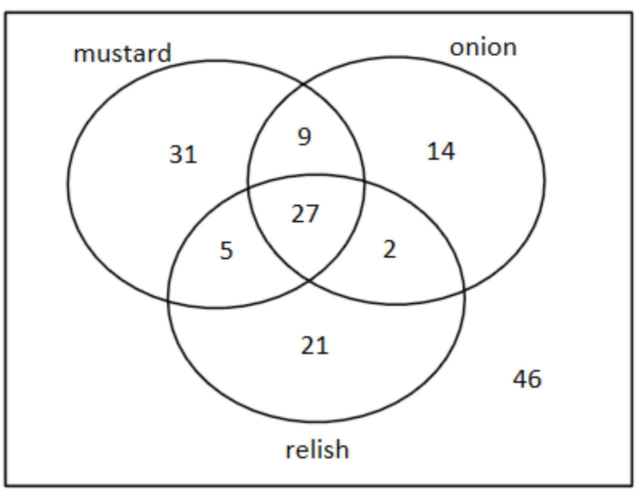


b. How many people use the stairs only? Answer: 27

c. How many people use neither the stairs nor the elevator? Answer: 12 (Maybe they stay on the first floor.)

YOU TRY IT 1.5.B SOLUTION:

a. Complete a Venn diagram using the survey information



b. How many people like mustard only? Answer: 31

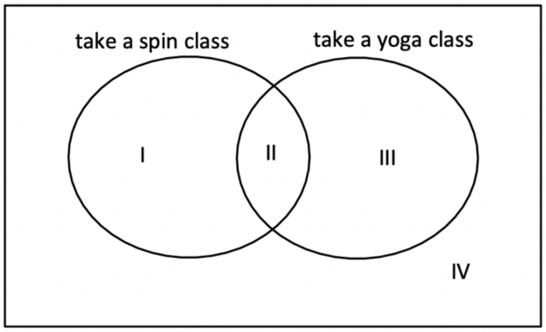
c. How many people like mustard or relish? Answer: 95

d. How many people do not like mustard? Answer: 83

e. How many people were surveyed? Answer: 155

Section 1.5 | Exercises

1. Describe what you know about the people within each section of the diagram below who completed the survey on their exercise last week.



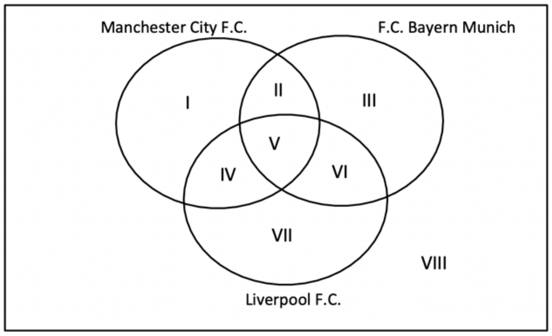
a. Describe the people in region I of the diagram.

b. Describe the people in region II of the diagram.

c. Describe the people in region III of the diagram.

d. Describe the people in region IV of the diagram.

2. Describe what you know about the people within each section of the diagram below who completed the survey on which soccer matches they watched over the weekend.



a. Describe the people in region I of the diagram.

b. Describe the people in region II of the diagram.

c. Describe the people in region III of the diagram.

d. Describe the people in region IV of the diagram.

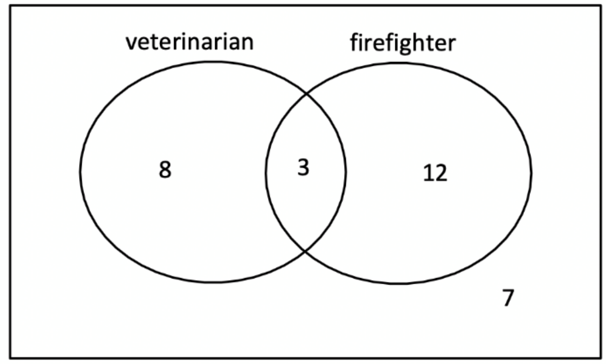
e. Describe the people in region V of the diagram.

f. Describe the people in region VI of the diagram.

g. Describe the people in region VII of the diagram.

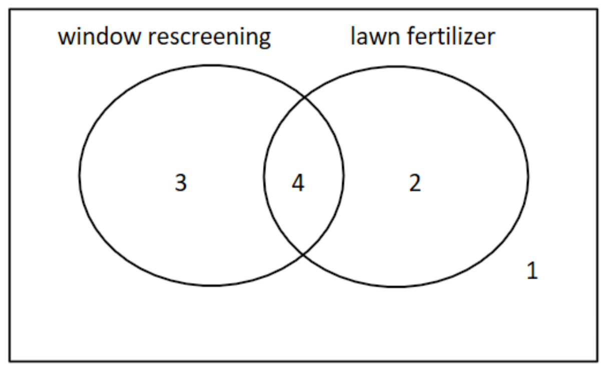
h. Describe the people in region VIII of the diagram.

3. A survey asked kindergarten students how they wanted to dress up for the costume party. Use the diagram provided to answer the following questions.



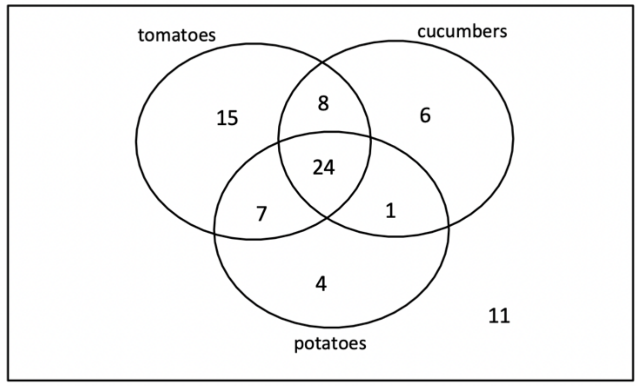
a. How many students were surveyed?
b. How many students want to dress up as a firefighter?
c. How many students want to dress up as a veterinarian and firefighter?
d. How many students want to dress up as a veterinarian or firefighter?
e. How many students do not want to dress up as a veterinarian?

4. One of the authors of this text has many hardware stores near her house. Suppose the author is interested in having windows rescreened and purchasing lawn fertilizer. Use the diagram provided to answer the following questions.



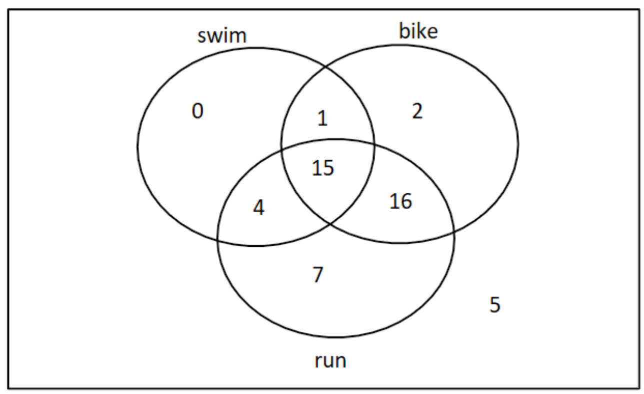
a. How many hardware stores does the author have near her house?
b. How many hardware stores do window rescreening?
c. How many hardware stores sell lawn fertilizer?
d. How many hardware stores don't do window rescreening?
e. How many hardware stores do both window rescreening and sell lawn fertilizer?

5. A survey asked local residents of Milford what they plant in their gardens. Use the diagram provided to answer the following questions.



a. How many residents were surveyed?
b. How many residents plant tomatoes and potatoes but not cucumbers?
c. How many residents plant cucumbers and potatoes?
d. How many residents plant tomatoes or potatoes?
e. How many residents plant tomatoes?
f. How many residents do not plant cucumbers?
g. How many residents plant tomatoes or cucumbers but not potatoes?

6. A survey asked triathletes what their favorite sport is – swim, bike, or run. The triathletes could select multiple sports. The following diagram summarizes the survey results. Use the diagram provided to answer the following questions.



a. How many triathletes were surveyed?
b. How many triathletes said swimming and running are their favorite sports?
c. How many triathletes said running but not biking is their favorite sport?
d. How many triathletes said biking but not running is their favorite sport?
e. What was the most common response from the triathletes? What was the least common response?
f. How many triathletes said biking or running or both is their favorite sport?

7. Use the information below to complete a Venn diagram and answer the questions.

A survey asked 150 people about the type of music they listened to yesterday.
62 people listened to country music
83 people listened to pop music
16 people listened to country music and pop music

a. How many people listened to pop music but not country music?
b. How many people listened to country music but not pop music?
c. How many people listen to country music or pop music?
d. How many people **did not** listen to country music or pop music?

8. A survey was taken of 86 students attending a lecture. Students were asked whether they plan to attend their college's football or basketball games this year. There were 34 students plan on attending football games, 23 of whom are also planning on attending basketball games. Twenty students plan on attending only basketball games.

a. How many students are planning on attending basketball games?
b. How many students are planning on only attending football games?
c. How many students are planning on attending either only a football game or only a basketball game?
d. How many students are not planning on attending a football game or a basketball game?

9. There are 125 students who have turned in an English paper. There are 67 students who used a laptop computer to complete the paper, 41 of whom also used a desktop computer. There are 98 students who used a desktop.

a. How many students used desktop computer but not a laptop?
b. How many students used either a laptop or desktop computer?
c. How many students used neither a desktop nor a laptop computer?
d. How many students did not use a laptop computer?

10. Use the information below to complete a Venn diagram and answer the questions.

A survey asked 600 college students if they prefer sweet or salty study snacks
324 students said they preferred sweet snacks
356 students said they preferred salty snacks
209 students said they preferred sweet snacks but did not like salty snacks

a. How many students liked sweet snacks and salty snacks?
b. How many students like sweet snacks or salty snacks?
c. How many students like salty snacks only?
d. How many students do not like sweet snacks?
e. How many students do not like sweet snacks and do not like salty snacks?

11. Use the information to complete a Venn Diagram and answer the following questions.

The Beverage Digest publishes market shares of the leading liquid refreshment brands (LRB) in the U.S. every year. (Beverage Digest Fact Book 25th edition, page 16, courtesy of statista.com)

In 2019:

5 of the 10 leading liquid refreshment brands were soda drinks
8 of the 10 leading liquid refreshment brands contained caffeine
4 of the 10 leading liquid refreshment brands were soda drinks that contained caffeine.

a. How many LRBs contained caffeine but weren't soda?
b. How many LRBs were soda but didn't contain caffeine?
c. How many LRBs were soda or contained caffeine?
d. How many LRBs weren't soda or didn't contain caffeine?

12. Thanksgiving is a major holiday for many Americans. Suppose 30 College of DuPage students were surveyed about their favorite potato to eat on Thanksgiving. Use the information to complete a Venn diagram and answer the following questions.

    30 people were surveyed
    20 people like mashed potatoes
    7 people like mashed potatoes but not sweet potatoes
    2 people didn't like mashed potatoes or sweet potatoes

a. How many people like sweet potatoes?
b. How many people like sweet potatoes but not mashed potatoes?
c. How many people like both sweet potatoes and mashed potatoes?

13. A survey of 1350 adults was taken to learn their cellphone preferences. Of those surveyed, 748 had invested an iPhone, 667 had an Android, and 217 had both an iPhone and an Android. Of those surveyed,

a. How many had an iPhone but not an Android?
b. How many had an Android but not an iPhone?
c. How many had an iPhone or an Android?
d. How many did not have either an iPhone or an Android?

14. A survey was taken of 67 people to determine whether they prefer shopping online or going to a brick-and-mortar store. Their shopping preferences are:

19 prefer shopping online and brick-and-mortar stores.
28 prefer shopping online.
14 prefer brick-and-mortar stores but not shopping online.

a. How many people prefer brick-and-mortar stores?
b. How many people prefer only shopping online?
c. How many people prefer shopping online or going to a brick-and-mortar store?
d. How many people did not like shopping online or going to a brick-and-mortar store?

15. Use the information below to complete a Venn diagram and answer the questions.

A survey asked 295 people about their preferred toppings for pizza.
126 people like mushrooms
132 people like sausage
73 people like onion
60 people like mushrooms and onion
52 people like mushrooms and sausage
47 people like sausage and onion
38 people like mushrooms, sausage, and onion

a. How many people like mushrooms and sausage but not onions?
b. How many people like mushrooms or onions?
c. How many people do not like mushrooms?
d. How many people do not like any of these toppings?
e. How many people like sausage or onions but not mushrooms?

16. A survey of 525 students was taken to determine what political issues are most important to them. Of those surveyed, 123 stated climate change, 112 stated education costs, and 73 stated racial wealth gap. Fifty-two students stated climate change and education costs, 33 stated climate change and racial wealth gap, and 24 stated education costs and racial wealth gap. Twenty students stated all three political issues were most important to them. Of those surveyed,

a. How many stated only education costs?
b. How many stated climate change and education costs, but not racial wealth gap?
c. How many stated climate change or education costs, but not racial wealth gap?
d. How many stated exactly two of these political issues?
e. How many did not state any of the three political issues?

17. A survey asked adults which female artist they heard on the radio today. Of the 36 adults who heard Adele, 13 heard Adele only. Of the 48 adults who heard Lizzo, 11 heard Lizzo only. There are 23 people who heard Adele, Lizzo, and Olivia Rodrigo on the radio today. From this information, find the following.

a. How many young adults heard Lizzo and Olivia Rodrigo today?
b. How many young adults heard Lizzo and Olivia Rodrigo but not Adele today?
c. How many young adults heard Adele and Lizzo but not Olivia Rodrigo today?
d. How many young adults heard Adele and Olivia today?
e. How many young adults heard Olivia Rodrigo?
f. How many young adults were surveyed?

18. Use the information below to complete a Venn diagram and answer the questions.

A survey asked people about their preferred method to complete a book.
80 people like to read a printed book
93 people like to read on a tablet
40 people like to listen to an audio book
15 people do not like to read a printed book and do not like to read on a tablet
33 people like to read a printed book and read on a tablet
12 people like to read on a tablet and listen to an audio book
25 people like to read a printed book and listen to an audio book
16 people like to read a printed book and listen to an audio book but do not like to use a tablet

a. How many people were surveyed?
b. How many people do not like to read on a tablet?
c. How many people do not like to listen to an audio book?
d. How many people like to only read a printed book?
e. How many people like to read a printed book or book on a tablet but not listen to an audio book?

19. A group of college students were surveyed about how they spent last Friday night:

37 attended a party, saw a movie, and went out to dinner.
128 saw a movie and went out to dinner.
153 went out to dinner
71 went to a party and saw a movie.
74 did not go to a party, see a movie, or go out to dinner.
44 went to a party but did not see a movie or go to dinner
18 went out to dinner only
188 went to a movie.

a. How many students were surveyed?
b. How many students went to a party and saw a movie but did not go out to dinner?
c. How many students went out to dinner and a movie but not a party?
d. How many students went to a party and out to dinner but did not see a movie?
e. How many students attended at least two of these things?

20. A survey asked 104 college students how they typically get to campus. 89 said by car. 17 said by car but not by bus. 68 said by walking. 49 said all three. 57 said car and by walking. 27 said by bus but not by walking. 12 said by bus but not by car.

a. How many said by bus in the survey?
b. How many said both by bus and walking?
c. How many said by walking only?
d. How many said by car only?
e. How many said by car or bus?
f. How many said by car or walking?

21. Trucks, electric vehicles, and non-American manufacturers are popular vehicle choices these days. A survey asked households about the vehicles they own. Use the survey results below to complete a Venn diagram and answer the following questions.

60 households were surveyed
8 households have a vehicle that is American made and electric
1 household had a vehicle that is American made and electric but not a truck
4 households have a vehicle that is electric and a truck, but not American made
11 households have a vehicle that is electric but not a truck and not American made
14 households have a vehicle that is a truck but not electric and not American made
10 households have a vehicle that is an American made truck
12 households had a vehicle that was not a truck, not electric and not American made

a. How many households had an electric vehicle?
b. How many households had a truck?
c. How many households had an American made vehicle?
d. How many households had vehicles that were American made electric trucks?
e. What's the most popular vehicle choice? What's the least popular

Section 1.5 | Exercise Solutions

1. Region I: People who took spin class(s) but not yoga last week

   Region II: People who took both spin class(s) and yoga class(s) last week

   Region III: People who took yoga class(s) but not spin classes last week

   Region IV: People who did not any take spin class and did not take any yoga class last week

2. Region I: People who only watched soccer games for Manchester City F.C.

   Region II: People who watched soccer games for Manchester City F.C. and F.C. Bayern Munich but did not watch games for Liverpool F.C.

   Region III: People who only watched soccer games for F.C. Bayern Munich

   Region IV: People who watched soccer games for Manchester City F.C. and Liverpool F.C. but did not watch games for F.C. Bayern Munich

   Region V: People who watched soccer games for all three teams; Manchester City F.C. and F.C. Bayern Munich and Liverpool F.C.

   Region VI: People who watched soccer games for F.C. Bayern Munich and Liverpool F.C. but did not watch games for Manchester City F.C.

   Region VII: People who only watched soccer games for F.C. Liverpool

   Region VIIII: People who did not watch soccer games for any of the three teams listed; Manchester City F.C. and F.C. Bayern Munich and Liverpool F.C.

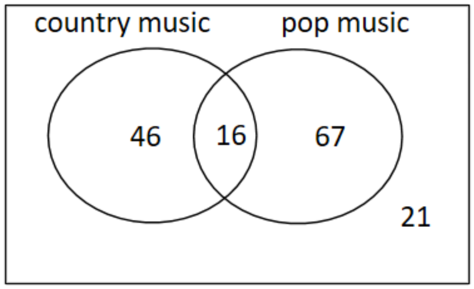
3. a. 30 b. 15 c. 3 d. 23 e. 19

4. a. 10. b. 7 c. 6. d. 3 e. 4

5. a. 76 b. 7 c. 25 d. 59 e. 54 f. 37 g. 29

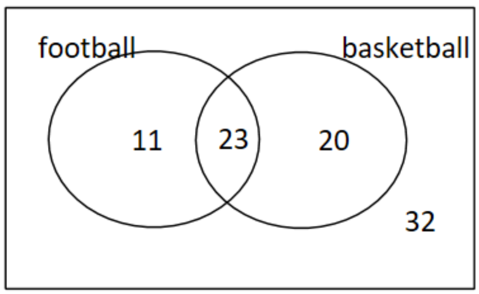
6. a. 50 b. 19 c. 11 d. 3
   e. most common – bike and run but not swim. Least common – swim but nothing else.
   f. 25

7.



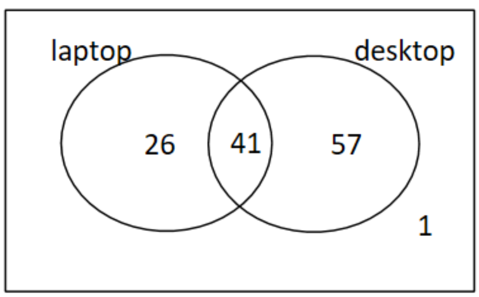
a. 67 b. 46 c. 129 d. 21

8.



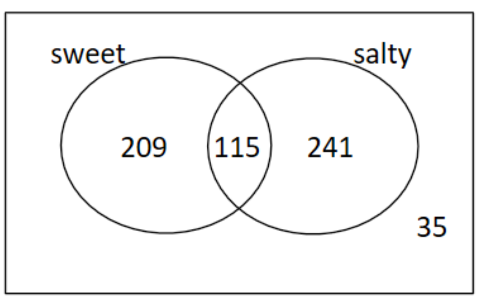
a. 43 b. 11 c. 31 d. 32

9.



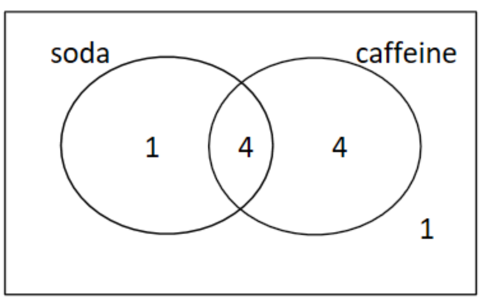
a. 57 b. 124 c. 1 d. 58

10.

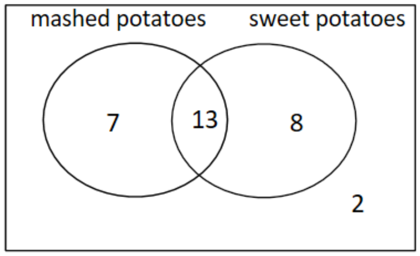


a. 115 b. 565 c. 241
d. 276 e. 35

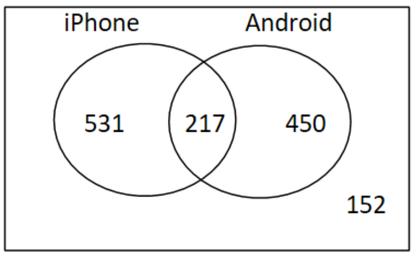
11.



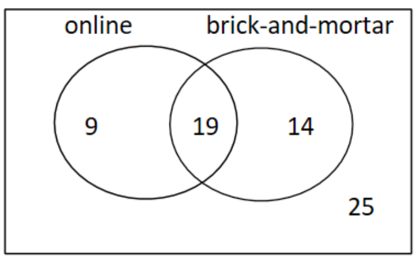
a. 4 b. 1 c. 9 d. 1

12. 

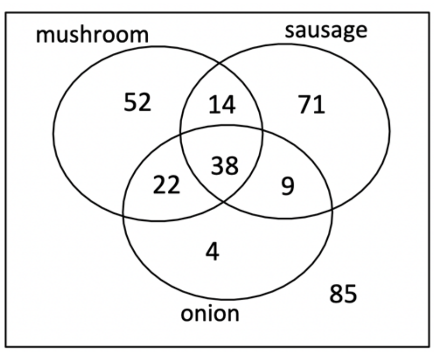
a. 21 b. 8 c. 13

13. 

a. 531 b. 450 c. 1198 d. 152

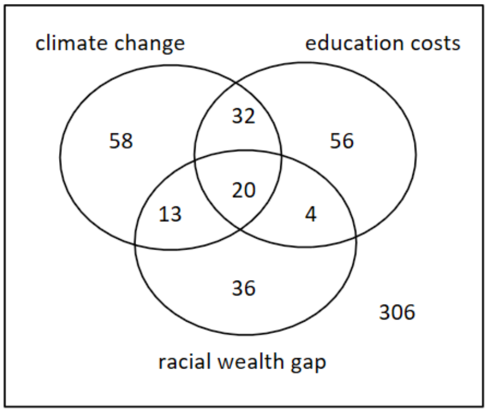
14. 

a. 33 b. 9 c. 42 d. 25

15. 

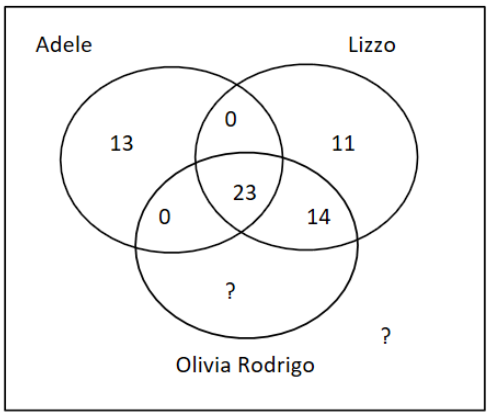
a. 14 b. 139 c. 169
d. 85 e. 84

16.



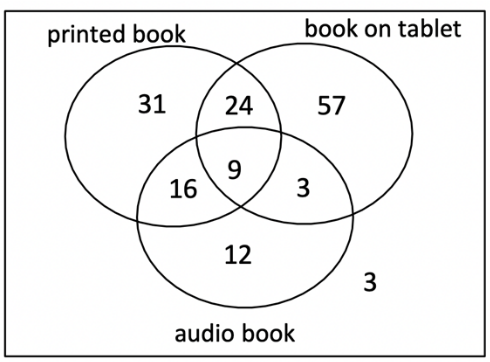
a. 56 b. 32 c. 146
d. 49 e. 306

17.



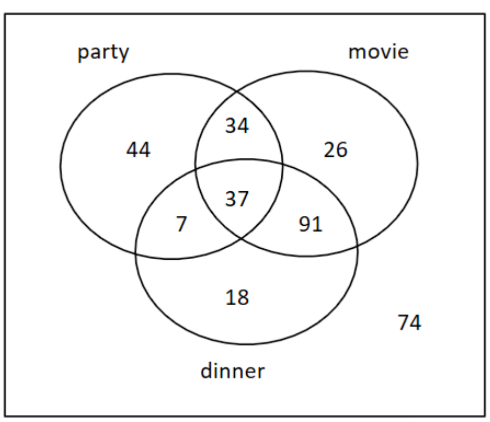
There is not enough information to complete the Venn diagram. Using the limited information provided, the following would be the answers:
a. 37 b. 14 c. 0
d. 23 e. cannot be determined
f. cannot be detemrined

18.



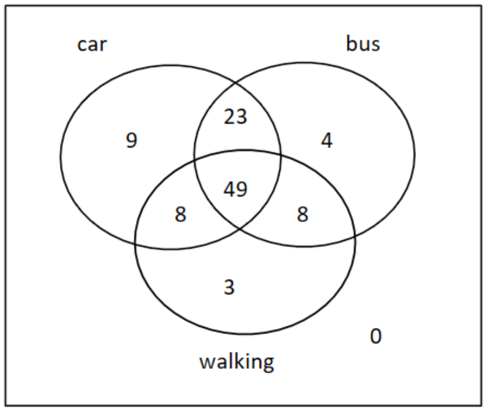
a. 155 b. 62 c. 115
d. 31 e. 112

19.



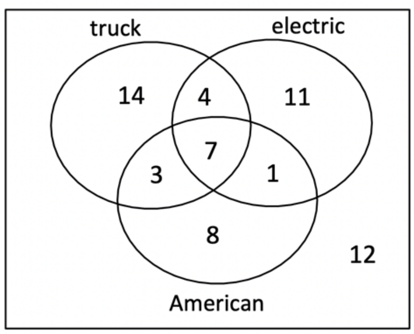
a. 331 b. 34 c. 91
d. 7 e. 169

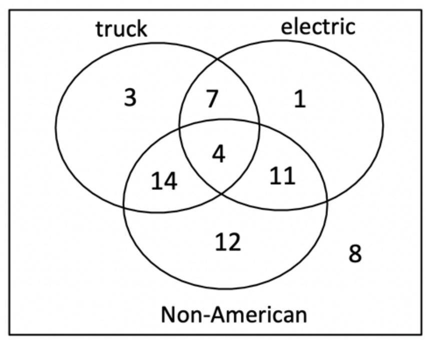
20.



a. 84 b. 57 c. 3
d. 9 e. 101 f. 100

21. Some students might have chosen one set to represent American made vehicles while other students might have chosen non-American vehicles for the set name. Both Venn diagrams are shown.





a. 23 b. 28 c. 19 d. 7
e. most popular: vehicles that are trucks that are not American and not electric
f. least popular: vehicles that are American and electric but not a truck

Chapter 2

Logic Theory

Section 2.1 | Statements, Negation, and Quantifiers

Objectives

- Identify simple statements in logic
- Write the negation of a simple statement in logic
- Identify compound and quantified statements in logic
- Write the negation of a quantified statement in logic

Logic is the branch of mathematics that deals with the truth of various statements and arguments. The rules of logic (together with a bit of binary code) govern how computers operate. Fuzzy logic, or logic with *true, false,* and *kind of true* values shows up in anti-lock braking, dishwashers, and elevators, for example.[1] Logic is also present in day-to-day life. When a mother tells her child "If you clean your room, you can have ice cream!" what is she really saying? Suppose a student argues "All welding classes are hard. This is a welding class; therefore, it will be hard." Is that valid? The study of logic begins with definitions then progresses through truth tables, verifying arguments, and Euler diagrams.

The study of logic begins with the definition of a few fundamental terms.

Statements in Logic

DEFINITION: A **statement in logic** is a declarative statement that is either true or false but not both. A statement in logic cannot be an opinion, question, exclamation, or command.

True or *false* is called the truth value of a statement in logic. For example, the statement "The Grand Canyon is in Arizona" has a truth value of *true*. The statement "Tuesday follows Friday in the days of the week" has a truth value of *false*.

➢ **EXAMPLE 2.1.1:** Are the following sentences statements in logic? If not, why not?

  a. That car is red.
  b. That car is the coolest car Honda makes.
  c. Pass the remote!
  d. Chicago is the capital of Illinois.
  e. Oh my gosh!

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[1] From https://www.cnet.com/news/google-doodle-honors-lotfi-zadeh-father-of-fuzzy-logic/

SOLUTION:

a. This is a statement in logic. The statement is either true (if the car is red) or false (if the car is any other color but red). The statement cannot be both true and false.

b. This is not a statement in logic. This is an opinion. That is, one person may find this statement to be true and another person may find this statement to be false. A statement in logic cannot be both true and false simultaneously.

c. This is not a statement in logic. This is a command and is neither true nor false.

d. This is a statement in logic. It happens to be false.

e. This is not a statement in logic. It is an exclamation and has no truth value.

Negations

DEFINITION: The **negation of a statement** is a statement with the opposite truth value from the original statement.

For example, the negation of “Today is Wednesday” is “Today is not Wednesday.” Notice that the negation retains as much of the original statement as possible. That is, the negation of “Today is Wednesday” is not “Today is Tuesday” even though, in some cases, those statements will have opposite truth values. The negation of a statement also doesn’t introduce any new information. For example, the negation of “Today is Wednesday” is not “The sky is blue.” The introduction of new information is not necessary.

- **EXAMPLE 2.1.2:** What is the negation of the following statements?
  - **a.** It is winter.
  - **b.** I have a cat.
  - **c.** Franklin Roosevelt was a president of the United States.

  SOLUTION:

  a. The negation of the statement “It is winter” is “It is not winter.” Notice the negation isn’t “It is spring.” While “It is winter” and “It is spring” sometimes have different truth values, they don’t always. For example, during summer, the statements “It is winter” and “It is spring” are both false. Thus “It is spring” cannot be the proper negation of “It is winter.” Also, the statement “It is spring” introduces new information which isn’t wise when writing a negation.

b. The negation of the statement “I have a cat” is “I do not have a cat.”

c. The negation of the statement “Franklin Roosevelt was a president of the United States” is “Franklin Roosevelt was not a president of the United States.”

Simple and Compound Statements

DEFINITION: A **simple statement** is a statement that contains only one idea. All the statements in logic presented thus far in this section are simple statements. Other simple statements include, for example:

- Simone Biles is a NASCAR driver.
- Betty White lived to be 99 years old.
- There is a Starbucks at College of DuPage.

DEFINITION: A **compound statement** is a statement that connects two or more ideas. In other words, a compound statement connects two or more simple statements. There are different ways to connect simple statements (and, or, if/then, etc.). These words that are used to connect simple statements are called **connectives**. Connectives will be explored more in Section 2.2. Some examples of compound statements include:

- If the moon is made of cheese, then tomorrow you’ll win the lottery.
- Simone Biles is a gymnast or a NASCAR driver.
- Today, I will complete a puzzle and I will go to the library.

➢ **EXAMPLE 2.1.3:** Are the following sentences statements in logic? If not, explain why not. If so, is the sentence a simple or a compound statement?

a. The best way to train for a marathon is to run far and run slow.
b. A marathon is 26.2 miles long and a half marathon is 13.1 miles long.
c. I need eggs, bread, and orange juice at the grocery store.

SOLUTION:

a. This is not a statement because it is an opinion. To some people, this statement may be true. To others, it may be false. A statement must have a single truth value.

b. This is a compound statement. It connects the idea “Simone Biles is a gymnast” with the idea “Simone Biles is a NASCAR driver.” It does not matter that one statement is true and one is false. The truth value of compound statements will be covered later in this chapter.

c. This is a compound statement. It connects the ideas "I need eggs," "I need bread," and "I need orange juice."

Negations of compound statements are more complex than negations of simple statements. Negations of compound statements will be studied in future sections.

Quantified Statements

DEFINITION: A **quantified statement** is a statement including a quantifier. Some examples of quantifiers include ***all, none, every, some,*** or ***at least one***.

DEFINITION: A **universal quantifier** is a quantifier such as ***all*** or ***every***. Universal quantifiers signify that all items share a certain characteristic.

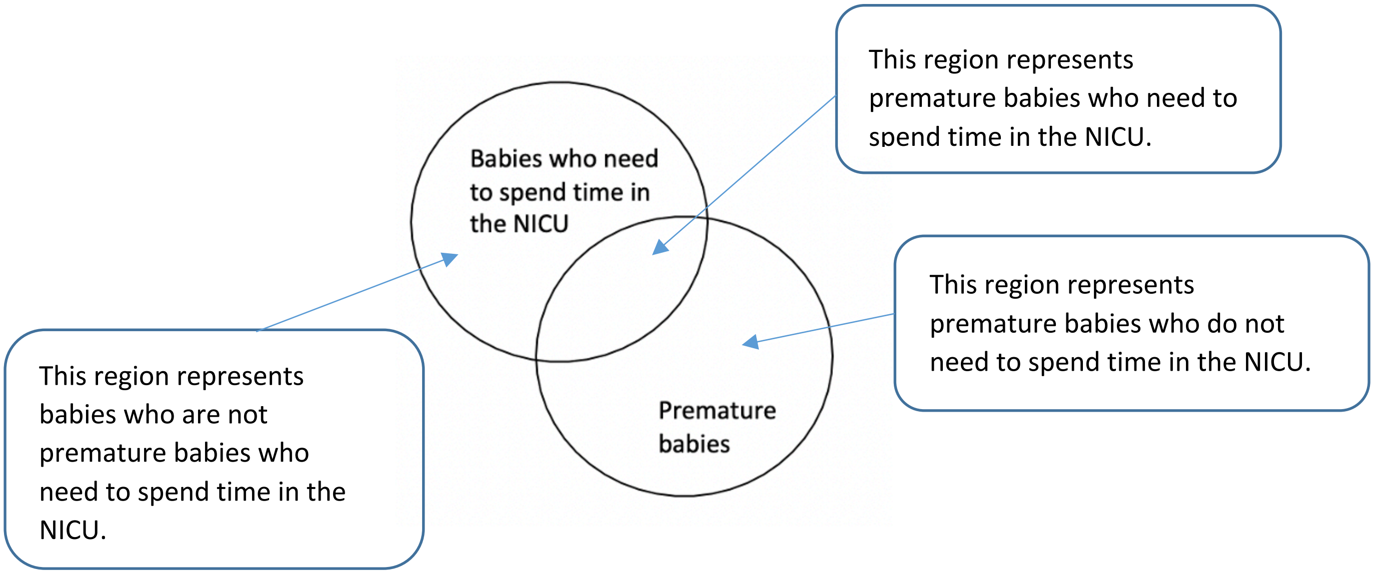
DEFINITION: An **existential quantifier** is a quantifier such as ***some*** or *a* ***few***. Existential quantifiers signify that at least one item exists that has a certain characteristic.

Examples of quantified statements include:

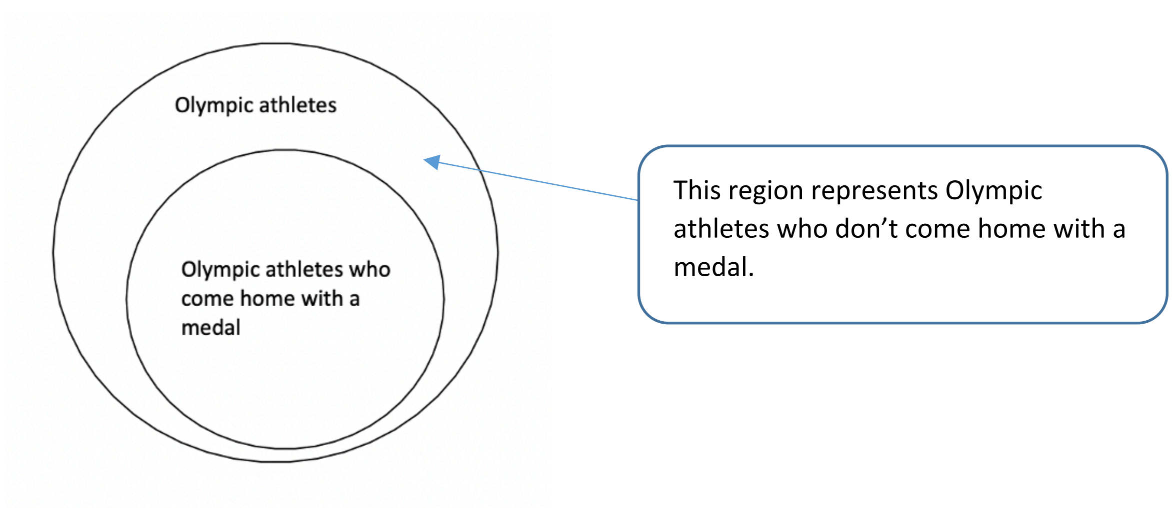
- Some premature babies need to spend time in the NICU.
- Not all Olympic athletes come home with a medal.
- No one is infected with smallpox in the 21st century.
- All humans belong to the mammal class.

A visual diagram such as an **Euler diagram** can be used to visualize statements involving quantifiers. An Euler diagram resembles a Venn diagram. An Euler diagram, however, helps show relationships between objects and certain characteristics.

One possible Euler diagram for the statement "Some premature babies need to spend time in the NICU" is shown on the next page. One circle represents the set of premature babies. The other circle represents the set of babies that need to spend time in the NICU. Since some, but not all, premature babies need to spend time in the NICU, the two circles overlap, but also have some areas not common to the two characteristics.

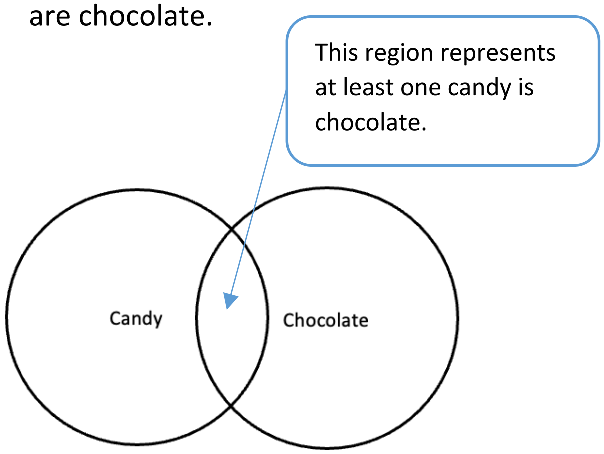


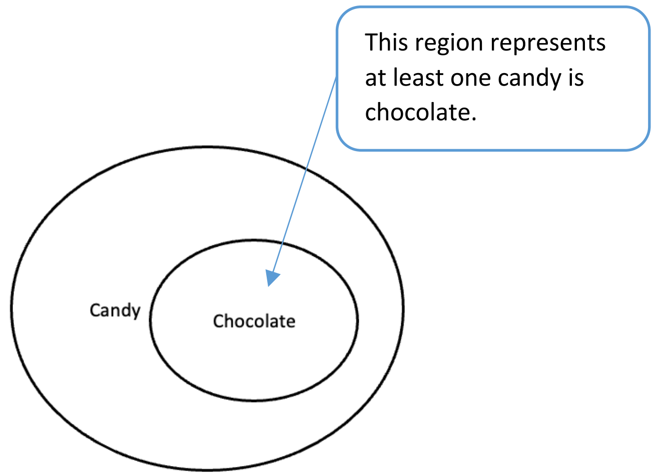
One possible Euler diagram for the statement “Not all Olympic athletes come home with a medal” is shown below. Note that another way to write this statement is “At least one Olympic athlete does not come home with a medal” or “Some Olympic athletes don’t come home with a medal.”

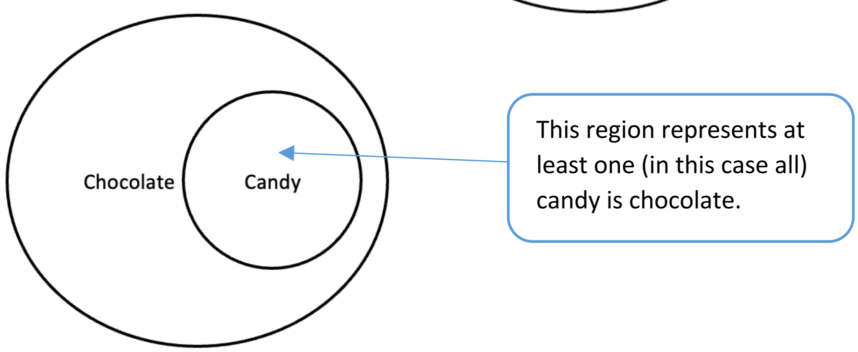


By definition, *some* means *at least one*. As a result, more than one Euler diagram might be appropriate in situations involving the quantifier some (at least one).

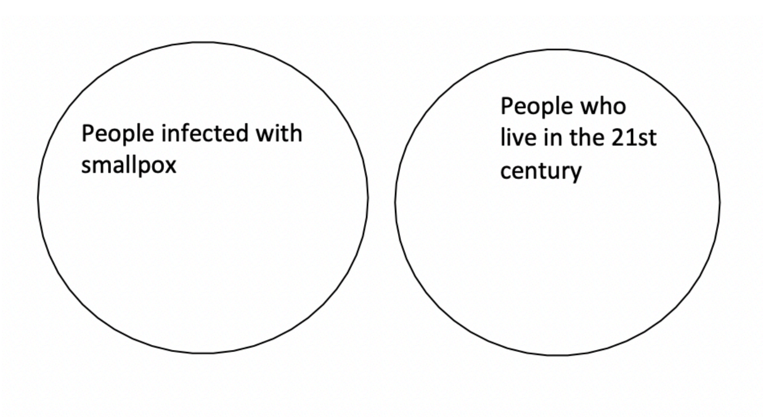
The following three Euler diagrams could represent the statement "Some candy is chocolate." Here, one circle represents the set of candy. The other circle represents the set of items that are chocolate.



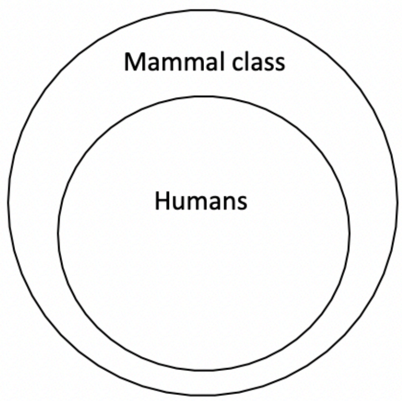




One possible Euler diagram for the statement "No one is infected with smallpox in the 21^{st} century" is shown below. Since there is nothing in common between the characteristics *smallpox* and *people in the* 21^{st} *century* those regions have no overlap. Recall, this is an example of disjoint sets as discussed in Section 1.2.



One possible Euler diagram for the statement "All humans belong to the mammal class" is shown below. Note that the *human* characteristic is completely contained within the *mammal class* characteristic.

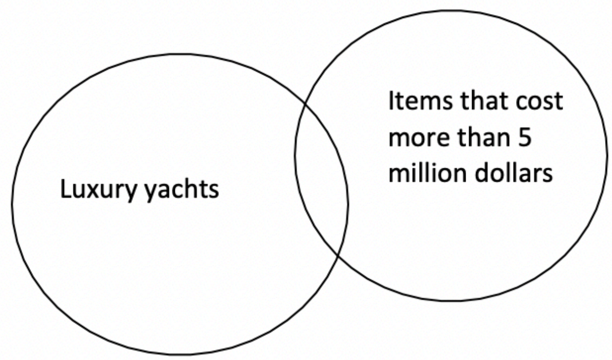


- **EXAMPLE 2.1.4:** Identify the quantifier in the following statements. State if the quantifier is a universal quantifier or an existential quantifier. Create one possible Euler diagram for the statement.
  - **a.** All grocery stores are less crowded before 10 am.
  - **b.** There exists a luxury yacht that costs more than 5 million dollars.
  - **c.** Not all college students are full-time students.

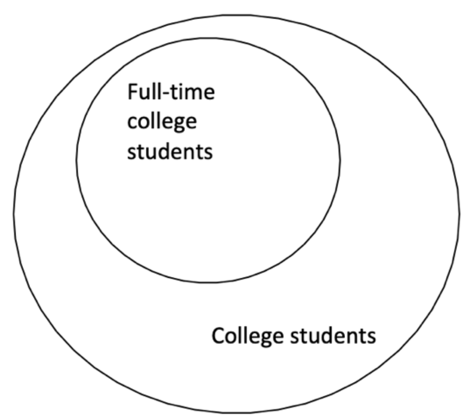
  SOLUTION:

  a. In the statement "All grocery stores are less crowded before 10 am," the quantifier is ***all***. This is a universal quantifier. It signifies that every grocery store is less crowded before 10 am.

b. For the statement, "There exists a luxury yacht that costs more than 5 million dollars" the quantifier is ***there exists.*** This is an existential quantifier. It signifies that at least one luxury yacht costs more than 5 million dollars.



c. For the statement "Not all college students are full-time students," the quantifier is ***not all***. It is an existential quantifier.

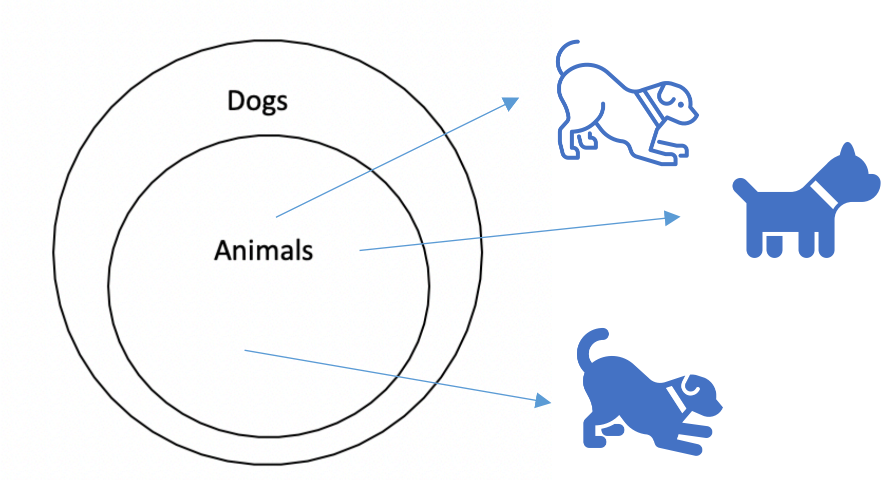


In this diagram, notice that there are some students outside the full-time circle; these would be the college students that are not full-time students (i.e., part-time students).

Negating quantified statements is also more complex than negating a simple statement. Recall a statement and its negation must have opposite truth values in all circumstances. One straightforward way to do this is to ask, "what does it take to prove the original statement false?"

Consider the statement “All animals are dogs” as an example.

One possible Euler diagram for the statement “All animals are dogs” is below.

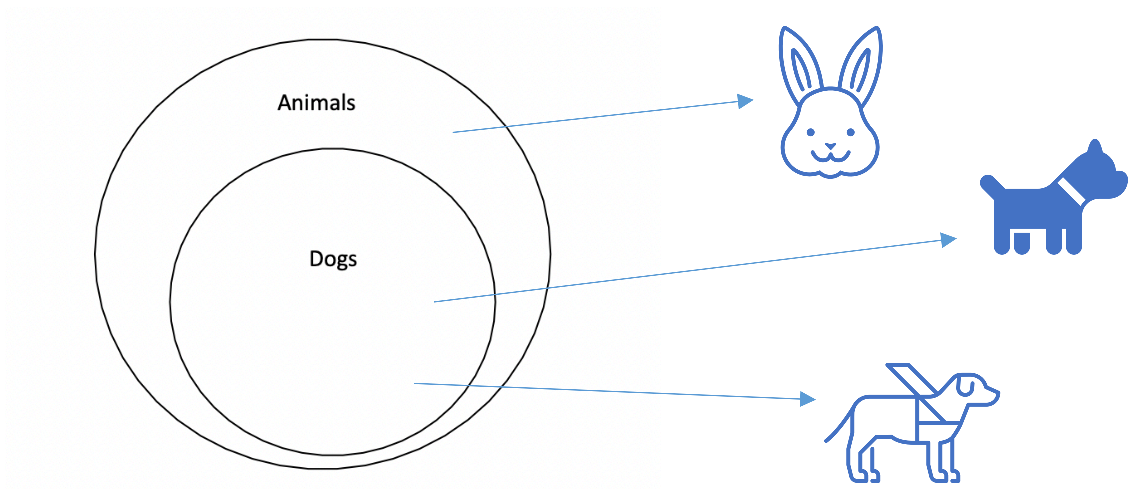


❖ **THINK ABOUT IT:** What does it take to prove the statement “All animals are dogs” is false?

  ANSWER: It would take just one animal that is not a dog. For instance, a cat, iguana, or rabbit is an animal that is not a dog.

Therefore, the negation of “All animals are dogs” is “At least one animal is not a dog” or “Not all animals are dogs.”

One possible Euler diagram for the statement “Not all animals are dogs” is below.



NOTE: The placement of the word ***not*** matters greatly, as does whether the quantifier is ***one*** or ***all***.

A summary of the general rules for negating quantified statements is in Figure 2.1.1.

FIGURE 2.1.1

Original statement	What is needed to prove it false	General form of negation
All A are B. **Ex:** All flowers are red.	At least one A is not B. **Ex:** At least one flower is not red.	Some A are not B. **Ex:** Some flowers are not red.
Some A are B. **Ex:** Some insects have 6 legs.	No A that are B. **Ex:** No insects that have 6 legs.	No A are B. **Ex:** No insects have 6 legs.
No A are B. **Ex:** No horses have horns.	One or more A that is B. **Ex:** One or more horses that have horns.	Some A are B. **Ex:** At least one horse has horns.
Some A are not B. **Ex:** Some beverages are not soda.	All A would have to be B. **Ex:** All beverages would have to be soda.	All A are B. **Ex:** All beverages are soda.

➢ **EXAMPLE 2.1.5:** Write the negation for each of the following quantified statements.

a. Some premature babies need to spend time in the NICU.
b. Not all Olympic athletes come home with a medal.
c. No one comes down with smallpox in the 21^{st} century.
d. All humans belong to the mammal class.
e. There exists a luxury yacht that costs more than 5 million dollars.

SOLUTION:

a. To prove this statement false, all premature babies would need to *not* spend time in the NICU. Since the original statement contains an existential quantifier (***some***), the negation needs to contain a universal quantifier (***all***). The negation of the original statement is "All premature babies don't need to spend time in the NICU" or "No premature babies need to spend time in the NICU."

b. To prove this statement false, all Olympic athletes would need to come home with a medal. The negation of the original statement is the statement "All Olympic athletes come home with a medal."

c. To prove this statement false, someone would have to come down with smallpox in the 21^{st} century. The negation of the original statement is "At least one person came down with smallpox in the 21^{st} century."

d. To prove this statement false, at least one human would have to belong to a different class. The negation of the original statement is "At least one human doesn't belong to the mammal class."

e. To prove this statement false, all luxury yachts would have to cost less than or equal to 5 million dollars. The negation of the original statement is the statement "All luxury yachts cost less than or equal to 5 million dollars." Or this could be said as "No luxury yachts cost more than 5 million dollars."

Quick Review

- A **simple statement** is a declarative sentence that is either true or false.
- A **compound statement** is two or more simple statements joined with a connective (and, or, if/then, etc.).
- A **quantifier** details how many using words such as *some, none, all,* or *there exists.*
- A **quantified statement** is a statement containing a quantifier.
- The **negation** of a statement is a statement with the opposite truth value.
- An **Euler diagram** is a visual representation of the relationship between objects and their characteristics.

Section 2.1 | Exercises

For exercises 1-10, identify whether the statement is a statement in logic and explain why or why not.

1. Math is the best subject.
2. Branches of mathematics include algebra, geometry, and calculus.
3. Gibberish is a spoken language.
4. I need help!
5. Take the dog for a walk.
6. Cats are the best pets.
7. The temperature of the sun is about 10,000 degrees Celsius.
8. Marie Currie is the only person to win the Nobel prize in two scientific fields.
9. Brush your teeth.
10. Katherine Johnson calculated the flight paths for the Apollo moon landing.

For exercises 11-15, write the negation of the given simple statement.

11. Brandon Sanderson is a famous fantasy and science fiction author.
12. Brandon Sanderson wrote *The Lord of the Rings.*
13. The *Mistborn* trilogy was written between 2006-2008.
14. There are seven books in the original *Harry Potter* series.
15. Harrison Ford wrote the song “Hey Jude”.

For exercises 16-20, identify the sentence as a simple or compound statement and explain.

16. An 18-month toddler should be able to say 10-15 words.
17. An 18-month toddler may throw lots of temper tantrums and give lots of sweet hugs.
18. Avocado production in the U.S. mainly occurs in California.
19. Avocados are thought to have originated from Mexico and Central America.
20. John Lennon and Paul McCartney were members of the Beatles.

For exercises 21-30, identify the quantifier in the following statements. State if the quantifier is a universal quantifier or an existential quantifier. Finally, create one possible Euler diagram for the statement.

21. Some coffee is sold in whole bean form.
22. All of the world’s coffee was affected by the drought in Brazil in 2021.
23. No coffee is completely caffeine free. (This is true; even decaf coffee contains a small amount of caffeine.)

24. There exists at least one college student who works full time.
25. All snakes are poisonous.
26. No videos on YouTube are educational.
27. All dogs have four legs.
28. Some flowers bloom in summer.
29. No one folds laundry at midnight.
30. Goldilocks found at least one chair comfortable.

For exercises 31-40, write the negation for each of the quantified statements.

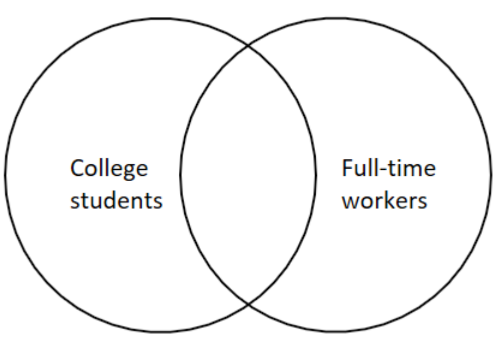
31. On some spring weekends, the zoo does a *Breakfast with a Bunny* event.
32. The zoo has some big cats on display.
33. On all spring days, the zoo is open until 7 pm.
34. All the world's music can be found on Spotify.
35. None of the main streets are open during the parade.
36. The store had no rolls of toilet paper.
37. Some birds cannot fly.
38. Some dinosaur skeletons have been discovered.
39. At least one person has swam the English Channel four times without stopping.
40. All dinosaurs were larger than the modern elephant.

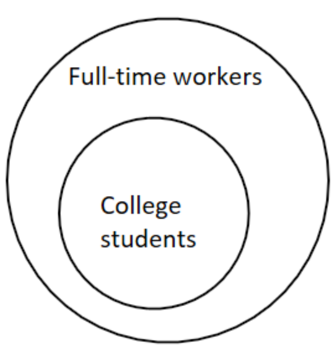
Concept Review

41. Draw an Euler diagram for the statement "All A are B." Draw an Euler diagram for the statement "All B are A." Are the Euler diagrams the same? Why or why not?
42. Is the statement "Not all animals are dogs" the same as the statement "All animals are not dogs?" Why or why not? An Euler diagram may help in your explanation.
43. Why is it incorrect to state that the negation of "All A are B" is "No A are B"? Give a counterexample to show why this is not the case.
44. Why is it incorrect to state that the negation of "Some A are B" is "Some A are not B"? Give a counterexample to show why this is not the case.

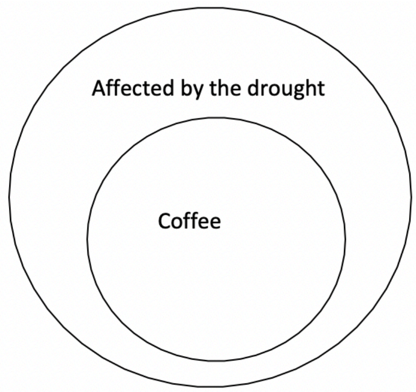
Section 2.1 | Exercise Solutions

1. No, this is an opinion. Opinions are not statements in logic.
2. This is a statement.
3. This is a statement.
4. No, this is an exclamation. Exclamations are not statements in logic.
5. No, this is a command. Commands are not statements in logic.
6. No, this is an opinion. Opinions are not statements in logic.
7. This is a statement.
8. This is a statement.
9. No, this is a command. Commands are not statements in logic.
10. This is a statement.
11. Brandon Sanderson is not a famous fantasy and science fiction author.
12. Brandon Sanderson did not write the *Lord of the Rings.*
13. The *Mistborn* trilogy was not written between 2006-2008.
14. There are not seven books in the original *Harry Potter* series.
15. Harrison Ford did not write the song "Hey Jude."
16. Simple. This sentence represents one simple statement or idea.
17. Compound. This sentence represents two simple statements or ideas – "tantrums" and "hugs."
18. Simple. This sentence represents one simple statement or idea.
19. Compound. This sentence represents two simple statements or ideas for the origin of avocados – "Mexico" and "Central America."
20. Compound. This sentence represents two simple statements or people as band members – "John Lennon" and "Paul McCartney."
21. The quantifier is *some*. This is an existential quantifier.
    Diagrams may vary. Two possibilities are shown.

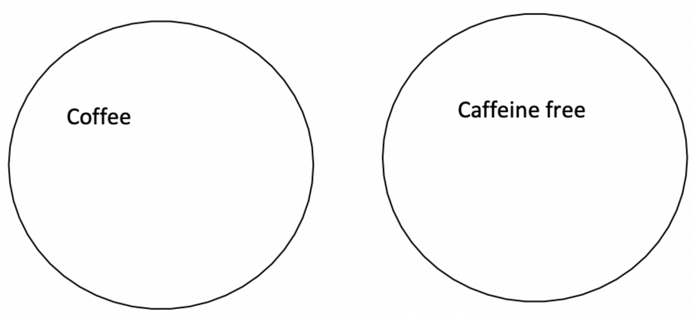




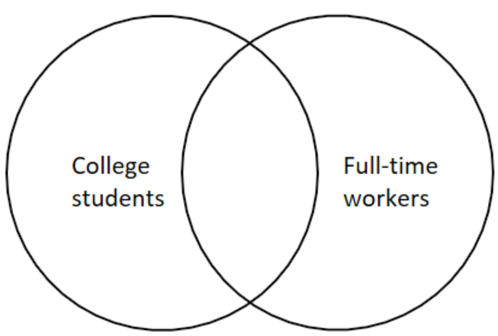
22. The quantifier is *all.* This is a universal quantifier.

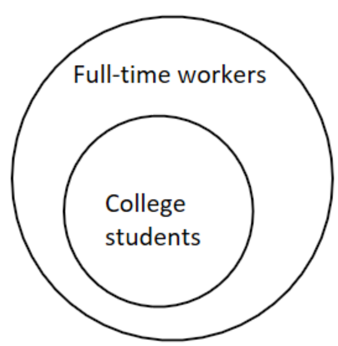


23. The quantifier is *no.* This is a universal quantifier.

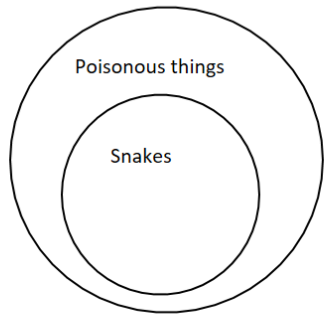


24. The quantifier is *at least one*. This is an existential quantifier.
Diagrams may vary. Two possibilities are shown.

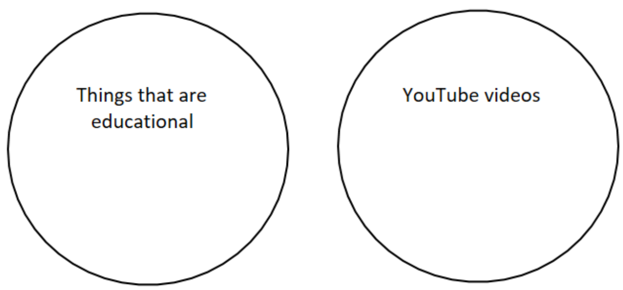




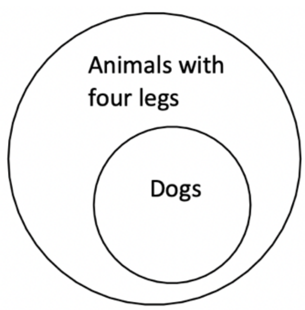
25. The quantifier is *all*. This is a universal quantifier.



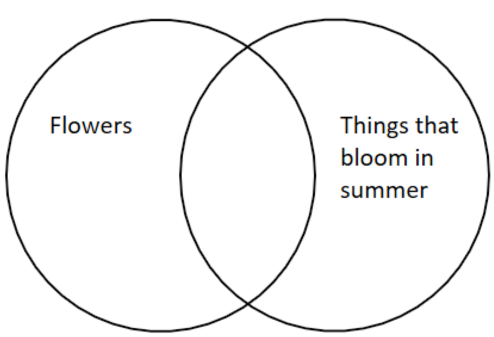
26. The quantifier is *no.* This is a universal quantifier.

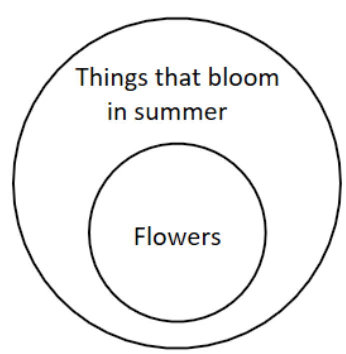


27. The quantifier is *all*. This is a universal quantifier.

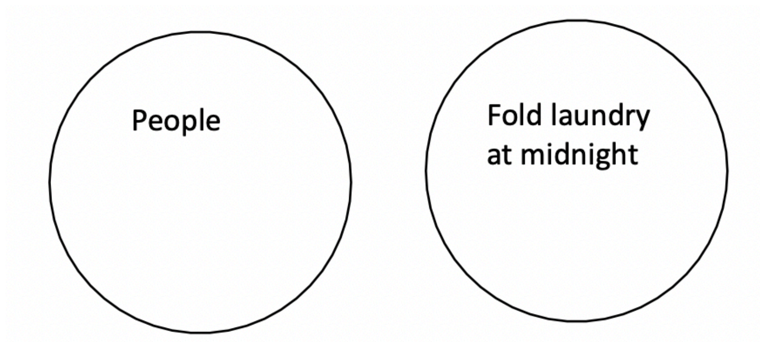


28. The quantifier is *some*. This is an existential quantifier.
Diagrams may vary. Two possibilities are shown.

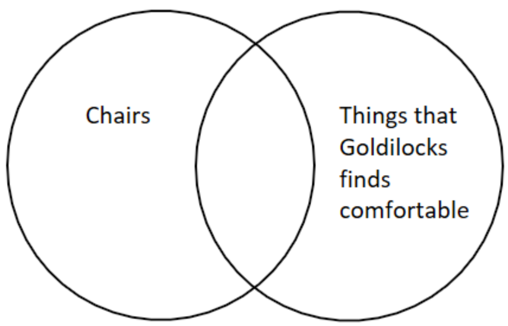


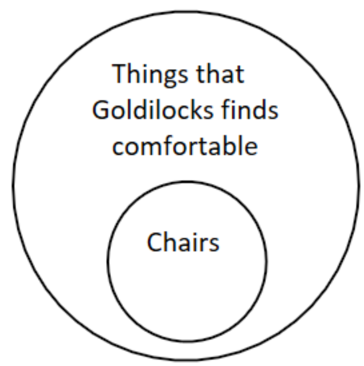


29. The quantifier is *no*. This is a universal quantifier.



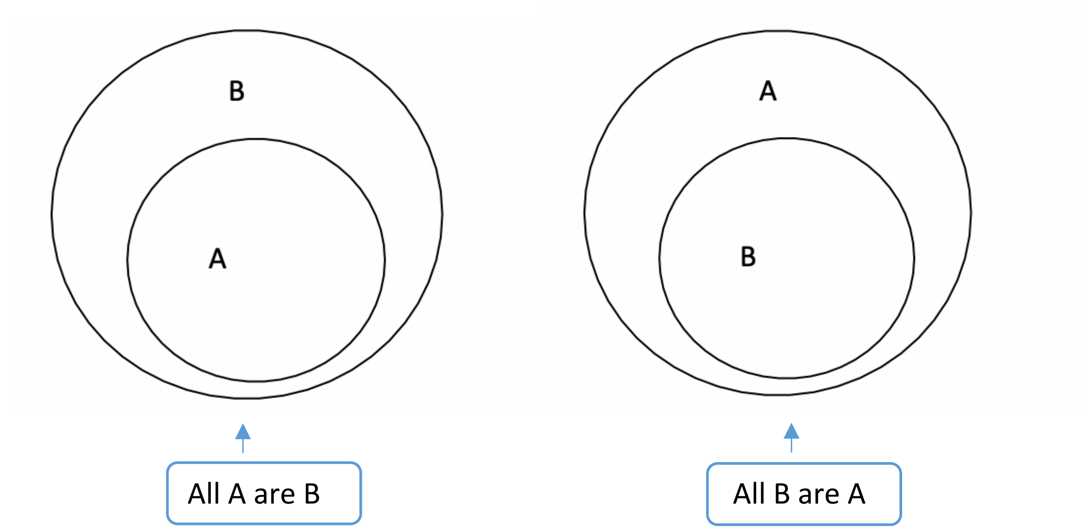
30. The quantifier is *at least one*. This is an existential quantifier.
Diagrams may vary. Two possibilities are shown.





31. On no spring weekends does the zoo do a *Breakfast with Bunny* event. OR
    On all spring weekends the zoo does not do a *Breakfast with Bunny* event.
32. The zoo has no big cats on display.
33. On some spring days, the zoo is not open until 7 pm.
34. Some of the world's music cannot be found on Spotify.
35. Some of the main streets are open during the parade.
36. The store had some rolls of toilet paper.
37. All birds can fly.
38. No dinosaur skeletons have been discovered.
39. No person has swum the English Channel four times without stopping.
40. At least one dinosaur was not larger than the modern elephant.

41. The Euler diagrams for the two are not the same. The first statement means that the A circle will be inside the B circle and the second statement means that the B circle is inside the A circle.



42. These statements do not mean the same thing. The first statement, “Not all animals are dogs,” means that at least one animal is not a dog. The second statement, “All animals are not dogs,” means that no animals are dogs.

43. It is possible for the statements “All A are B” and “No A are B” to have the same truth value, which means that one cannot be the negation of the other. For example, the statements “All apples are red” and “No apples are red” are both false. Since they have the same truth value, one cannot be the negation of the other.

44. It is possible for the statements “Some A are B” and “Some A are not B” to have the same truth value, which means that one cannot be the negation of the other. For example, the statements, “Some cars are blue” and “Some cars are not blue” are both true. Since they have the same truth value, one cannot be the negation of the other.

Section 2.2 | Operations on Simple Statements

Objectives

- Represent simple statements and negations symbolically
- Represent compound statements symbolically using negations, conjunctions, disjunctions, conditionals and biconditionals
- Express symbolic statements in English

Variables and symbols can be used to represent logic statements. The use of variables and symbols can take long, complicated sentences and reduce them to only a few variables and symbols. Symbolic notations can help simplify the process of finding a truth value for a simple statement or compound statement.

Typically, statements are represented as lowercase letters such as p, q or r. For example, let variable p represent the simple statement "Today is Friday" and variable q represent the simple statement "I went bowling." An alternate method used to define statements is shown below:

$\boldsymbol{p}$: Today is Friday.
$\boldsymbol{q}$: I went bowling.

Notice that in each simple statement, only one idea is presented.

Negations

In the previous section, the negation of a statement was defined as having the opposite truth value from the original statement while maintaining much of the focus of the original statement.

DEFINITION: A **negation** is a statement with the opposite truth value from the original statement. The symbol $\sim$ is used to represent the negation of a statement in logic theory. For example, if p represents the statement "Today is Friday" then the negation "Today is not Friday" can be represented using $\sim p$.

➢ **EXAMPLE 2.2.1:** Given the statements q, r, s, t and u below, write the symbolic notation for the negation of the statement. Then write the negation using words.

$\boldsymbol{q}$: I went bowling.
$\boldsymbol{r}$: Batman is a D.C. Comics superhero.
$\boldsymbol{s}$: A leap year does not have 365 days.
$\boldsymbol{t}$**:** Some cars are red.
$\boldsymbol{u}$**:** All libraries contain books.

SOLUTION:

In each example, the symbolic notation is shown first, followed by the English sentence.

$\sim\boldsymbol{q}$: I did not go bowling.
$\sim\boldsymbol{r}$: Batman is not a D.C. Comics superhero.
$\sim\boldsymbol{s}$: A leap year **does** have 365 days.
$\sim\boldsymbol{t}$: All cars are not red. This could also be written as "No cars are red."
$\sim\boldsymbol{u}$**:** At least one library does not contain books.

NOTE: Statement s contained the expression "does not have 365 days," therefore the negation, $\sim s$, utilizes the expression "does have 365 days."

❖ **YOU TRY IT 2.2.A:** Given the statements p, q, and r below, write the symbolic notation for the negation of the statement. Then write the negation using words.

a. $\boldsymbol{p}$: My computer software is not up to date.
b. $\boldsymbol{q}$: All of the cars in the parking lot are red.
c. $\boldsymbol{r}$: Some squirrels are red.

Connectives

More complex sentences such as "Today is Friday and I did not go bowling" can be created using words that connect simple statements, called **connectives**.

DEFINITION: The words used to connect two or more simple statements are called **connectives**. There will be four connectives discussed in this chapter: ***and***, ***or***, ***if ... then***, and ***if and only if***.

For example, given the simple statements p, q, r, and s above, many compound statements can be created. Some examples are shown below:

I went bowling and Superman is a D.C. Comics superhero.
A leap year does not have 365 days or Superman is a D.C. Comics superhero.
If today is Friday, then I went bowling.
I went bowling if and only if today is Friday.

Conjunctions

The discussion will begin with connecting two simple statements using the word ***and***, such as "I went bowling and Superman is a D.C. Comics superhero."

DEFINITION: The logic operation ***and*** is called a **conjunction**. Two simple statements can be connected with the logic operation ***and***, such as "I went bowling and Superman is a D.C. Comics superhero." The symbol ∧ is used to represent the connective ***and*** in logic theory. The compound statement "p and q" can be represented using $p \wedge q$.

- **EXAMPLE 2.2.2:** Given the statements $p, q, r, s,$ and t, write the sentences using words.

$\boldsymbol{p}$: Shaun White competed in five winter Olympic Games.
$\boldsymbol{q}$: The University of Michigan is located in Tampa, Florida.
$\boldsymbol{r}$: Annapolis is the capital of Maryland.
$\boldsymbol{s}$: Kobe Bryant was an ice skater.
$\boldsymbol{t}$: Michael Phelps competed in five summer Olympic Games.

a. $p \wedge t$
b. $p \wedge {\sim}s$
c. ${\sim}q \wedge r$
d. $p \wedge t \wedge {\sim}s$

SOLUTION:

a. Shaun White competed in five winter Olympic Games and Michael Phelps competed in five summer Olympic Games.
b. Shaun White competed in five winter Olympic Games and Kobe Bryan was not an ice skater.
c. The University of Michigan is not located in Tampa, Florida and Annapolis is the capital of Maryland.
d. Shaun White completed in five winter Olympic Games and Michael Phelps competed in five summer Olympic Games and Kobe Bryan was not an ice skater.

➢ **EXAMPLE 2.2.3:** Given the statements p, q, r, s, and t, represent the following using symbolic notation.

$\boldsymbol{p}$: Niagara Falls is located in Canada.
$\boldsymbol{q}$: Niagara Falls is located in the United States of America.
$\boldsymbol{r}$: A van Gogh Museum is located in Amsterdam.
$\boldsymbol{s}$: The Arc de Triumph is located in Mexico.
$\boldsymbol{t}$: The world's tallest building, Burj Khalifa, is located in Dubai.

a. Niagara Falls is located in Canada and the United States.
b. A van Gogh Museum is located in Amsterdam and the Arc de Triumph is not located in Mexico.
c. The Arc de Triumph is not located in Mexico and the world's tallest building, Burj Khalifa, is located in Dubai.

SOLUTION:
a. $p \wedge q$
b. $r \wedge \sim s$
c. $\sim s \wedge t$

Disjunctions

Two simple statements can be connected with the logic operation ***or,*** such as "I went bowling or I went to the movies."

DEFINITION: The logic operation ***or*** is called a **disjunction**. Two simple statements can be connected with the logic operation ***or***, such as "I went bowling or I went to the movies." The symbol ∨ is used to represent the connective ***or*** in logic theory. The compound statement "p or q" can be represented using $p \vee q$.

The word ***or*** can have two meanings, the exclusive meaning of ***or*** compared to the inclusive meaning of ***or***. For example, in the disjunction, "I went bowling or I went to the movies," the exclusive meaning indicates that only one of the statements is true. Either the person went bowling or went to the movies; however, the person did not both go bowling and to the movies. The inclusive meaning of ***or*** indicates that one or both of the statements is true. The person went bowling, or went to the movies, or the person did both. **In this textbook and the study of logic, the inclusive meaning of *or* is used.**

➢ **EXAMPLE 2.2.4:** Given the statements p, q, r and s, write the sentences using words.

$\boldsymbol{p}$: *Mary Poppins* is a Disney movie.
$\boldsymbol{q}$: *Toy Story* is a Pixar movie.
$\boldsymbol{r}$: Steven Smith directed the movie, *Jaws*.
$\boldsymbol{s}$: Steven King wrote the book *The Shining*.

a. $s \vee q$
b. $\sim r \ \vee s$
c. $s \vee \sim p \wedge q$

SOLUTION:

a. Steven King wrote the book *The Shining* or *Toy Story* is a Pixar Movie.
b. Steven Smith did not direct the movie *Jaws* or Steven King wrote the book *The Shining*.
c. Steven King wrote the book *The Shining* or *Mary Poppins* is not a Disney movie and *Toy Story* is a Pixar movie.

➢ **EXAMPLE 2.2.5:** Given the statements p, q, r and s, represent the following using symbolic notation.

$\boldsymbol{p}$: Maya Angelou is a famous ice skater.
$\boldsymbol{q}$: The poem “The Road Not Taken” was written by Robert Frost.
$\boldsymbol{r}$: “The Hill We Climb” is a poem written by Amanda Gorman.
$\boldsymbol{s}$: Shel Silverstein wrote the Johnny Cash track “A Boy Named Sue.”

a. Maya Angelou is not a famous ice skater or “The Hill We Climb” is a poem written by Amanda Gorman.
b. The poem “The Road Not Taken” was written by Robert Frost or “The Hill We Climb” is a poem written by Amanda Gorman.
c. Shel Silverstein wrote the Johnny Cash track “A Boy Named Sue” or the poem “The Road Not Taken” was not written by Robert Frost and “The Hill We Climb” is a poem written by Amanda Gorman.

SOLUTION:

a. $\sim p \vee r$
b. $q \vee r$
c. $s \vee \sim q \wedge r$

❖ **YOU TRY IT 2.2.B:** Given the statements p, q, r and s, write the sentences using words.

$\boldsymbol{p}$: Charles Dickens wrote *Oliver Twist.*
$\boldsymbol{q}$: Charles Dickens did not write *The Odyssey.*
$\boldsymbol{r}$: J.K. Rowling wrote the Harry Potter books.
$\boldsymbol{s}$: Agatha Christie is famous for writing history novels.

a. $p \wedge q$
b. $\sim s \vee q$
c. $(p \wedge q) \vee \sim s$

❖ **YOU TRY IT 2.2.C:** Given the statements p, q, r and s, represent the following using symbolic notation.

$\boldsymbol{p}$: Albert Einstein was a physicist.
$\boldsymbol{q}$: Marie Curie is known for discovering radium.
$\boldsymbol{r}$: Louis Pasteur is famous for building rockets.
$\boldsymbol{s}$: Isaac Newton discovered the laws of gravity.
$\boldsymbol{t}$: Isaac Newton did not invent calculus.
$\boldsymbol{u}$: Marie Curie is known for finding treatments for cancer.

a. Albert Einstein was a physicist or Louis Pasteur is famous for building rockets.
b. Isaac Newton discovered the laws of gravity and invented calculus.
c. Louis Pasteur is not famous for building rockets and Marie Curie is known for finding treatments for cancer.
d. Marie Curie is known for discovering radium and finding treatments for cancer and Albert Einstein was a physicist.

Conditionals

Two simple statements can be connected with the logic operation ***if ... then,*** such as "If it is Friday, then the cafeteria serves pizza." Sometimes in the English language the word *then* is omitted, such as "If it is Friday, the cafeteria serves pizza."

DEFINITION: The logic operation ***if ... then*** is called a **conditional.** Two simple statements can be connected with the logic operation ***if ... then***, such as "If it is Friday, then the cafeteria serves pizza." The symbol → is used to represent the connective ***if … then*** in logic theory. The compound statement "If p, then q" can be represented using $p \rightarrow q$.

NOTE: The arrow used to represent a conditional statement points in only one direction. The direction of the arrow is important. For example, in the compound statement $p \rightarrow q$, statement p is pointing towards statement q and indicates that statement p implies statement q. The p statement (before the arrow) is called the **antecedent** while the q statement (after the arrow) is called the **consequent** or **consequence**.

➢ **EXAMPLE 2.2.6:** Given the statements p, q, r and s, write the sentences using words.

$\boldsymbol{p}$**:** It is Tuesday.
$\boldsymbol{q}$**:** It is 9 pm.
$\boldsymbol{r}$**:** Bill watches the news.
$\boldsymbol{s}$**:** Bill eats tacos.

a. $p \rightarrow s$
b. $q \rightarrow r$
c. $r \rightarrow {\sim}s$

SOLUTION:

a. If it is Tuesday, then Bill eats tacos.
b. If it is 9 pm, then Bill watches the news.
c. If Bill watches the news, then he doesn't eat tacos.

There are many ways to write a conditional so that it has the same meaning as "If p, then q." For each example below, use the given simple statements:

$\boldsymbol{p}$**:** It is raining.
$\boldsymbol{q}$**:** There are clouds.

Consider the conditional: "If it is raining, then there are clouds."

Note: You might have learned that clouds are made of water droplets. As the droplets combine together and grow, they become too heavy to remain in the cloud and fall to Earth[2].

---

[2]Source: https://scijinks.gov/rain/

Several ways to represent the conditional "If p, then q" are shown in Figure 2.2.1. They are all represented symbolically as $p \rightarrow q$.

FIGURE 2.2.1

Alternate Ways to Represent the Conditional Statement $p \rightarrow q$	
If p, then q	Here the "if" is associated with p. The antecedent is p and the consequent is q. Ex: **If it is raining, then there are clouds.**
p implies q	The antecedent is p and the consequent is q. Ex: **Seeing rain implies there are clouds.** Remember: Clouds are needed to make the raindrops.
p only if q	The antecedent is p and the consequent is q. Ex: **It is raining only if there are clouds.** Note: Here the meaning remains the same as "if it is raining, then there are clouds." There must be clouds to create the raindrops.
p is sufficient for q	The sufficient condition is the antecedent. The antecedent is p and the consequent is q. Ex: **It is raining is sufficient information to know there are clouds.** Remember: Clouds are needed to make the raindrops.
q if p	Here the "if" is associated with p. The antecedent is p and the consequent is q. Ex: **There are clouds if it is raining.**
q is necessary for p	The necessary condition is the consequent. The antecedent is p and the consequent is q. Ex: **Seeing clouds is necessary for rain.** Remember: Clouds are needed to make the raindrops.
q is a consequence of p	Here the sentence emphasizes q is the consequence. The antecedent is p and the consequent is q. Ex: **Seeing clouds is a consequence of it raining.**
q follows p	The antecedent is p and the consequent is q. Ex: **Seeing clouds follows having rain.** Note: Here the meaning remains the same as "if it is raining, then there are clouds."
q whenever p	The antecedent is p and the consequent is q. Ex: **There are clouds whenever it is raining.**

➢ **EXAMPLE 2.2.7:** Given the statements p, q, r and s, represent the following using symbolic notation.

$\boldsymbol{p}$**:** It is cold.
$\boldsymbol{q}$**:** It is not snowing.
$\boldsymbol{r}$**:** It is summer.
$\boldsymbol{s}$**:** Miles went swimming.

a. If it is summer, then Miles went swimming.
b. It is not summer, if it is snowing.
c. Being cold is necessary for snow.
d. It is summer is sufficient for it is not snowing.

SOLUTION:

a. $r \rightarrow s$
b. Rewrite the sentence in ***if … then*** format.
If it is snowing, then it is not summer. This would have the symbolic notation: $\sim q \rightarrow \sim r$
c. Rewrite the sentence in ***if … then*** format.
If it is snowing, then it is cold. This would have the symbolic notation: $\sim q \rightarrow p$
d. Rewrite the sentence in ***if … then*** format.
If it is summer, then it is not snowing. This would have the symbolic notation: $r \rightarrow q$

Biconditionals

Connecting two simple statements using the connection ***if and only if***, such as "It measures a yard if and only if it measures 36 inches" is called a biconditional.

DEFINITION: The logic operation ***if and only if*** is called a **biconditional.** Two simple statements can be connected with the logic operation ***if and only if***, such as "It measures a yard if and only if it measures 36 inches." The symbol $\leftrightarrow$ is used to represent the connective ***if and only if*** in logic theory. The compound statement "p if and only if q" can be represented using $p \leftrightarrow q$.

NOTE: The arrow used to represent a biconditional points in both directions, indicating that p implies q, **and** also that q implies p. Symbolically, this means that $p \rightarrow q$ ***and*** $q \rightarrow p$. This could also be written as $(p \rightarrow q) \land (q \rightarrow p)$.

- **EXAMPLE 2.2.8:** Given the statements $p, q, r,$ and $s,$ write the sentences using words.

  $\boldsymbol{p}$: Pigs can fly.
  $\boldsymbol{q}$: I go to the amusement park.
  $\boldsymbol{r}$: I clean my room.
  $\boldsymbol{s}$: I am not dreaming.

  a. $q \leftrightarrow r$
  b. $p \leftrightarrow \sim s$
  c. $r \leftrightarrow p$

  SOLUTION:

  a. I go to the amusement park if and only if I clean my room.
  b. Pigs can fly if and only if I am dreaming.
  c. I clean my room if and only if pigs can fly.

- **EXAMPLE 2.2.9:** Given the statements $p, q, r, s,$ and $t,$ represent the following using symbolic notation.

  $\boldsymbol{p}$: I did not study.
  $\boldsymbol{q}$: I passed the test.
  $\boldsymbol{r}$: It is raining.
  $\boldsymbol{s}$: Cam walks to school.
  $\boldsymbol{t}$: I do not carry an umbrella.

  a. I did not pass the test if and only if I did not study.
  b. Cam walks to school if and only if it is not raining.
  c. I carry an umbrella if and only if it is raining.

  SOLUTION:

  a. $\sim q \leftrightarrow p$. Recall, the arrow is pointing in both directions, so another answer could be $p \leftrightarrow \sim q$
  b. $s \leftrightarrow \sim r$
  c. $\sim t \leftrightarrow r$

❖ **YOU TRY IT 2.2.D:** Given the statements p, q, r and s, write the sentences using words.

$\boldsymbol{p}$: It is Monday.
$\boldsymbol{q}$: Sharon gets 10% off her bill.
$\boldsymbol{r}$: Sharon goes out to eat.
$\boldsymbol{s}$: The restaurant offers free dessert.

a. $p \rightarrow r$
b. $q \leftrightarrow \sim s$
c. $p \rightarrow (q \vee s)$

❖ **YOU TRY IT 2.2.E:** Given the statements p, q, r and s, represent the following using symbolic notation.

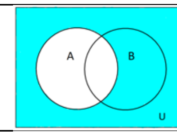
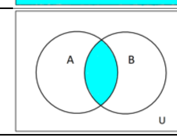
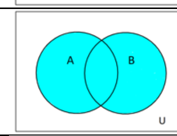
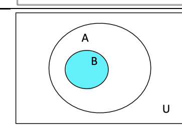
$\boldsymbol{p}$: Carl makes pancakes.
$\boldsymbol{q}$: Carl washes the dishes.
$\boldsymbol{r}$: Carl drinks coffee.
$\boldsymbol{s}$: Carl drinks orange juice.

a. If Carl makes pancakes, then he does not wash the dishes.
b. Carl drinks orange juice if and only if he doesn't drink coffee.
c. If Carl makes pancakes, then he drinks coffee and orange juice.
d. Carl drinks orange juice and washes the dishes if and only if he doesn't make pancakes.

Connecting Set Theory and Logic Concepts

There are many similarities between concepts discussed in Chapter 1 on Set Theory and concepts discussed in Chapter 2 on Logic Theory as shown in Figure 2.2.2.

FIGURE 2.2.2

Similarities in Set Theory and Logic Theory					
Logic Theory			**Set Theory**		
Negation	not	$\sim p$	Complement	A'	
Conjunction	and	$p \wedge q$	Intersection	$A \cap B$	
Disjunction	or	$p \vee q$	Union	$A \cup B$	
Conditional	If … then	$p \rightarrow q$	Subset	$B \subseteq A$	

Compound Statements with Grouping Symbols

Grouping symbols are used to signify which portion of the problem should be completed first or grouped together. For example, in the symbolic expression $\sim(p \wedge q)$, it would indicate the negation of the conjunction $(p \wedge q)$. As a specific example, let

$\boldsymbol{p}$: It is snowing.
$\boldsymbol{q}$: I go sledding.

Then $\sim(p \wedge q)$ would be read as the negation of $(p \wedge q)$. The parentheses show that $p \wedge q$ need to be grouped together with the connector ***and*** first. Next, the negation of the conjunction within the parentheses should be found. In words, $\sim(p \wedge q)$ would be written as "It is not true that it is snowing and I go sledding." Notice that the phrase "it is not true" would indicate the negation for the conjunction "It is snowing and I go sledding."

➢ **EXAMPLE 2.2.10:** Given the statements p, q, r, and s, write the sentences using words.

$\boldsymbol{p}$: It is Saturday.
$\boldsymbol{q}$: I go to work.
$\boldsymbol{r}$: It is raining.
$\boldsymbol{s}$: The sun is shining.

a. $\sim p \wedge r$
b. $\sim(p \wedge r)$
c. $\sim(s \vee \sim r)$
d. $\sim(r \rightarrow \sim s)$

SOLUTION:

a. It is not Saturday and it is raining.
b. It is not true that it is Saturday and it is raining.
c. It is not true that the sun is shining or it is not raining.
d. It is not true that if it is raining then the sun is not shining.

➢ **EXAMPLE 2.2.11:** Given the statements $p, q, r, s\ t$, and u, represent the following using symbolic notation.

$\boldsymbol{p}$: I took a math class.
$\boldsymbol{q}$: I passed the final exam.
$\boldsymbol{r}$: It is March.
$\boldsymbol{s}$: It is not snowing.
$\boldsymbol{t}$: The trees are green.
$\boldsymbol{u}$: Pigs fly.

a. It is not true that it is March and it is snowing.
b. It is March, and I took a math class and passed the final exam.
c. If the trees are green, then it is not March and it is not snowing.
d. I took a math class and failed the final exam, if and only if pigs fly.

SOLUTION:

a. $\sim(r \wedge \sim s)$. Because of the phrase "it is not true" the information after that phrase should be in parentheses.

b. $r \wedge (p \wedge q)$. A comma separates the sentence into two parts "It is March" and "I took a math class and passed the final exam." Therefore, parentheses are used to group together the conjunction $(p \wedge q)$ after the comma.

c. $t \rightarrow (\sim r \wedge s)$. The conditional statement is separated into two parts. The hypothesis after the word *if* would be "The trees are green" and the consequent after the word *then* would be "It is not March and it is not snowing." Therefore, parentheses are used to group together the entire consequent $(\sim r \wedge s)$.

d. $(p \wedge \sim q) \leftrightarrow u$. The biconditional is separated into two parts "I took a math class and failed the final exam" before the *if and only if* connective and "pigs fly" after the *if and only if* connective. Therefore, parentheses are used to group together the conjunction before the *if and only if* phrase.

Dominance of Connectives

Consider the compound statement $p \rightarrow \sim q \wedge r$. Without parentheses or the dominance of connectives, this statement would be ambiguous. It could indicate the main idea is a conditional $p \rightarrow (\sim q \wedge r)$ such as "If I study, then I do not make dinner and order a pizza." Or the conditional could have a negation of the compound conjunction $p \rightarrow \sim(q \wedge r)$ which would be written as "If I study, then it is not true that I make dinner and order a pizza. As a third option, the statement could have the main idea as a conjunction $(p \rightarrow \sim q) \wedge r$ meaning "If I study then I do not make dinner, and I order a pizza."

To avoid ambiguous statements the dominance of logic connectives should be used to highlight the main connective of the statement. The logic connectives are listed below from the most dominant to the least dominant. Parentheses are used before and after the dominant connective to highlight the main connective of the statement.

(1) biconditional
(2) conditional
(3) conjunction and disjunction (same dominance level)
(4) negation

➢ **EXAMPLE 2.2.12:** Given the compound statements, rewrite the statement using parentheses before and after the dominant connective to emphasize the logic operation that dominates the statement.

a. $\sim p \wedge r \rightarrow q$
b. $q \leftrightarrow p \rightarrow \sim r$
c. $t \vee p \rightarrow \sim q \wedge s$
d. $p \rightarrow r \wedge \sim t \leftrightarrow q$

SOLUTION:

a. The dominant connective is the conditional ***if … then***. The statement could be rewritten with parentheses before the arrow as $(\sim p \wedge r) \rightarrow q$ to highlight that the main idea of the statement is the conditional ***if … then***.

b. The dominant connective is the biconditional ***if and only if***. The statement could be rewritten with parentheses after the double-sided arrow as $q \leftrightarrow (p \rightarrow \sim r)$ to highlight that the main idea of the statement is the biconditional ***if and only if***.

c. The dominant connective is the conditional ***if … then***. The statement could be rewritten with parentheses before and after the arrow as $(t \vee p) \rightarrow (\sim q \wedge s)$ to highlight that the main idea of the statement is the conditional ***if … then***.

d. The dominant connective is the conditional ***if … then***. The statement could be rewritten with parentheses before the double-sided arrow as $(p \rightarrow r \wedge \sim t) \leftrightarrow q$ to highlight that the main idea of the statement is the biconditional ***if and only if***. Further, the next logic operation of dominance would be the conditional ***if … then***. To highlight this conditional, the expression could be written with parentheses after the arrow as $[p \rightarrow (r \wedge \sim t)] \leftrightarrow q$.

Quick Review

- A **connective** is a word used to connect two or more simple statements.
- A **conjunction** is the logic operation ***and*** that can be used to connect two simple statements. It is represented symbolically as $p \wedge q$.
- A **disjunction** is the logic operation ***or*** that can be used to connect two simple statements. It is represented symbolically as $p \vee q$.
- A **conditional** is the logic operation ***if…then*** that can be used to connect two simple statements. It is represented symbolically as $p \rightarrow q$.
- A **biconditional** is the logic operation ***if and only if*** that can be used to connect two simple statements. It is represented symbolically as $p \longleftrightarrow q$.
- The dominance of logic operations is (1) biconditional, (2) conditional, (3) conjunction and disjunction moving left to right, and finally (4) negation.

YOU TRY IT 2.2.A SOLUTION:

- **a.** $\sim\boldsymbol{p}$**:** My computer software is up to date.
- **b.** $\sim\boldsymbol{q}$**:** At least one care in the parking lot is not red.
- **c.** $\sim\boldsymbol{r}$**:** No squirrels are red.

YOU TRY IT 2.2.B SOLUTION:

- **a.** Charles Dickens wrote *Oliver Twist* and he did not write *The Odyssey.*
- **b.** Agatha Christie is not famous for writing history novels or Charles Dickens did not write *They Odyssey.*
- **c.** Charles Dickens wrote *Oliver Twist* and he did not write The Odyssey or Agatha Christie is not famous for writing history novels.

YOU TRY IT 2.2.C SOLUTION:

- **a.** $p \vee r$
- **b.** $s \wedge t$
- **c.** $\sim r \wedge u$
- **d.** $q \wedge u \wedge p$

YOU TRY IT 2.2.D SOLUTION:

- **a.** If it is Monday, then Sharon goes out to eat.
- **b.** Sharon gets 10% off her bill if and only if the restaurant does not offer free dessert.
- **c.** If it is Monday, then Sharon gets 10% off her bill or the restaurant offers free dessert.

YOU TRY IT 2.2.E SOLUTION:

- **a.** $p \rightarrow \sim q$
- **b.** $s \leftrightarrow \sim r$
- **c.** $p \rightarrow (r \wedge s)$
- **d.** $(s \wedge q) \leftrightarrow \sim p$

Section 2.2 | Exercises

In Exercises 1 – 12, write the symbolic notation for the negation of the statement. Then write the negation using words.

1. p: Tom Holland plays Spiderman in the 2021 movie *Spiderman: No Way Home*.
2. q: *Encanto* won the best animated feature film at the 2022 Oscars.
3. r: New York City has over 10 million visitors each year.
4. s: Born in 1918, Robert Wadlow is considered the tallest man at 8 feet and 11.1 inches.
5. t: All flowers have petals.
6. p: Some soccer fans like Lionel Messi.
7. q: Laura likes hot air balloon rides.
8. r: Arizona does not have daylight savings time.
9. s: Mandarin is not the most spoken language in the world.
10. t: Some cats don't sleep 15 hours or more each day.
11. u: No person has climbed Gangkhar Puensum.
12. v: Apple pie did not originate in the United States.

In Exercises 13 – 26, write the sentences using words.

$\boldsymbol{p}$**:** France is the top tourist destination in the world.
$\boldsymbol{q}$**:** The Eifel Tower is a symbol of Paris.
$\boldsymbol{r}$**:** The Louvre is found in Barcelona.
$\boldsymbol{s}$: In a 2019 fire the Notre Dame Cathedral spire collapsed.
$\boldsymbol{t}$**:** The abbey Mont Saint-Michel is not on a 17-acre island in Normandy.

13. $p \wedge q$
14. $q \wedge \sim t$
15. $p \vee r$
16. $s \vee \sim r$
17. $p \wedge q \wedge \sim r$
18. $p \wedge (r \vee s)$
19. $\sim(r \wedge t)$

$\boldsymbol{p}$: Serena Williams is a world champion tennis player.
$\boldsymbol{q}$: Tom Brady plays football for the Chicago Bears.
$\boldsymbol{r}$: Megan Rapinoe is a professional soccer player who is not a goalie.
$\boldsymbol{s}$: Michael Jordan was not a professional basketball player.
$\boldsymbol{t}$: Michael Jordan was a professional baseball player.

20. $\sim s \wedge p$
21. $r \vee \sim q$
22. $t \wedge \sim(q \vee s)$
23. $p \wedge \sim q \wedge \sim t$
24. $p \vee (s \wedge \sim q)$
25. $\sim(q \wedge p) \vee r$
26. $(p \vee s) \wedge (\sim q \vee t)$

In Exercises 27 – 36, represent the sentences using symbolic notation.

$\boldsymbol{p}$: Over 8 million people live in New York City.
$\boldsymbol{q}$: The Statue of Liberty was a gift from Mexico.
$\boldsymbol{r}$: Central Park covers 200 acres.
$\boldsymbol{s}$: More than 12 million immigrants came to the United States through Ellis Island.
$\boldsymbol{t}$: The Empire State Building is 102 stories tall.

27. Over 8 million people live in New York City and more than 12 million immigrants came to the Unites States through Ellis Island.
28. The Empire State Building is 102 stories tall and the Statue of Liberty was not a gift from Mexico.
29. Central Park does not cover 200 acres or the Statue of Liberty was a gift from Mexico.
30. Over 8 million people live in New York City, and the Empire State Building is 102 stories tall or Central Park covers 200 acres.
31. It is not true that Central Park covers 200 acres or the Statue of Liberty was a gift from Mexico.

$\boldsymbol{p}$: Drake wrote the song “Hotline Bling.”
$\boldsymbol{q}$: The Jonas Brothers wrote the “Star – Spangled Banner.”
$\boldsymbol{r}$: Dua Lipa sang the song “Levitating.”
$\boldsymbol{s}$: Childish Gambino wrote the song “This is America.”
$\boldsymbol{t}$: Post Malone sang the song “Diamonds.”

32. Drake wrote the song “Hotline Bling” and Childish Gambino wrote the song “This is America.”

33. It is not true that Post Malone sang the song “Diamonds” or the Jonas Brothers wrote the “Star – Spangled Banner.”
34. Dua Lipa sang the song “Levitating” or the Jonas Brothers did not write the “Star – Spangled Banner.”
35. Post Malone did not sing the song “Diamonds,” or Dua Lipa sang the song “Levitating” and Childish Gambino wrote the song “This is America.”
36. The Jonas Brothers didn’t write the “Star – Spangled Banner,” and it is not the case that Post Malone sang the song “Diamonds” and Dua Lipa sang the song “Levitating”.

In Exercises 37 – 46, write the conditionals in ***if … then*** form.

37. Paying tuition is necessary to attend college.
38. You must pass through the security check whenever you travel in an airplane.
39. Being awarded a gold medal follows winning an Olympic event.
40. Good coffee is made only if hot water is used.
41. Carry an umbrella if it is raining.
42. Drinking water is necessary to live.
43. I will drink the coffee if it has no sugar.
44. The candidate becoming president is a consequence of having the most electoral votes.
45. Running a mile is sufficient for raising your heart rate.
46. A shootout in a regular season NHL game occurs whenever the game is still tied after overtime.

In Exercises 47 – 55, write the sentences using words.

$\boldsymbol{p}$: It is March.
$\boldsymbol{q}$: Monday was 70 degrees.
$\boldsymbol{r}$: It snowed on Wednesday.
$\boldsymbol{s}$: Sharon travels to Chicago.
$\boldsymbol{t}$: Sharon visits Navy Pier.

47. $s \rightarrow t$
48. $p \rightarrow t$
49. $p \rightarrow \sim q$
50. $\sim s \rightarrow \sim t$
51. $s \rightarrow (q \wedge r)$
52. $(p \wedge s) \rightarrow (q \vee r)$
53. $t \leftrightarrow \sim p$
54. $\sim s \leftrightarrow r$
55. $r \leftrightarrow (p \wedge q)$

In Exercises 56 – 62, write the sentences using words.

$\boldsymbol{p}$: The ocean was pink.
$\boldsymbol{q}$: The sky was green.
$\boldsymbol{r}$: The rainbow was colorful.
$\boldsymbol{s}$: The tiger was friendly.
$\boldsymbol{t}$: My dream was vivid.

56. $(p \vee q) \rightarrow t$
57. $\sim p \wedge \sim q \wedge \sim s$
58. $r \leftrightarrow t$
59. $\sim(s \wedge q) \leftrightarrow t$
60. $(\sim t \rightarrow \sim r) \vee q$
61. $(s \wedge \sim p) \rightarrow (\sim t \vee r)$
62. $(t \wedge \sim p) \leftrightarrow (r \vee \sim q)$

In Exercises 63 – 72, represent the sentences using symbolic notation.

$\boldsymbol{p}$: You skip class.
$\boldsymbol{q}$: You watch all of the Indiana Jones movies.
$\boldsymbol{r}$: You study.
$\boldsymbol{s}$: You practice homework questions.
$\boldsymbol{t}$: You pass the exam.

63. If you study, then you will practice homework questions.
64. If you skip class, then you will not pass the exam.
65. You pass the exam if and only if you practice homework questions.
66. You watch all of the Indiana Jones movies if and only if you skip class.
67. If you skip class and watch all of the Indiana Jones movies, then you will not practice homework questions.
68. Studying is necessary to pass the exam.
69. You will pass the exam if and only if you do not watch all of the Indiana Jones movies.
70. You watch all of the Indiana Jones movies if you skip class.
71. You study and you practice homework questions and you do not skip class, or you do not pass the exam.
72. It is not the case that you study or pass the exam, if and only if you skip class and don't practice homework questions.

In Exercises 73 – 82, given statements $p, q, r, s\ t$, and u, represent the following using symbolic notation.

$\boldsymbol{p}$**:** The horse is blue.
$\boldsymbol{q}$**:** A pink cow makes strawberry milk.
$\boldsymbol{r}$**:** The mouse scared the lion.
$\boldsymbol{s}$**:** The horse eats green eggs and ham.
$\boldsymbol{t}$**:** The sheep watch television.
$\boldsymbol{u}$**:** Pigs fly.

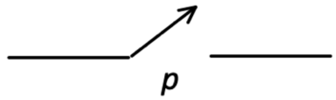
73. Pigs fly and the horse does not eat green eggs and ham.
74. It is not true that the horse is blue and sheep watch television.
75. It is not true that the horse eats green eggs and ham or pigs cannot fly.
76. A pink cow makes strawberry milk, and it is not true that pigs fly or the mouse scared the lion.
77. If the mouse scared the lion, then it is not true that sheep do not watch television or the horse is blue.
78. The mouse scared the lion if and only if pigs fly.
79. The sheep watch television and a pink cow makes strawberry milk, if the horse is blue.
80. The horse is blue whenever a pink cow makes strawberry milk and the sheep watch television.
81. The sheep watch television and the mouse scared the lion, if and only if pigs don't fly or the horse is not blue.
82. Pigs being able to fly is sufficient for the horse eating green eggs and ham or the mouse scaring the lion.

In Exercises 83 – 92, given the compound statements, determine the dominant connective and rewrite the statement using parentheses to emphasize the logic operation that dominates the statement. The parentheses are used before and/or after the dominant connective to highlight the main connective of the statement.

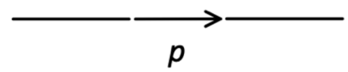
83. $r \leftrightarrow p \vee s$
84. $p \wedge t \rightarrow \sim q$
85. $\sim r \rightarrow q \wedge \sim s$
86. $q \wedge \sim p \leftrightarrow t \vee \sim s$
87. $\sim s \rightarrow q \rightarrow \sim r \vee p$
88. $r \rightarrow s \wedge \sim p \leftrightarrow \sim r$
89. $s \vee \sim r \rightarrow p \leftrightarrow \sim t$
90. $t \wedge \sim u \rightarrow \sim t \wedge u$
91. $\sim p \leftrightarrow q \wedge \sim r \wedge q$
92. $p \rightarrow \sim q \vee r \leftrightarrow \sim p$

Applications I

Electric circuits are one application of logic theory. Electricity can flow through a closed switch; however, electricity is stopped and cannot flow through an open switch. An electric switch represents a simple statement that can either be open (turned off) or closed (turned on).

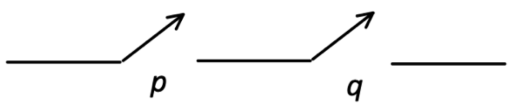


open switch
electricity cannot flow through

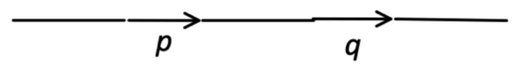


closed switch
electricity can flow through

Connecting two or more switches in a line represents a series circuit. In logic, this would represent a conjunction $p \wedge q$ represented by the series circuits shown below. For electricity to flow through a series circuit, all switches must be closed.

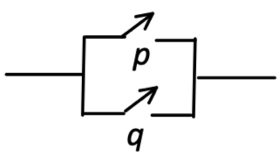


electricity cannot flow through

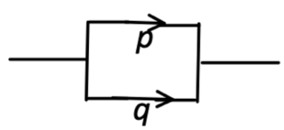


electricity can flow through

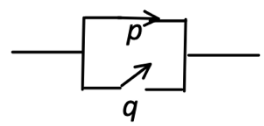
Connecting two or more switches in parallel lines represents a parallel circuit. In logic, this would represent a disjunction $p \vee q$ represented by the parallel circuits shown below. For electricity to flow through a parallel circuit, at least one switch must be closed.



electricity cannot flow through



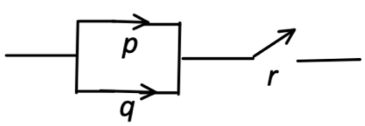
electricity can flow through



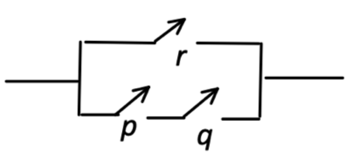
electricity can flow through

In Exercises 93 – 96, write the symbolic form the corresponds to the circuit.

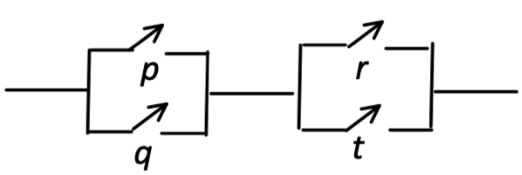
93.

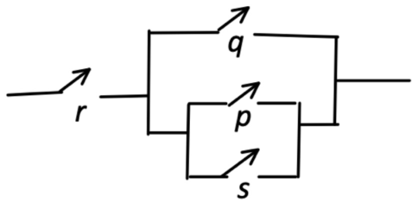


94.



95.



96. 

In Exercises 97 – 100, draw the circuit that corresponds to the symbolic form.

97. $p \wedge (r \vee q)$
98. $(p \wedge r) \vee q$
99. $(p \wedge r) \vee (r \wedge q)$
100. $q \vee (p \vee r)$

Computer Programming Application

Computer programming is another application of logic theory. In algebra, order of operations is followed: (1) parentheses, (2) exponents, (3) multiplication and division moving left to right, and finally (4) addition and subtraction moving left to right. When parentheses are not provided, order of operations is still used to ensure everyone reads and solves the problem in the same way, finishing with the same answer. This is a referred to as a **precedence** of arithmetic operations or order of operations.

In the same manner, there is a **precedence** to evaluate logic operations. The order of logic operations follows: (1) negation, (2) conjunction, (3) disjunction, (4) conditional, and finally (4) biconditional. Computer programming is one application that uses the precedence of evaluating logic operations.

NOTE: Precedence of logic operations is different than dominance of logic connectives. Dominance of logic connectives uses parentheses to highlight the dominant connective (**and, or, if … then,** as well as **if and only if**) or main connective of the sentence. For example, is the main connective a conjunction making the main idea an OR statement? On the other hand, precedence of logic operations is used when evaluating the truth value for a compound statement that doesn't have parentheses. The precedence of logic operations determines the order the logic operations should be completed.

For example, evaluate $p \wedge {\sim}q \rightarrow p \vee r$ when p is true, q is true, and r is false.

$p \wedge {\sim}q \rightarrow p \vee r$	
$T \wedge {\sim}T \rightarrow T \vee F$	First evaluate negations.
$T \wedge F \rightarrow T \vee F$	Second evaluate conjunctions.
$F \rightarrow T \vee F$	Then evaluate disjunctions.
$F \rightarrow T$	Finally evaluate the conditional.
T	The final truth value is true.

In Exercises 1 – 5, evaluate the logic statement using the **precedence** of logic operations when p is false, q is true, and r is false.

1. $p \wedge {\sim}r \vee q$
2. ${\sim}q \rightarrow r \vee p$
3. $p \vee {\sim}r \leftrightarrow q \wedge {\sim}p$
4. $r \rightarrow p \leftrightarrow {\sim}q \wedge {\sim}r$
5. ${\sim}p \leftrightarrow q \wedge {\sim}r \rightarrow {\sim}q \vee p$

Section 2.2 | Exercise Solutions

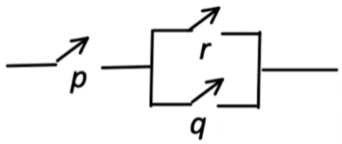
1. $\sim p$: Tom Holland does not play Spider-Man in the 2021 movie *Spider-Man: No Way Home*.
2. $\sim q$: *Encanto* did not win the best animated feature film at the 2022 Oscars.
3. $\sim r$: New York City does not have over 10 million visitors each year.
4. $\sim s$: Born in 1918, Robert Wadlow is not considered the tallest man at 8 feet and 11.1 inches.
5. $\sim t$: At least one flower does not have petals.
6. $\sim p$: No soccer fans like Lionel Messi.
7. $\sim q$: Laura does not like hot air balloon rides.
8. $\sim r$: Arizona has daylight savings time.
9. $\sim s$: Mandarin is the most spoken language in the world.
10. $\sim t$: All casts sleep 15 hours or more each day.
11. $\sim u$: Some people have climbed Gangkhar Puensum.
12. $\sim v$: Apple pie did originate in the United States.
13. France is the top tourist destination in the world and the Eifel Tower is a symbol of Paris.
14. The Eifel Tower is a symbol of Paris and the abbey Mont Saint-Michel is on a 17-acre island in Normandy.
15. France is the top tourist destination in the world or the Louvre is found in Barcelona.
16. In a 2019 fire the Notre Dame Cathedral spire collapsed or the Louvre is not found in Barcelona.
17. France is the top tourist destination in the world and the Eifel Tower is a symbol of Paris or the Louvre is not found in Barcelona.
18. France is the top tourist destination in the world, and the Louvre is found in Barcelona or in a 2019 fire the Notre Dame Cathedral spire collapsed.
19. It is not true that the Louvre is found in Barcelona and the abbey Mont Saint-Michel is not on a 17-acre island in Normandy.
20. Michael Jordan was a professional basketball player and Serena Williams is a world – champion tennis player.
21. Megan Rapinoe is a professional soccer player who is not a goalie or Tom Brady does not play football for the Chicago Bears.
22. Michael Jordan was a professional baseball player, and it is not true that Tom Brady plays football for the Chicago Bears or Michael Jordan was not a professional basketball player.

23. Serena Williams is a world champion tennis player and Tom Brady does not play football for the Chicago Bears and Michael Jordan was not a professional baseball player.
24. Serena Williams is a world-champion tennis player, or Michael Jordan was not a professional basketball player and Tom Brady does not play football for the Chicago Bears.
25. It is not true that Tom Brady plays football for the Chicago Bears and Serena Williams is a world-champion tennis player, or Megan Rapinoe is a professional soccer player who is not a goalie.
26. Serena Williams is a world champion tennis player or Michael Jordan was not a professional basketball player, and Tom Brady does not play football for the Chicago Bears or Michael Jordan was a professional baseball player.
27. $p \wedge s$
28. $t \wedge \sim q$
29. $\sim r \vee q$
30. $p \wedge (t \vee r)$
31. $\sim(r \vee q)$
32. $p \wedge s$
33. $\sim(t \vee q)$
34. $r \vee \sim q$
35. $\sim t \vee (r \wedge s)$
36. $\sim q \wedge \sim(t \wedge r)$
37. If you attend college, then you pay tuition.
38. If a person travels in an airplane, then she must pass through the security check.
39. If a person wins an Olympic event, then she is awarded a gold medal.
40. If good coffee is made, then hot water is used.
41. If it is raining, then you should carry an umbrella.
42. If a person is alive, then she drinks water.
43. If it has no sugar, then I will drink the coffee.
44. If the candidate has the most electoral votes, then they become president.
45. If you run a mile, then your heart rate will raise.
46. If a regular season NHL game is still tied after overtime, then a shootout occurs.
47. If Sharon travels to Chicago, then she visits Navy Pier.
48. If it is March, then Sharon visits Navy Pier.
49. If it is March, then Monday was not 70 degrees.
50. If Sharon does not travel to Chicago, then she does not visit Navy Pier.
51. If Sharon travels to Chicago, then Monday was 70 degrees and it snowed on Wednesday.
52. If it is March and Sharon travels to Chicago, then Monday was 70 degrees or it snowed on Wednesday.
53. Sharon visits Navy Pier if and only if it is not March.

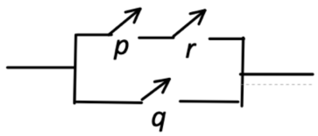
54. Sharon did not travel to Chicago if and only if it snowed on Wednesday.
55. It snowed on Wednesday if and only if it is March and Monday was 70 degrees.
56. If the ocean was pink or the sky was green, then my dream was vivid.
57. The ocean was not pink and the sky was not green and the tiger was not friendly.
58. The rainbow was colorful if and only if my dream was vivid.
59. It is not the case that the tiger was friendly and the sky was green, if and only if my dream was vivid.
60. If my dream was not vivid then the rainbow was not colorful, or the sky was green.
61. If the tiger was friendly and the ocean was not pink, then my dream was not vivid or the rainbow was colorful.
62. My dream was vivid and the ocean was not pink, if and only if the rainbow was colorful or the sky was not green.
63. $r \rightarrow s$
64. $p \rightarrow \sim t$
65. $t \leftrightarrow s$
66. $q \leftrightarrow p$
67. $(p \wedge q) \rightarrow s$
68. $t \rightarrow r$
69. $t \leftrightarrow \sim q$
70. $p \rightarrow q$
71. $(r \wedge s \wedge \sim p) \vee \sim t$
72. $\sim(r \vee t) \leftrightarrow (p \wedge \sim s)$
73. $u \wedge \sim s$
74. $\sim(p \wedge t)$
75. $\sim(s \vee \sim u)$
76. $q \wedge \sim(u \vee r)$
77. $r \rightarrow \sim(\sim t \vee p)$
78. $r \leftrightarrow u$
79. $p \rightarrow (t \wedge q)$
80. $(q \wedge t) \rightarrow p$
81. $(t \wedge r) \leftrightarrow (\sim u \vee \sim p)$
82. $u \rightarrow (s \vee r)$
83. $r \leftrightarrow (p \vee s)$
84. $(p \wedge t) \rightarrow \sim q$
85. $\sim r \vee (q \wedge \sim s)$
86. $(q \wedge \sim p) \leftrightarrow (t \vee \sim s)$
87. $\sim s \rightarrow [q \rightarrow (\sim r \vee p)]$
88. $[r \rightarrow (s \wedge \sim p)] \leftrightarrow \sim r$
89. $[(s \vee \sim r) \rightarrow p] \leftrightarrow \sim t$
90. $(t \wedge \sim u) \rightarrow (\sim t \wedge u)$
91. $\sim p \leftrightarrow [q \wedge \sim r \wedge q]$
92. $[p \rightarrow (\sim q \vee r)] \leftrightarrow \sim p$

93. $(p \vee q) \wedge r$
94. $r \vee (p \wedge q)$
95. $(p \vee q) \wedge (r \vee t)$
96. $r \wedge [\, q \ \vee (p \vee s)]$

97.



98.



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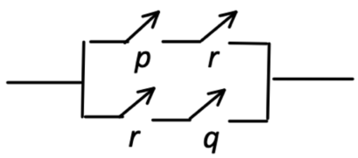
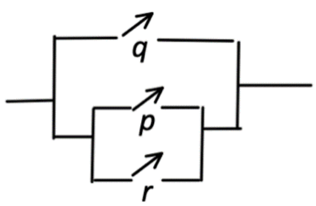


100.



Computer Programming Application Solutions

1. True
2. False
3. True
4. False
5. False

Section 2.3 Truth Tables: Negations, Conjunctions & Disjunctions

Objectives

- Construct truth tables with two simple statements that include negations, conjunctions, and disjunctions
- Construct truth tables with three simple statements that include negations, conjunctions, and disjunctions
- Determine the truth values for specific cases

Truth tables can be used to identify the truth values of statements. Statements can either be true (T) or false (F) and can be simple or complex. Our discussion will start with simple statements then progress to complex statements.

Given a simple statement p, p can be either true or false, but not both. Below is a truth table for the simple statement p.

p
T
F

Recall that the **negation** of a statement has the opposite truth value from the original statement. The negation of p is often read as "not p". Statements p and $\sim p$ can be represented using a truth table. If statement p is true, then statement $\sim p$ must be false. If statement p is false, then statement $\sim p$ must be true. The truth table for the negation $\sim p$ is shown below.

Truth Table for the Negation Statement

p	$\sim p$
T	F
F	T

Truth Tables for Two Statements

Given two statements p and q, there are four ways to represent the truth values. This can be represented using a truth table.

p	q
T	T
T	F
F	T
F	F

Each row of the truth table is referred to as a case. In the above truth table, for the first case p and q are both true. The second case is p is true and q is false. For the third case p is false and q is true. The fourth case is both p and q are false. When working with two simple statements, there will be four cases.

NOTE: Although the four cases (rows) can be presented in any order, for the remainder of this chapter, we will use the truth values for two statements in the order they are represented in the table above.

Recall that a statement using the connective ***and*** is called a **conjunction**. The conjunction of statements p and q can be represented as $p \wedge q$. The truth table for the conjunction $p \wedge q$ is shown below. In order for the statement $p \wedge q$ to be true, both statements p and q must be true. If one of the statements p or q is false, then the conjunction $p \wedge q$ is false. Notice in the following truth table, $p \wedge q$ is only true for the first case when p and q are both true. The final truth values of the other cases are false.

Truth Table for Conjunction Statements

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

The following is an example of a conjunction. The truth values of the conjunction are based on the truth values of the simple statements p and q.

Suppose you are given the following statements:

$\boldsymbol{p}$**:** Sam went to Paris.
$\boldsymbol{q}$**:** Sam went to London.

There are four possible cases for the statement $p \wedge q$.

CASE 1: Sam went to Paris and Sam went to London.

In this case, statements p and q are both true. Therefore, when Sam makes the statement "Sam went to Paris and London," he is telling the truth. This would mean that the statement $p \wedge q$ is true.

CASE 2: Sam went to Paris and Sam did not go to London.

In this case, statement p is true and statement q is false. Therefore, when Sam makes the statement "Sam went to Paris and London," he is NOT telling the truth. This would mean that the statement $p \wedge q$ is false.

CASE 3: Sam did not go to Paris and Sam went to London.

In this case, statement p is false and statement q is true. Therefore, when Sam makes the statement "Sam went to Paris and London," he is NOT telling the truth. This would mean that the statement $p \wedge q$ is false.

CASE 4: Sam did not go to Paris and Sam did not go to London.

In this case, statements p and q are both false. Therefore, when Sam makes the statement "Sam went to Paris and London," he is NOT telling the truth. This would mean that the statement $p \wedge q$ is false.

Recall that a statement using the connective ***or*** is called a **disjunction**. The disjunction of statements p or q can be represented as $p \vee q$. The truth table for the disjunction $p \vee q$ is shown below. In order for the statement $p \vee q$ to be true either p or q must be true. Notice in the following truth table, the first three cases will be true. The only time the statement $p \vee q$ is false is when both p and q are false. In the study of logic, the inclusive definition of ***or*** is used**.** This means that the truth value of $p \vee q$ is true when either p is true, q is true, or both p and q are true.

Truth Table for Disjunction Statements

$\boldsymbol{p}$	$\boldsymbol{q}$	$\boldsymbol{p \vee q}$
T	T	T
T	F	T
F	T	T
F	F	F

The following is an example of a disjunction. The truth values of the disjunction are based on the truth values of the simple statements p and q.

Suppose you are given the following statements:

$\boldsymbol{p}$**:** I will take a math class.
$\boldsymbol{q}$**:** I will take a speech class.

There are four possible cases for the statement $p \vee q$.

CASE 1: I will take a math class or I will take a speech class.

In this case, statements p and q are both true. Therefore, when I make the statement "I will take a math class or I will take a speech class," I am telling the truth. This would mean that the statement $p \vee q$ is true.

RECALL: In Section 2.2, logic uses the inclusive definition of OR indicating that at least one of the statements is true.

CASE 2: I will take a math class or I will not take a speech class.

In this case, statement p is true and statement q is false. Therefore, when I make the statement "I will take a math class or I will take a speech class," I am telling the truth. This would mean that the statement $p \vee q$ is true.

CASE 3: I will not take a math class or I will take a speech class.

In this case, statement p is false and statement q is true. Therefore, when I make the statement "I will take a math class or I will take a speech class," I am telling the truth. This would mean that the statement $p \vee q$ is true.

CASE 4: I will not take a math class or I will not take a speech class.

In this case, both statements p and q are false. Therefore, when I make the statement "I will take a math class or I will take a speech class," I am NOT telling the truth. This would mean that the statement $p \vee q$ is false.

➢ **EXAMPLE 2.3.1:** Given the statements p and q, create a truth table for $\sim p$, $\sim q$, $p \wedge q$, and $p \vee q$.

SOLUTION:

$\boldsymbol{p}$	$\boldsymbol{q}$	$\boldsymbol{\sim p}$	$\boldsymbol{\sim q}$	$\boldsymbol{p \wedge q}$	$\boldsymbol{p \vee q}$
T	T	F	F	T	T
T	F	F	T	F	T
F	T	T	F	F	T
F	F	T	T	F	F

➢ **EXAMPLE 2.3.2:** Given the statements p and q, create a truth table for $\sim p \wedge q$.

SOLUTION:

$\boldsymbol{p}$	$\boldsymbol{q}$	$\boldsymbol{\sim p}$	$\boldsymbol{\sim p \wedge q}$
T	T	F	F
T	F	F	F
F	T	T	T
F	F	T	F

➢ **EXAMPLE 2.3.3:** Given the statements p and q, create a truth table for $p \vee \sim q$.

SOLUTION:

p	q	$\sim q$	$p \vee \sim q$
T	T	F	T
T	F	T	T
F	T	F	F
F	F	T	T

NOTE: In the prior examples 2 and 3, notice that $\sim p$ and $\sim q$ are included in the truth tables. Although the examples do not explicitly ask for either separately, it is necessary to find each of the statements in order to find the truth values for $\sim p \wedge q$ and $p \vee \sim q$.

❖ **YOU TRY IT 2.3.A:** Given the statements p and q, create a truth table for $\sim p \wedge \sim q$ and $\sim p \vee \sim q$.

➢ **EXAMPLE 2.3.4:** Given the statements p and q, create a truth table for $(p \vee q) \vee \sim p$.

SOLUTION:

p	q	$\sim p$	$p \vee q$	$(p \vee q) \vee \sim p$
T	T	F	T	T
T	F	F	T	T
F	T	T	T	T
F	F	T	F	T

❖ **YOU TRY IT 2.3.B:** Given the statements p and q, create a truth table for $(\sim p \wedge \sim q) \vee \sim p$.

➢ **EXAMPLE 2.3.5:** Given the statements p and q, create a truth table for $\sim(\sim p \vee \sim q) \wedge \sim p$.

SOLUTION:

p	q	$\sim p$	$\sim q$	$\sim p \vee \sim q$	$\sim(\sim p \vee \sim q)$	$\sim(\sim p \vee \sim q) \wedge \sim p$
T	T	F	F	F	T	F
T	F	F	T	T	F	F
F	T	T	F	T	F	F
F	F	T	T	T	F	F

➢ **EXAMPLE 2.3.6:** Construct a truth table for the statement $\sim[(p \vee \sim q) \wedge (\sim p \wedge q)]$.

SOLUTION:

p	q	$\sim p$	$\sim q$	$p \vee \sim q$	$\sim p \wedge q$	$(p \vee \sim q) \wedge (\sim p \wedge q)$	$\sim[(p \vee \sim q) \wedge (\sim p \wedge q)]$
T	T	F	F	T	F	F	T
T	F	F	T	T	F	F	T
F	T	T	F	F	T	F	T
F	F	T	T	T	F	F	T

Truth Tables for Three Statements

Truth tables can be created with one or more simple statements. For this textbook, both two and three simple statements will be discussed. Given three simple statements p, q, and r, below is a truth table.

p	q	r
T	T	T
T	T	F
T	F	T
T	F	F
F	T	T
F	T	F
F	F	T
F	F	F

NOTE: Although the eight cases (rows) can be represented in any order, for the remainder of this chapter, we will use the truth values for three statements in the order they are represented in the previous table. To set up the column containing the truth values for statement p (going vertically) start with four Ts, then four Fs. For the truth values for the statement q column (again vertically), there are two Ts then two Fs, then two Ts, and two Fs. The truth values for the statement r column alternate (vertically) between Ts and Fs starting with T.

➢ **EXAMPLE 2.3.7:** Given the statements p, q, and r create a truth table for $p \wedge q \vee r$.

SOLUTION: Since parenthesis are not indicated, read the statements from left to right. This means find $p \wedge q$, and then find $(p \wedge q) \vee r$.

p	q	r	$p \wedge q$	$p \wedge q \vee r$
T	T	T	T	T
T	T	F	T	T
T	F	T	F	T
T	F	F	F	F
F	T	T	F	T
F	T	F	F	F
F	F	T	F	T
F	F	F	F	F

❖ **YOU TRY IT 2.3.C:** Given the statements p, q, and r create a truth table for $p \vee {\sim}q \vee r$.

➢ **EXAMPLE 2.3.8:** Given the statements p, q, and r create a truth table for $(p \wedge {\sim}q) \wedge (r \wedge {\sim}p)$.

SOLUTION:

p	q	r	${\sim}p$	${\sim}q$	$p \wedge {\sim}q$	$r \wedge {\sim}p$	$(p \wedge {\sim}q) \wedge (r \wedge {\sim}p)$
T	T	T	F	F	F	F	F
T	T	F	F	F	F	F	F
T	F	T	F	T	T	F	F
T	F	F	F	T	T	F	F
F	T	T	T	F	F	T	F
F	T	F	T	F	F	F	F
F	F	T	T	T	F	T	F
F	F	F	T	T	F	F	F

➢ **EXAMPLE 2.3.9:** Construct a truth table for the statement $\sim[(\sim p \vee \sim r) \wedge (\sim p \wedge q)] \vee (r \wedge \sim q)$.

SOLUTION:

p	q	r	$\sim p$	$\sim q$	$\sim r$	$\sim p \vee \sim r$	$\sim p \wedge q$	*	**	$r \wedge \sim q$	***
T	T	T	F	F	F	F	F	F	T	F	T
T	T	F	F	F	T	T	F	F	T	F	T
T	F	T	F	T	F	F	F	F	T	T	T
T	F	F	F	T	T	T	F	F	T	F	T
F	T	T	T	F	F	T	T	T	F	F	F
F	T	F	T	F	T	T	T	T	F	F	F
F	F	T	T	T	F	T	F	F	T	T	T
F	F	F	T	T	T	T	F	F	T	F	T

* $(\sim p \vee \sim r) \wedge (\sim p \wedge q)$
** $\sim[(p \vee \sim r) \wedge (\sim p \wedge q)]$
*** $\sim[(\sim p \vee \sim r) \wedge (\sim p \wedge q)] \vee (r \wedge \sim q)$.

❖ **YOU TRY IT 2.3.D:** Construct a truth table for the statement $\sim[(\sim q \wedge \sim r) \vee (p \vee \sim r)] \wedge (p \vee \sim r)$.

DEFINITION: A **tautology** is a complex statement where the final truth values are true for every possible case. Examples 2.3.4 and 2.3.6 are tautologies.

DEFINITION: A **contradiction** is a complex statement where the final truth values are false for every possible case. Examples 2.3.5 and 2.3.8 are contradictions.

DEFINITION: A **contingency** is a complex statement where the final truth values are a mix of true and false. Examples 2.3.1, 2.3.2, 2.3.3, 2.3.7, and 2.3.9 are contingencies.

Determining the Truth Values of Specific Cases

➢ **EXAMPLE 2.3.10:** Create a truth table for the statement "It is 60 degrees outside and the sun is out."

SOLUTION: Assign letters to each of the simple statements given. One example is as follows.

> $\boldsymbol{p}$: It is 60 degrees outside.
> $\boldsymbol{q}$: The sun is out.

The given statement is a conjunction because the simple statements are combined with the word ***and***.

$\boldsymbol{p}$	$\boldsymbol{q}$	$\boldsymbol{p \wedge q}$
T	T	T
T	F	F
F	T	F
F	F	F

- **EXAMPLE 2.3.11:** Given the above compound statement and using the truth table in Example 2.3.10, answer the following questions.

a. Suppose it is not 60 degrees outside and the sun is out. Is this case true or false?
b. Suppose it is 60 degrees outside and the sun is not out. Is this case true or false?

SOLUTION:

a. This case is false. Specifically, it is not 60 degrees out, so p is false and the sun is out, so q is true. The cases to the right of the arrow are the truth values for the simple statements and the compound statement. The last column of the third row (circled) is the truth value for the compound statement.

$\boldsymbol{p}$	$\boldsymbol{q}$	$\boldsymbol{p \wedge q}$
T	T	T
T	F	F
F	T	F
F	F	F

b. This case is false. Specifically, it is 60 degrees outside, so p is true and the sun is not out, so q is false. The cases to the right of the arrow are the truth values for the simple statements and the compound statement. The last column of the second row (circled) is the truth value for the compound statement.

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

➢ **EXAMPLE 2.3.12:** Create a truth table for the statement "I work out and I eat healthy, or I gain weight."

SOLUTION: Assign letters to each of the simple statements given. One example is as follows.

$\boldsymbol{p}$: I work out.
$\boldsymbol{q}$: I eat healthy.
$\boldsymbol{r}$: I gain weight.

This complex statement can be represented as $(p \wedge q) \vee r$. Notice there is a comma after the statement "I work out and I eat healthy." This indicates there are parenthesis around the statements p and q.

p	q	r	$p \wedge q$	$(p \wedge q) \vee r$
T	T	T	T	T
T	T	F	T	T
T	F	T	F	T
T	F	F	F	F
F	T	T	F	T
F	T	F	F	F
F	F	T	F	T
F	F	F	F	F

➢ **EXAMPLE 2.3.13:** Given the statement and using the truth table in Example 2.3.12, answer the following questions.

a. Determine the overall truth value of the statement in the case that:
- I do work out.
- I do not eat healthy.
- I do gain weight.

b. Determine the overall truth value of the statement in the case that:
- I do not work out.
- I do not eat healthy.
- I do not gain weight.

SOLUTION:

a. This case is true. Specifically, I work out, means p is true. I do not eat healthy, means q is false, and I gain weight means r is true. The case to the right of the arrow are the truth values for the simple statements and the complex statement. The last column of the third row (circled) is the truth value for the complex statement.

$\boldsymbol{p}$	$\boldsymbol{q}$	$\boldsymbol{r}$	$\boldsymbol{p \wedge q}$	$\boldsymbol{(p \wedge q) \vee r}$
T	T	T	T	T
T	T	F	T	T
T	F	T	F	T
T	F	F	F	F
F	T	T	F	T
F	T	F	F	F
F	F	T	F	T
F	F	F	F	F

b. This case is false. Specifically, I do not work out means p is false. I do not eat healthy means q is false, and I do not gain weight means r is false. The cases to the right of the arrow are the truth values for the simple statements and the complex statement. The last column of the last row (circled) is the truth value for the complex statement.

p	q	r	$p \wedge q$	$(p \wedge q) \vee r$
T	T	T	T	T
T	T	F	T	T
T	F	T	F	T
T	F	F	F	F
F	T	T	F	T
F	T	F	F	F
F	F	T	F	T
F	F	F	F	F

❖ **YOU TRY IT 2.3.E:** Create a truth table for the statement: "John goes to Hawaii, or he goes to Miami and Key West." Then, given the following simple statements, are the complex statements true or false?

a. Using the truth table that was created, determine the overall truth value of the statement in the case that:

- John does not go to Hawaii.
- John does not go to Miami.
- John does not go to Key West.

b. Using the truth table that was created, determine the overall truth value of the statement in the case that:

- John does not go to Hawaii.
- John does not go to Miami.
- John goes to Key West.

It is possible to determine the truth value of a specific case without a truth table. For instance, using the compound statement from Example 2.3.10, "It is 60 degrees outside and the sun is out," for the specific case where p is false and q is true, a truth value can be determined without completing all four rows of the truth table. The given truth values of the simple statements can be inserted directly into the statement $p \wedge q$ to determine the truth value for that specific case.

$$p \wedge q$$
$$F \wedge T$$
$$F$$

Since at least one of the simple statements of the conjunction is false, the compound statement $p \wedge q$ is false.

- **EXAMPLE 2.3.14:** Given the statement "I study for my exam or I fail my exam" the compound statement is represented symbolically as $p \vee q$, with p and q as follows:

  $\boldsymbol{p}$: I study for my exam.
  $\boldsymbol{q}$: I fail my exam.

  Find the overall truth value for the specific case given

  - I do not study for my exam.
  - I fail my exam.

  SOLUTION: For the specific case "I do not study for my exam or I fail my exam," the value of p is false and q is true. These truth values can be inserted into the symbolic notation for the compound statement $p \vee q$ to determine the overall truth value of the compound statement.

$$p \vee q$$
$$F \vee T$$
$$T$$

  Since at least one of the simple statements is true, the compound statement $p \vee q$ is true.

- **EXAMPLE 2.3.15:** Given the statement "She likes rap and pop music, or she likes country music" the complex statement is represented symbolically as $(p \wedge q) \vee r$ with p, q, and r as follows:

  $\boldsymbol{p}$: She likes rap music.
  $\boldsymbol{q}$: She likes pop music.
  $\boldsymbol{r}$**:** She likes country music.

  Find the overall truth value for the specific case given

  - She likes rap music.
  - She does not like pop music.
  - She likes country music.

  SOLUTION: For the specific case that "She likes rap music and does not like pop music, or she likes country music," the value of p is true, q is false, and r is true. These truth values can be inserted into the symbolic notation for the complex statement $(p \wedge q) \vee r$ to determine the overall truth value of the complex statement.

$$(p \wedge q) \vee r$$
$$(T \wedge F) \vee T$$
$$F \vee T$$
$$T$$

The complex statement $(p \wedge q) \vee r$ is true.

In the following section, additional truth tables will be studied. Below is a review of the negation, conjunction, and disjunction and their corresponding truth tables.

Quick Review

- The **negation** of a true statement is a false statement. The **negation** of a false statement is a true statement.
- The **conjunction** $p \wedge q$ is true when both simple statements p and q are true. In all other cases $p \wedge q$ is false.
- The **disjunction** $p \vee q$ is false when both simple statements p and q are false. In all other cases $p \vee q$ is true.
- Basic truth table for negation, conjunction, and disjunction statements:

p	q	$\sim p$	$p \wedge q$	$p \vee q$
T	T	F	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	F

YOU TRY IT 2.3.A SOLUTION:

p	q	$\sim p$	$\sim q$	$\sim p \wedge \sim q$	$\sim p \vee \sim q$
T	T	F	F	F	F
T	F	F	T	F	T
F	T	T	F	F	T
F	F	T	T	T	T

YOU TRY IT 2.3.B SOLUTION:

p	q	$\sim p$	$\sim q$	$\sim p \wedge \sim q$	$(\sim p \wedge \sim q) \vee \sim p$
T	T	F	F	F	F
T	F	F	T	F	F
F	T	T	F	F	T
F	F	T	T	T	T

YOU TRY IT 2.3.C SOLUTION:

p	q	r	$\sim q$	$p \vee \sim q$	$(p \vee \sim q) \vee r$
T	T	T	F	T	T
T	T	F	F	T	T
T	F	T	T	T	T
T	F	F	T	T	T
F	T	T	F	F	T
F	T	F	F	F	F
F	F	T	T	T	T
F	F	F	T	T	T

YOU TRY IT 2.3.D SOLUTION:

p	q	r	$\sim q$	$\sim r$	$\sim q \wedge \sim r$	$p \vee \sim r$	*	**	***
T	T	T	F	F	F	T	T	F	F
T	T	F	F	T	F	T	T	F	F
T	F	T	T	F	F	T	T	F	F
T	F	F	T	T	T	T	T	F	F
F	T	T	F	F	F	F	F	T	F
F	T	F	F	T	F	T	T	F	F
F	F	T	T	F	F	F	F	T	F
F	F	F	T	T	T	T	T	F	F

* $(\sim q \wedge \sim r) \vee (p \vee \sim r)$
** $\sim[(\sim q \wedge \sim r) \vee (p \vee \sim r)]$
*** $\sim[(\sim q \wedge \sim r) \vee (p \vee \sim r)] \wedge (p \vee \sim r)$

This is a contradiction.

YOU TRY IT 2.3.E SOLUTION:

Assign letters to each of the simple statements given. One example is as follows.

$\boldsymbol{p}$: John goes to Hawaii.
$\boldsymbol{q}$: John goes to Miami.
$\boldsymbol{r}$: John goes to Key West.

This complex statement can be represented as $p \vee (q \wedge r)$. Notice there is a comma after the statement John goes to Hawaii. This indicates there are parenthesis around the statements q and r.

$\boldsymbol{p}$	$\boldsymbol{q}$	$\boldsymbol{r}$	$\boldsymbol{q \wedge r}$	$\boldsymbol{p \vee (q \wedge r)}$
T	T	T	T	T
T	T	F	F	T
T	F	T	F	T
T	F	F	F	T
F	T	T	T	T
F	T	F	F	F
F	F	T	F	F
F	F	F	F	F

a. This case is false. Specifically, John does not go to Hawaii, so p is false. John does not go to Miami, so q is false. John does not go to Key West, so r is false. The case to the right of the arrow are the truth values for the simple statements and the complex statement. The last column of the last row (circled) is the truth value for the complex statement.

$\boldsymbol{p}$	$\boldsymbol{q}$	$\boldsymbol{r}$	$\boldsymbol{q \wedge r}$	$\boldsymbol{p \vee (q \wedge r)}$
T	T	T	T	T
T	T	F	F	T
T	F	T	F	T
T	F	F	F	T
F	T	T	T	T
F	T	F	F	F
F	F	T	F	T
F	F	F	F	F

b. This case is false. Specifically, John does not go to Hawaii, so p is false. John does not go to Miami, so q is false. John does go to Key West, so r is true. The case to the right of the arrow are the truth values for the simple statements and the complex statement. The last column of the second to last row (circled) is the truth value for the complex statement.

p	q	r	$q \wedge r$	$p \vee (q \wedge r)$
T	T	T	T	T
T	T	F	F	T
T	F	T	F	T
T	F	F	F	T
F	T	T	T	T
F	T	F	F	F
F	F	T	F	F
F	F	F	F	F

Section 2.3 | Exercises

In exercises 1 – 8, create a truth table for the given statement.

1. $\sim p \vee (q \wedge \sim q)$
2. $(p \vee \sim q) \wedge \sim q$
3. $\sim[(p \vee q) \wedge (\sim p \wedge \sim q)]$
4. $\sim[q \vee (\sim p \vee \sim q)]$
5. $\sim(p \wedge q) \vee \sim(p \vee q)$
6. $(\sim p \vee p) \wedge (\sim q \vee q)$
7. $(\sim p \wedge \sim q) \wedge \sim(q \wedge p)$
8. $\sim(\sim p \wedge \sim q)$

In exercises 9 – 19, create a truth table for the given statement.

9. $p \wedge \sim q \wedge \sim r$
10. $\sim r \vee \sim p \vee q$
11. $\sim(p \vee q) \vee r$
12. $(p \wedge q) \vee r$
13. $\sim(\sim q \vee \sim r) \wedge (p \wedge r)$
14. $(r \wedge q) \vee (\sim p \wedge \sim r)$
15. $\sim[(q \wedge \sim p) \vee (q \wedge \sim r)]$
16. $[(\sim p \wedge r) \vee \sim q] \wedge \sim p$
17. $\sim[(q \wedge \sim p) \vee (q \wedge \sim r)] \wedge (p \vee q \vee r)$
18. $(p \wedge \sim p) \vee (q \wedge r)$
19. $(p \vee r) \vee \sim[(p \vee q) \wedge (q \vee r)]$

In exercises 20 – 26, create a truth table and then find the truth value for each specific case.

20. a. Create a truth table for the statement "I am late to the airport and I miss my flight."
    Let $\boldsymbol{p}$: I am late to the airport. Let $\boldsymbol{q}$: I miss my flight.
    b. Determine the overall truth value of the statement in the case that:
    - I am not late to the airport.
    - I do not miss my flight.

21. a. Create a truth table for the statement "It is not true that it is April and snowing in Chicago." Let $\boldsymbol{p}$: It is April. Let $\boldsymbol{q}$: It is snowing in Chicago.
    b. Determine the overall truth value of the statement in the case that:
    - It is April.
    - It is not snowing in Chicago.

22. a. Create a truth table given the statement "I wear my swimsuit or I go skinny dipping." Let $\boldsymbol{p}$: I wear my swimsuit. Let $\boldsymbol{q}$: I go skinny dipping.
    b. Determine the overall truth value of the statement in the case that:
    - I wear my swimsuit.
    - I go skinny dipping.

23. a. Create a truth table given the statement "I get a summer job or I can't afford to take summer courses." Let $\boldsymbol{p}$: I get a summer job. Let $\boldsymbol{q}$: I can afford to take summer courses.
    b. Determine the overall truth value of the statement in the case that:
    - I don't get a summer job.
    - I can afford to take summer courses.

24. a. Create a truth table given the statement "I lose my keys and my car gets stolen." Let $\boldsymbol{p}$: I lose my keys. Let $\boldsymbol{q}$: My car gets stolen.
    b. Determine the overall truth value of the statement in the case that:
    - I do not lose my keys.
    - My car does not get stolen.

25. a. Create a truth table for the statement "I had soup or salad with dinner tonight." Let $\boldsymbol{p}$: I had soup with dinner tonight. Let $\boldsymbol{q}$: I had salad with dinner tonight.
    b. Determine the overall truth value of the statement in the case that:
    - I had soup with dinner tonight.
    - I had salad with dinner tonight.

26. a. Create a truth table for the statement "I wasn't sore and I lifted weights, or I was sore and I didn't lift weights." Let $\boldsymbol{p}$: I lifted weights. Let $\boldsymbol{q}$: I was sore.
    b. Determine the overall truth value of the statement in the case that:
    - I was sore.
    - I lifted weights.

In exercises 27 – 35, create a truth table and then find the truth value for each specific case.

27. a. Create a truth table for the statement "I like the Red Hot Chili Peppers and Phish and Dua Lipa."
    b. Determine the overall truth value of the statement in the case that:
    - I like the Red Hot Chili Peppers.
    - I do not like Phish.
    - I like Dua Lipa.

28. a. Create a truth table for the statement "It is not the case that I play tennis or go to lunch, or I don't work an extra shift today."
    b. Determine the overall truth value of the statement in the case that:
    - I play tennis.
    - I do not go to lunch.
    - I work an extra shift today.

29. a. Create a truth table for the statement "Yoga makes me relax and Pilates stresses me out, or running makes me relax."
    b. Determine the overall truth value of the statement in the case that:
    - Yoga does not make me relax.
    - Pilates does not stress me out.
    - Running makes me relax.

30. a. Create a truth table for the statement "I pay off my credit card bill and the credit card company doesn't charge me lots of interest, or my parents get upset with me."
    b. Determine the overall truth value of the statement in the case that:
    - I pay off my credit card bill.
    - The credit card company charges me lots of interest.
    - My parents don't get upset with me.

31. a. Create a truth table for the statement "I drink coffee or I drink tea, and I drink water in the morning."
    b. Determine the overall truth value of the statement in the case that:
    - I do not drink coffee in the morning.
    - I do not drink tea in the morning.
    - I drink water in the morning.

32. a. Create a truth table for the statement "It is not the case that: there is a window or natural light in my office. And my office has fake plants."
    b. Determine the overall truth value of the statement in the case that:
    - There isn't a window in my office.
    - There isn't natural light in my office.
    - There are fake plants in my office.

33. a. Create a truth table for the statement "I follow Instagram and TikTok, or I follow Twitter."
    b. Determine the overall truth value of the statement in the case that:
    - I do not follow Instagram.
    - I do not follow TikTok.
    - I do not follow Twitter.

34. a. Create a truth table for the statement "I met the prerequisite and registered for the course, or I didn't meet the prerequisite and took the placement exam."
    b. Determine the overall truth value of the statement in the case that:
    - I met the prerequisite.
    - I registered for the course.
    - I didn't take the placement exam.

35. a. Create a truth table for the statement "I drive a Maserati, a Bentley, and a Rolls Royce."
    b. Determine the overall truth value of the statement in the case that:
    - I drive a Maserati.
    - I do not drive a Bentley.
    - I do not drive a Rolls Royce.

In exercises 36 – 42, determine the truth value of each statement for the specific case given.

36. $\sim(p \vee q);\ p{:}\, true,\ q{:}\, false$
37. $p \wedge (q \vee \sim p);\ p{:}\, true,\ q{:}\, true$
38. $(p \wedge \sim q) \wedge (p \vee \sim q);\ p{:}\, true,\ q{:}\, false$
39. $(\sim p \wedge \sim q) \vee (p \wedge q); p$: true, q: false
40. $(\sim p \wedge q) \vee (p \vee \sim q);\ p{:}\, false,\ q{:}\, false$
41. $[(q \vee \sim p) \wedge \sim p] \wedge \sim q;\ p{:}\, false,\ q{:}\, true$
42. $[(p \vee q) \wedge \sim(p \wedge q)] \vee \sim p; p$: false, q: true

In exercises 43 – 50, determine the truth value of each statement for the specific case given.

43. $\sim(p \vee r) \vee \sim q$; $p: true$, $q: true$, $r: false$
44. $\sim p \vee (q \wedge r)$; p: true, q: false, r: true
45. $(p \wedge \sim q) \vee \sim(r \wedge \sim q)$; $p: true$, $q: false$, $r: true$
46. $[(q \vee \sim p) \wedge \sim r] \wedge \sim p$; $p: true$, $q: false$, $r: false$
47. $[(p \vee \sim r) \wedge (q \vee r)] \wedge \sim(q \vee p)$; $p: true$, $q: false$, $r: true$
48. $\sim((p \vee q) \wedge \sim r) \vee (\sim r \wedge \sim q)$; p: true, q: true, r: true
49. $\sim p \vee [\sim(q \wedge r) \vee (q \wedge \sim p)]$; $p: true$, $q: true$, $r: false$
50. $[(\sim r \vee q) \wedge (\sim p \vee r)] \wedge (r \vee p)$; $p: false$, $q: false$, $r: true$

Section 2.3 Exercise Solutions

1.

p	q	$\sim p$	$\sim q$	$q \wedge \sim q$	$\sim p \vee (q \wedge \sim q)$
T	T	F	F	F	F
T	F	F	T	F	F
F	T	T	F	F	T
F	F	T	T	F	T

2.

p	q	$\sim q$	$p \vee \sim q$	$(p \vee \sim q) \wedge \sim q$
T	T	F	T	F
T	F	T	T	T
F	T	F	F	F
F	F	T	T	T

3.

p	q	$\sim p$	$\sim q$	$p \vee q$	$\sim p \wedge \sim q$	$(p \vee q) \wedge (\sim p \wedge \sim q)$	$\sim[(p \vee q) \wedge (\sim p \wedge \sim q)]$
T	T	F	F	T	F	F	T
T	F	F	T	T	F	F	T
F	T	T	F	T	F	F	T
F	F	T	T	F	T	F	T

This is a tautology.

4.

p	q	$\sim p$	$\sim q$	$\sim p \vee \sim q$	$q \vee (\sim p \vee \sim q)$	$\sim[q \vee (\sim p \vee \sim q)]$
T	T	F	F	F	T	F
T	F	F	T	T	T	F
F	T	T	F	T	T	F
F	F	T	T	T	T	F

This is a contradiction.

5.

p	q	$p \wedge q$	$\sim(p \wedge q)$	$p \vee q$	$\sim(p \vee q)$	$\sim(p \wedge q) \vee \sim(p \vee q)$
T	T	T	F	T	F	F
T	F	F	T	T	F	T
F	T	F	T	T	F	T
F	F	F	T	F	T	T

6.

p	q	$\sim p$	$\sim q$	$\sim p \vee p$	$\sim q \vee q$	$(\sim p \vee p) \wedge (\sim q \vee q)$
T	T	F	F	T	T	T
T	F	F	T	T	T	T
F	T	T	F	T	T	T
F	F	T	T	T	T	T

This is a tautology.

7.

p	q	$\sim p$	$\sim q$	$\sim p \wedge \sim q$	$q \wedge p$	$\sim(q \wedge p)$	$(\sim p \wedge \sim q) \wedge \sim(q \wedge p)$
T	T	F	F	F	T	F	F
T	F	F	T	F	F	T	F
F	T	T	F	F	F	T	F
F	F	T	T	T	F	T	T

8.

p	q	$\sim p$	$\sim q$	$\sim p \wedge \sim q$	$\sim(\sim p \wedge \sim q)$
T	T	F	F	F	T
T	F	F	T	F	T
F	T	T	F	F	T
F	F	T	T	T	F

9.

p	q	r	$\sim q$	$\sim r$	$p \wedge \sim q$	$(p \wedge \sim q) \wedge \sim r$
T	T	T	F	F	F	F
T	T	F	F	T	F	F
T	F	T	T	F	T	F
T	F	F	T	T	T	T
F	T	T	F	F	F	F
F	T	F	F	T	F	F
F	F	T	T	F	F	F
F	F	F	T	T	F	F

10.

p	q	r	$\sim p$	$\sim r$	$\sim r \vee \sim p$	$(\sim r \vee \sim p) \vee q$
T	T	T	F	F	F	T
T	T	F	F	T	T	T
T	F	T	F	F	F	F
T	F	F	F	T	T	T
F	T	T	T	F	T	T
F	T	F	T	T	T	T
F	F	T	T	F	T	T
F	F	F	T	T	T	T

11.

p	q	r	$p \vee q$	$\sim(p \vee q)$	$\sim(p \vee q) \vee r$
T	T	T	T	F	T
T	T	F	T	F	F
T	F	T	T	F	T
T	F	F	T	F	F
F	T	T	T	F	T
F	T	F	T	F	F
F	F	T	F	T	T
F	F	F	F	T	T

12.

p	q	r	$p \wedge q$	$(p \wedge q) \vee r$
T	T	T	T	T
T	T	F	T	T
T	F	T	F	T
T	F	F	F	F
F	T	T	F	T
F	T	F	F	F
F	F	T	F	T
F	F	F	F	F

13.

p	q	r	$\sim q$	$\sim r$	$\sim q \vee \sim r$	$\sim(\sim q \vee \sim r)$	$p \wedge r$	$\sim(\sim q \vee \sim r) \wedge (p \wedge r)$
T	T	T	F	F	F	T	T	T
T	T	F	F	T	T	F	F	F
T	F	T	T	F	T	F	T	F
T	F	F	T	T	T	F	F	F
F	T	T	F	F	F	T	F	F
F	T	F	F	T	T	F	F	F
F	F	T	T	F	T	F	F	F
F	F	F	T	T	T	F	F	F

14.

p	q	r	$\sim p$	$\sim r$	$r \wedge q$	$\sim p \wedge \sim r$	$(r \wedge q) \vee (\sim p \wedge \sim r)$
T	T	T	F	F	T	F	T
T	T	F	F	T	F	F	F
T	F	T	F	F	F	F	F
T	F	F	F	T	F	F	F
F	T	T	T	F	T	F	T
F	T	F	T	T	F	T	T
F	F	T	T	F	F	F	F
F	F	F	T	T	F	T	T

15.

p	q	r	$\sim p$	$\sim r$	$q \wedge \sim p$	$q \wedge \sim r$	$(q \wedge \sim p) \vee (q \wedge \sim r)$	$\sim[(q \wedge \sim p) \vee (q \wedge \sim r)]$
T	T	T	F	F	F	F	F	T
T	T	F	F	T	F	T	T	F
T	F	T	F	F	F	F	F	T
T	F	F	F	T	F	F	F	T
F	T	T	T	F	T	F	T	F
F	T	F	T	T	T	T	T	F
F	F	T	T	F	F	F	F	T
F	F	F	T	T	F	F	F	T

16.

p	q	r	$\sim p$	$\sim q$	$\sim p \wedge r$	$(\sim p \wedge r) \vee \sim q$	$[(\sim p \wedge r) \vee \sim q] \wedge \sim p$
T	T	T	F	F	F	F	F
T	T	F	F	F	F	F	F
T	F	T	F	T	F	T	F
T	F	F	F	T	F	T	F
F	T	T	T	F	T	T	T
F	T	F	T	F	F	F	F
F	F	T	T	T	T	T	T
F	F	F	T	T	F	T	T

17.

p	q	r	$\sim p$	$\sim r$	$q \wedge \sim p$	$q \wedge \sim r$	$(q \wedge \sim p) \vee (q \wedge \sim r)$	*	**	***
T	T	T	F	F	F	F	F	T	T	T
T	T	F	F	T	F	T	T	F	T	F
T	F	T	F	F	F	F	F	T	T	T
T	F	F	F	T	F	F	F	T	T	T
F	T	T	T	F	T	F	T	F	T	F
F	T	F	T	T	T	T	T	F	T	F
F	F	T	T	F	F	F	F	T	T	T
F	F	F	T	T	F	F	F	T	F	F

* $\sim[(q \wedge \sim p) \vee (q \wedge \sim r)]$

** $(p \vee q \vee r)$

*** $\sim[(q \wedge \sim p) \vee (q \wedge \sim r)] \wedge (p \vee q \vee r)$

18.

$\boldsymbol{p}$	$\boldsymbol{q}$	$\boldsymbol{r}$	$\boldsymbol{\sim p}$	$\boldsymbol{p \wedge \sim p}$	$\boldsymbol{q \wedge r}$	$\boldsymbol{(p \wedge \sim p) \vee (q \wedge r)}$
T	T	T	F	F	T	T
T	T	F	F	F	F	F
T	F	T	F	F	F	F
T	F	F	F	F	F	F
F	T	T	T	F	T	T
F	T	F	T	F	F	F
F	F	T	T	F	F	F
F	F	F	T	F	F	F

19.

$\boldsymbol{p}$	$\boldsymbol{q}$	$\boldsymbol{r}$	$\boldsymbol{p \vee r}$	$\boldsymbol{p \vee q}$	$\boldsymbol{q \vee r}$	*	**	***
T	T	T	T	T	T	T	F	T
T	T	F	T	T	T	T	F	T
T	F	T	T	T	T	T	F	T
T	F	F	T	T	F	F	T	T
F	T	T	T	T	T	T	F	T
F	T	F	F	T	T	T	F	F
F	F	T	T	F	T	F	T	T
F	F	F	F	F	F	F	T	T

*$(p \vee q) \wedge (q \vee r)$
**$\sim[(p \vee q) \wedge (q \vee r)]$
*** $(p \vee r) \vee \sim[(p \vee q) \wedge (q \vee r)]$

20.

a.

$\boldsymbol{p}$	$\boldsymbol{q}$	$\boldsymbol{p \wedge q}$
T	T	T
T	F	F
F	T	F
F	F	F

b. False

21.

a.

p	q	$p \wedge q$	$\sim(p \wedge q)$
T	T	T	F
T	F	F	T
F	T	F	T
F	F	F	T

b. True

22.

a.

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

b. True

23.

a.

p	q	$\sim q$	$p \vee \sim q$
T	T	F	T
T	F	T	T
F	T	F	F
F	F	T	T

b. False

24.

a.

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

b. False

25.

a.

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

b. True

26.

a.

p	q	$\sim p$	$\sim q$	$\sim q \wedge p$	$q \wedge \sim p$	$(\sim q \wedge p) \vee (q \wedge \sim p)$
T	T	F	F	F	F	F
T	F	F	T	T	F	T
F	T	T	F	F	T	T
F	F	T	T	F	F	F

b. False

For exercises 27 – 35, answers can vary but one solution is shown.

27. Answers can vary. One way to assign statements p, q and r is provided.
a. Let p: I like the Red Hot Chili Peppers. Let q: I like Phish. Let r: I like Dua Lipa.

p	q	r	$p \wedge q$	$(p \wedge q) \wedge r$
T	T	T	T	T
T	T	F	T	F
T	F	T	F	F
T	F	F	F	F
F	T	T	F	F
F	T	F	F	F
F	F	T	F	F
F	F	F	F	F

b. False

28. Answers can vary. One way to assign statements p, q and r is provided.
a. Let p: I play tennis. Let q: I go to lunch. Let r: I work an extra shift today.

p	q	r	$\sim r$	$p \vee q$	$\sim(p \vee q)$	$\sim(p \vee q) \vee \sim r$
T	T	T	F	T	F	F
T	T	F	T	T	F	T
T	F	T	F	T	F	F
T	F	F	T	T	F	T
F	T	T	F	T	F	F
F	T	F	T	T	F	T
F	F	T	F	F	T	T
F	F	F	T	F	T	T

b. False

29. Answers can vary. One way to assign statements p, q and r is provided.
a. Let p: Yoga makes me relax. Let q: Pilates stresses me out. Let r: Running makes me relax.

p	q	r	$p \wedge q$	$(p \wedge q) \vee r$
T	T	T	T	T
T	T	F	T	T
T	F	T	F	T
T	F	F	F	F
F	T	T	F	T
F	T	F	F	F
F	F	T	F	T
F	F	F	F	F

b. True

30. Answers can vary. One way to assign statements p, q and r is provided.
a. Let p: I payoff my credit card bill. Let q: The credit card company charges me lots of interest. Let r: My parents get upset with me.

p	q	r	$\sim q$	$p \wedge \sim q$	$(p \wedge \sim q) \vee r$
T	T	T	F	F	T
T	T	F	F	F	F
T	F	T	T	T	T
T	F	F	T	T	T
F	T	T	F	F	T
F	T	F	F	F	F
F	F	T	T	F	T
F	F	F	T	F	F

b. False

31. Answers can vary. One way to assign statements p, q and r is provided.
a. Let p: I drink coffee in the morning. Let q: I drink tea in the morning. Let r: I drink water in the morning.

p	q	r	$p \vee q$	$(p \vee q) \wedge r$
T	T	T	T	T
T	T	F	T	F
T	F	T	T	T
T	F	F	T	F
F	T	T	T	T
F	T	F	T	F
F	F	T	F	F
F	F	F	F	F

b. False

32. Answers can vary. One way to assign statements p, q and r is provided.
a. Let p: There is a window in my office. Let q: There is natural light in my office.
Let r: There are fake plants in my office.

p	q	r	$p \vee q$	$\sim(p \vee q)$	$\sim(p \vee q) \wedge r$
T	T	T	T	F	F
T	T	F	T	F	F
T	F	T	T	F	F
T	F	F	T	F	F
F	T	T	T	F	F
F	T	F	T	F	F
F	F	T	F	T	T
F	F	F	F	T	F

b. True

33. Answers can vary. One way to assign statements p, q and r is provided.
a. Let p: I follow Instagram. Let q: I follow TikTok. Let r: I follow Twitter.

p	q	r	$p \wedge q$	$(p \wedge q) \vee r$
T	T	T	T	T
T	T	F	T	T
T	F	T	F	T
T	F	F	F	F
F	T	T	F	T
F	T	F	F	F
F	F	T	F	T
F	F	F	F	F

b. False

34. Answers can vary. One way to assign statements p, q and r is provided.
a. Let p: I met the prerequisite. Let q: I registered for the course. Let r: I took the placement exam.

$\boldsymbol{p}$	$\boldsymbol{q}$	$\boldsymbol{r}$	$\boldsymbol{\sim p}$	$\boldsymbol{p \wedge q}$	$\boldsymbol{\sim p \wedge r}$	$\boldsymbol{(p \wedge q) \vee (\sim p \wedge r)}$
T	T	T	F	T	F	T
T	T	F	F	T	F	T
T	F	T	F	F	F	F
T	F	F	F	F	F	F
F	T	T	T	F	T	T
F	T	F	T	F	F	F
F	F	T	T	F	T	T
F	F	F	T	F	F	F

b. True

35. Answers can vary. One way to assign statements p, q and r is provided.
a. Let p: I drive a Maserati. Let q: I drive a Bentley. Let r: I drive a Rolls Royce.

$\boldsymbol{p}$	$\boldsymbol{q}$	$\boldsymbol{r}$	$\boldsymbol{p \wedge q}$	$\boldsymbol{p \wedge q \wedge r}$
T	T	T	T	T
T	T	F	T	F
T	F	T	F	F
T	F	F	F	F
F	T	T	F	F
F	T	F	F	F
F	F	T	F	F
F	F	F	F	F

b. False

36. False
37. True
38. True
39. False
40. True
41. False
42. True
43. False
44. False
45. True
46. False
47. False
48. True
49. True
50. False

Section 2.4 Truth Tables: Conditional & Biconditional Statements

Objectives

- Construct truth tables with two simple statements that include conditional and biconditional statements
- Construct truth tables with three simple statements that include conditional and biconditional statements
- Determine the truth value for a specific case

Conditional statements are often used in everyday life. For example, a parent may declare, "If you eat your vegetables, then you can have dessert!" The return policy for a store may read, "If you have a receipt that is dated within the past 30 days, then you can return the item." A sign at an amusement park might state, "If you are less than 48 inches tall, then you cannot ride this roller coaster." Suppose a boss promises her workers that if they work overtime today, they can have tomorrow off. When would this promise be broken? Now suppose the workers are told that they can have tomorrow off if and only if they work overtime today. Does that mean the same as the original promise? It turns out that logic can be used to answer these questions! In this section, the truth values of ***if … then*** and ***if and only if*** statements will be examined. Basic truth tables will be developed for these connectives, and they will be used to analyze compound statements involving these connectives.

Truth Tables for Two Simple Statements

Recall that an ***if … then*** statement is called a **conditional** statement. If written in ***if … then*** form, the first part of the statement is called the **antecedent** and the second part of the statement is called the **consequent**. In logic, the statement *"If p, then q"* is represented $p \rightarrow q$.

Suppose a boss promises her workers that if they work overtime today, then they can have tomorrow off. In what case would this promise be broken?

This conditional statement can be represented symbolically as $p \rightarrow q$, where p represents "They work overtime today" and q represents "They have tomorrow off." There are four different cases to examine:

CASE 1: They work overtime today and end up having tomorrow off.

In this case the promise was not broken. So, the conditional statement $p \rightarrow q$ is true.

CASE 2: They work overtime today and **DON'T** end up having tomorrow off.

In this case, the promise was broken. So, the conditional statement $p \rightarrow q$ is false.

CASE 3: They don't work overtime today and end up having tomorrow off.

The workers were only promised something if they work overtime today. Nothing was ever promised if they don't work overtime today. So, in this case the promise was not broken. Thus, the conditional statement $p \rightarrow q$ is true.

CASE 4: They don't work overtime today and don't end up having tomorrow off.

Just as in Case 3, nothing was promised if they don't work overtime today. So, in this case the promise was not broken. Thus, the conditional statement $p \rightarrow q$ is true.

By examining the cases above, the promise was broken only in the case in which the antecedent was true (they worked overtime today) and the consequent was false (they didn't end up having off tomorrow). In all other cases, the promise was not broken.

This information can be summarized using a truth table. The truth table for the conditional statement $p \rightarrow q$ is represented as follows. For the statement $p \rightarrow q$ to be false, the antecedent (p) must be true and the consequent (q) must be false. This case appears in the second row of the truth table. In every other case, the statement $p \rightarrow q$ is true.

Truth Table for Conditional Statements

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

➢ **EXAMPLE 2.4.1:** Given the statements p and q, create truth tables for $\sim p \rightarrow q$ and $q \rightarrow \sim p$.

SOLUTION:

p	q	$\sim p$	$\sim p \rightarrow q$
T	T	F	**T**
T	F	F	**T**
F	T	T	**T**
F	F	T	**F**

$\sim p \rightarrow q$

F → T is **True**

F → F is **True**

T → T is **True**

T → F is **False**

p	q	$\sim p$	$q \rightarrow \sim p$
T	T	F	**F**
T	F	F	**T**
F	T	T	**T**
F	F	T	**T**

$q \rightarrow \sim p$

T → F is **False**

F → F is **True**

T → T is **True**

F → T is **True**

NOTE: Order matters when determining truth values of a conditional statement. That is, the final truth values of $\sim p \rightarrow q$ and $q \rightarrow \sim p$ are not the same in every case! This idea will be examined further in Section 2.6.

Recall that an ***if and only if*** statement is called a **biconditional** statement. In logic, the statement "p if and only if q" is represented $p \leftrightarrow q$.

In a biconditional statement $p \leftrightarrow q$, it is implied that $p \rightarrow q$ AND $q \rightarrow p$ at the same time. See Figure 2.4.1 for an example.

FIGURE 2.4.1

Equivalent Way to Represent the Biconditional Statement $p \leftrightarrow q$	
$p \leftrightarrow q$	$(p \rightarrow q) \wedge (q \rightarrow p)$
Ex: A polygon is a triangle **if and only** if it has three sides.	**Ex:** If a polygon is a triangle then it has three sides **and** if a polygon has three sides then it is a triangle.

The truth table for the biconditional statement $p \leftrightarrow q$ can be constructed using the idea that $p \leftrightarrow q$ will have the same truth values as $(p \rightarrow q) \wedge (q \rightarrow p)$ in every single case.

p	q	$p \rightarrow q$	$q \rightarrow p$	$(p \rightarrow q) \wedge (q \rightarrow p)$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

The last column shows that the statement $(p \rightarrow q) \wedge (q \rightarrow p)$ is true when the original truth values of p and q are the same, that is when both p and q are true or when both p and q are false. Alternatively, the statement $(p \rightarrow q) \wedge (q \rightarrow p)$ is false when the original truth values of p and q differ, that is when p is true and q is false or when p is false and q is true. The same rules will apply when determining the truth values of the biconditional statement $p \leftrightarrow q$.

Truth Table for Biconditional Statements

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

➢ **EXAMPLE 2.4.2:** Given the statements p and q, create truth tables for $\sim p \leftrightarrow q$ and $\sim q \leftrightarrow p$.

SOLUTION:

p	q	$\sim p$	$\sim p \leftrightarrow q$
T	T	F	F
T	F	F	T
F	T	T	T
F	F	T	F

p	q	$\sim q$	$\sim q \leftrightarrow p$
T	T	F	F
T	F	T	T
F	T	F	T
F	F	T	F

❖ **YOU TRY IT 2.4.A:** Given the statements p and q, create truth tables for $\sim q \rightarrow \sim p$ and $\sim q \leftrightarrow \sim p$.

➢ **EXAMPLE 2.4.3:** Given the statements p and q, create a truth table for $(p \rightarrow \sim q) \wedge \sim p$.

SOLUTION:

p	q	$\sim p$	$\sim q$	$p \rightarrow \sim q$	$(p \rightarrow \sim q) \wedge \sim p$
T	T	F	F	F	F
T	F	F	T	T	F
F	T	T	F	T	T
F	F	T	T	T	T

➢ **EXAMPLE 2.4.4:** Given the statements p and q, create a truth table for $\sim q \leftrightarrow \sim(\sim q \vee p)$.

SOLUTION:

p	q	$\sim q$	$\sim q \vee p$	$\sim(\sim q \vee p)$	$\sim q \leftrightarrow \sim(\sim q \vee p)$
T	T	F	T	F	T
T	F	T	T	F	F
F	T	F	F	T	F
F	F	T	T	F	F

❖ **YOU TRY IT 2.4.B:** Given the statements p and q, create a truth table for $(\sim q \wedge p) \leftrightarrow \sim p$.

➢ **EXAMPLE 2.4.5:** Given the statements p and q, create a truth table for $(\sim q \wedge p) \leftrightarrow \sim(p \vee q)$.

SOLUTION:

p	q	$\sim q$	$\sim q \wedge p$	$p \vee q$	$\sim(p \vee q)$	$(\sim q \wedge p) \leftrightarrow \sim(p \vee q)$
T	T	F	F	T	F	T
T	F	T	T	T	F	F
F	T	F	F	T	F	T
F	F	T	F	F	T	F

❖ **YOU TRY IT 2.4.C:** Given the statements p and q, create a truth table for $(\sim p \rightarrow q) \vee (\sim q \leftrightarrow p)$.

Truth Tables for Three Simple Statements

Instead of two simple statements, three simple statements might be given. In this case, the simple statements are often labeled as p, q, and r. Refer to Section 2.3 for the eight possible cases for three simple statements and the standard order in which these cases appear.

➢ **EXAMPLE 2.4.6:** Given the statements p, q, and r create a truth table for $\sim(\sim p \leftrightarrow q) \rightarrow \sim r$.

SOLUTION:

$\boldsymbol{p}$	$\boldsymbol{q}$	$\boldsymbol{r}$	$\boldsymbol{\sim p}$	$\boldsymbol{\sim r}$	$\boldsymbol{\sim p \leftrightarrow q}$	$\boldsymbol{\sim(\sim p \leftrightarrow q)}$	$\boldsymbol{\sim(\sim p \leftrightarrow q) \rightarrow \sim r}$
T	T	T	F	F	F	T	F
T	T	F	F	T	F	T	T
T	F	T	F	F	T	F	T
T	F	F	F	T	T	F	T
F	T	T	T	F	T	F	T
F	T	F	T	T	T	F	T
F	F	T	T	F	F	T	F
F	F	F	T	T	F	T	T

❖ **YOU TRY IT 2.4.D:** Given the statements p, q, and r create a truth table for $\sim p \rightarrow (\sim q \wedge r)$.

➢ **EXAMPLE 2.4.7:** Given the statements p, q, and r create a truth table for $\sim(p \rightarrow \sim r) \wedge (q \leftrightarrow \sim r)$.

SOLUTION:

$\boldsymbol{p}$	$\boldsymbol{q}$	$\boldsymbol{r}$	$\boldsymbol{\sim r}$	$\boldsymbol{p \rightarrow \sim r}$	$\boldsymbol{\sim(p \rightarrow \sim r)}$	$\boldsymbol{q \leftrightarrow \sim r}$	$\boldsymbol{\sim(p \rightarrow \sim r) \wedge (q \leftrightarrow \sim r)}$
T	T	T	F	F	T	F	F
T	T	F	T	T	F	T	F
T	F	T	F	F	T	T	T
T	F	F	T	T	F	F	F
F	T	T	F	T	F	F	F
F	T	F	T	T	F	T	F
F	F	T	F	T	F	T	F
F	F	F	T	T	F	F	F

➢ **EXAMPLE 2.4.8:** Given the statements p, q, and r create a truth table for $[(\sim p \leftrightarrow q) \rightarrow (\sim r \wedge p)] \rightarrow (\sim p \vee q)$.

SOLUTION:

p	q	r	$\sim p$	$\sim r$	$\sim p \leftrightarrow q$	$\sim r \wedge p$	$(\sim p \leftrightarrow q) \rightarrow (\sim r \wedge p)$	$\sim p \vee q$	*
T	T	T	F	F	F	F	T	T	T
T	T	F	F	T	F	T	T	T	T
T	F	T	F	F	T	F	F	F	T
T	F	F	F	T	T	T	T	F	F
F	T	T	T	F	T	F	F	T	T
F	T	F	T	T	T	F	F	T	T
F	F	T	T	F	F	F	T	T	T
F	F	F	T	T	F	F	T	T	T

* $[(\sim p \leftrightarrow q) \rightarrow (\sim r \wedge p)] \rightarrow (\sim p \vee q)$

❖ **YOU TRY IT 2.4.E:** Given the statements p, q, and r create a truth table for $(\sim q \vee r) \leftrightarrow \sim[(r \rightarrow q) \rightarrow (p \wedge \sim q)]$.

Determining the Truth Values of Specific Cases

Just as in Section 2.3, truth tables can be created for compound statements involving conditionals and biconditionals. These truth tables can be used the determine truth values of specific cases. In Section 2.5 truth tables will be used to determine if statements are equivalent. Furthermore, in Section 2.7 truth tables will be used to determine the validity of arguments.

➢ **EXAMPLE 2.4.9:**

a. Create a truth table for the statement "If they go to the movies, then they won't have time to finish their homework."

b. Using the truth table created in part a, determine the overall truth value of the statement in the case that:
- They go to the movies.
- They have time to finish their homework.

SOLUTION:

a. Assign letters to each of the simple statements given. One way to do so is as follows.

> $\boldsymbol{p}$: They go to the movies.
> $\boldsymbol{q}$: They have time to finish their homework.

This statement from part a can be represented symbolically as $p \to {\sim}q$.

$\boldsymbol{p}$	$\boldsymbol{q}$	$\boldsymbol{\sim q}$	$\boldsymbol{p \to \sim q}$
T	T	F	F
T	F	T	T
F	T	F	T
F	F	T	T

b. In this specific case, they go to the movies which means that statement p is true. They also have time to finish their homework which means that statement q is true. To determine the truth value of the statement "If they go to the movies, then they won't have time to finish their homework," use the row of the table where p and q are both true (highlighted). The final truth value in this row is circled and indicates that the compound statement is false in this case.

$\boldsymbol{p}$	$\boldsymbol{q}$	$\boldsymbol{\sim q}$	$\boldsymbol{p \to \sim q}$
T	T	F	F
T	F	T	T
F	T	F	T
F	F	T	T

➢ EXAMPLE 2.4.10:

a. Create a truth table for the statement "A student is eligible to take Precalculus if and only if they have a proper placement test score, or a student is not eligible to take Precalculus if they did not pass the prerequisite course."

b. Using the truth table created in part a, determine the overall truth value of the statement in the case that:

- The student is not eligible to take Precalculus.
- The student does not have a proper placement test score.
- The student passed the prerequisite course.

SOLUTION:

a. Assign letters to each of the simple statements given. One way to do so is as follows.

$\boldsymbol{p}$: A student is eligible to take Precalculus.
$\boldsymbol{q}$: A student has a proper placement test score.
$\boldsymbol{r}$: A student passed the prerequisite course.

The statement from part a can be represented as $(p \leftrightarrow q) \vee (\sim r \rightarrow \sim p)$.

(Recall that the statement "A student is not eligible to take Precalculus if they did not pass the prerequisite course" can be rewritten "If a student did not pass the prerequisite course, then they are not eligible to take Precalculus.")

$\boldsymbol{p}$	$\boldsymbol{q}$	$\boldsymbol{r}$	$\boldsymbol{\sim p}$	$\boldsymbol{\sim r}$	$\boldsymbol{p \leftrightarrow q}$	$\boldsymbol{\sim r \rightarrow \sim p}$	$\boldsymbol{(p \leftrightarrow q) \vee (\sim r \rightarrow \sim p)}$
T	T	T	F	F	T	T	T
T	T	F	F	T	T	F	T
T	F	T	F	F	F	T	T
T	F	F	F	T	F	F	F
F	T	T	T	F	F	T	T
F	T	F	T	T	F	T	T
F	F	T	T	F	T	T	T
F	F	F	T	T	T	T	T

b. In this specific case, the student is not eligible to take Precalculus which means that statement p is false. Additionally, the student does not have a proper placement score which means that statement q is false. Finally, the student passed the prerequisite course which means that statement r is true. To determine the truth value of the statement "A student is eligible to take Precalculus if and only if they have a proper placement test score, or a student is not eligible to take Precalculus if they did not pass the prerequisite course," use the row of the table where p and q are both false and r is true (highlighted). The final truth value in this row is circled and indicates that the compound statement is true in this case.

p	q	r	$\sim p$	$\sim r$	$p \leftrightarrow q$	$\sim r \rightarrow \sim p$	$(p \leftrightarrow q) \vee (\sim r \rightarrow \sim p)$
T	T	T	F	F	T	T	T
T	T	F	F	T	T	F	T
T	F	T	F	F	F	T	T
T	F	F	F	T	F	F	F
F	T	T	T	F	F	T	T
F	T	F	T	T	F	T	T
F	F	T	T	F	T	T	T
F	F	F	T	T	T	T	T

❖ **YOU TRY IT 2.4.F:**

a. Create a truth table for the statement "If a video is longer than three minutes then it cannot be shown on TikTok, or it can be shown on YouTube."

b. Determine the overall truth value of the statement in the case that:

- The video is longer than three minutes.
- The video cannot be shown on TikTok.
- The video cannot be shown on YouTube.

It is possible to determine the truth value of a specific case without a truth table. In Example 2.4.9, the statement "If they go to the movies, then they won't have time to finish their homework" was represented symbolically as $p \rightarrow \sim q$, with p and q as follows:

$\boldsymbol{p}$: They go to the movies.
$\boldsymbol{q}$: They have time to finish their homework.

In the specific case that they go to the movies and have time to finish their homework, the values of p and q are both true. These truth values can be inserted into the statement $p \rightarrow \sim q$ to determine the overall truth value of the statement.

$$p \rightarrow \sim q$$
$$T \rightarrow \sim T$$
$$T \rightarrow F$$
$$F$$

Since the antecedent is true and the consequent is false, the ***if … then*** statement is false in this specific case. Note this same result was obtained using a truth table in Example 2.4.9.

If using a truth table to determine the truth value for a particular case, then it is only necessary to create the row that applies. In the example above p and q are both true. Thus, it is only necessary to create the row in the table that reflects these values. From this row, it can be determined that the statement $p \rightarrow \sim q$ is false in the case when p and q are both true.

$\boldsymbol{p}$	$\boldsymbol{q}$	$\boldsymbol{\sim q}$	$\boldsymbol{p \rightarrow \sim q}$
T	T	F	F

➢ **EXAMPLE 2.4.11:** Determine the truth value of the statement "She is granted a scholarship and is able to take classes in the fall, or if she doesn't get a summer job then she won't take classes in the fall" in the case that:

- She is not granted a scholarship.
- She takes classes in the fall.
- She gets a summer job.

SOLUTION: Assign letters to each of the simple statements given in the statement "She is granted a scholarship and is able to take classes in the fall, or if she doesn't get a summer job then she won't take classes in the fall." One way to do so is as follows.

$\boldsymbol{p}$: She is granted a scholarship.
$\boldsymbol{q}$: She takes classes in the fall.
$\boldsymbol{r}$: She gets a summer job.

The compound statement can be represented as $(p \wedge q) \vee (\sim r \rightarrow \sim q)$. Looking at the specific case that is provided in the bullet points above, it can be determined that p is false, q is true, and r is true. These truth values can be inserted into the statement to determine its overall truth value.

$$(p \wedge q) \vee (\sim r \rightarrow \sim q)$$
$$(F \wedge T) \vee (\sim T \rightarrow \sim T)$$
$$(F \wedge T) \vee (F \rightarrow F)$$
$$F \vee T$$
$$T$$

The original compound statement is true for this specific case.

This result could have also been accomplished by creating the row of the truth table for the original compound statement in which p is false, q is true, and r is true:

$\boldsymbol{p}$	$\boldsymbol{q}$	$\boldsymbol{r}$	$\boldsymbol{\sim r}$	$\boldsymbol{\sim q}$	$\boldsymbol{p \wedge q}$	$\boldsymbol{\sim r \rightarrow \sim q}$	$\boldsymbol{(p \wedge q) \vee (\sim r \rightarrow \sim q)}$
F	T	T	F	F	F	T	T

- **YOU TRY IT 2.4.G:** Determine the truth value of the statement "If it is below freezing, then it is not true that he will ride his bike to school or wear his new sneakers" in the case that:
  - It is below freezing.
  - He rides his bike to school.
  - He doesn't wear his new sneakers.

Quick Review

- The **conditional** $p \rightarrow q$ is only false when the antecedent, p, is true and the consequent, q, is false.
- The **biconditional** $p \leftrightarrow q$ is true only when p and q have the same truth value, that is when they are either both true or when they are both false.
- Basic truth table for conditional and biconditional statements:

p	q	$p \rightarrow q$	$p \leftrightarrow q$
T	T	T	T
T	F	F	F
F	T	T	F
F	F	T	T

YOU TRY IT 2.4.A SOLUTION:

p	q	$\sim p$	$\sim q$	$\sim q \rightarrow \sim p$
T	T	F	F	T
T	F	F	T	F
F	T	T	F	T
F	F	T	T	T

p	q	$\sim p$	$\sim q$	$\sim q \leftrightarrow \sim p$
T	T	F	F	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

YOU TRY IT 2.4.B SOLUTION:

p	q	$\sim p$	$\sim q$	$\sim q \wedge p$	$(\sim q \wedge p) \leftrightarrow \sim p$
T	T	F	F	F	T
T	F	F	T	T	F
F	T	T	F	F	F
F	F	T	T	F	F

YOU TRY IT 2.4.C SOLUTION:

p	q	$\sim p$	$\sim q$	$\sim p \rightarrow q$	$\sim q \leftrightarrow p$	$(\sim p \rightarrow q) \vee (\sim q \leftrightarrow p)$
T	T	F	F	T	F	T
T	F	F	T	T	T	T
F	T	T	F	T	T	T
F	F	T	T	F	F	F

YOU TRY IT 2.4.D SOLUTION:

p	q	r	$\sim p$	$\sim q$	$\sim q \wedge r$	$\sim p \rightarrow (\sim q \wedge r)$
T	T	T	F	F	F	T
T	T	F	F	F	F	T
T	F	T	F	T	T	T
T	F	F	F	T	F	T
F	T	T	T	F	F	F
F	T	F	T	F	F	F
F	F	T	T	T	T	T
F	F	F	T	T	F	F

YOU TRY IT 2.4.E SOLUTION:

p	q	r	$\sim q$	$\sim q \vee r$	$r \rightarrow q$	$p \wedge \sim q$	$(r \rightarrow q) \rightarrow (p \wedge \sim q)$	*	**
T	T	T	F	T	T	F	F	T	T
T	T	F	F	F	T	F	F	T	F
T	F	T	T	T	F	T	T	F	F
T	F	F	T	T	T	T	T	F	F
F	T	T	F	T	T	F	F	T	T
F	T	F	F	F	T	F	F	T	F
F	F	T	T	T	F	F	T	F	F
F	F	F	T	T	T	F	F	T	T

* $\sim[(r \rightarrow q) \rightarrow (p \wedge \sim q)]$
** $(\sim q \vee r) \leftrightarrow \sim[(r \rightarrow q) \rightarrow (p \wedge \sim q)]$

YOU TRY IT 2.4.F SOLUTION:

a. Assign letters to each of the simple statements given. One way to do so is as follows.

$\boldsymbol{p}$: A video is longer than three minutes.
$\boldsymbol{q}$: A video can be shown on TikTok.
$\boldsymbol{r}$: A video can be shown on YouTube.

The compound statement can be represented symbolically as $(p \rightarrow \sim q) \vee r$.

$\boldsymbol{p}$	$\boldsymbol{q}$	$\boldsymbol{r}$	$\sim\boldsymbol{q}$	$\boldsymbol{p} \rightarrow \sim\boldsymbol{q}$	$(\boldsymbol{p} \rightarrow \sim\boldsymbol{q}) \vee \boldsymbol{r}$
T	T	T	F	F	T
T	T	F	F	F	F
T	F	T	T	T	T
T	F	F	T	T	T
F	T	T	F	T	T
F	T	F	F	T	T
F	F	T	T	T	T
F	F	F	T	T	T

b. In this specific case, p is true and q and r are both false. The final truth value in the highlighted row above indicates that the compound statement is true in this case.

YOU TRY IT 2.4.G SOLUTION: Assign letters to each of the simple statements given in the compound statement. One way to do so is as follows.

$\boldsymbol{p}$: It is below freezing.
$\boldsymbol{q}$: He will ride his bike to school.
$\boldsymbol{r}$: He will wear his new sneakers.

The compound statement can be represented as $p \rightarrow \sim(q \vee r)$. Looking at the specific case that is provided in the bullet points, it can be determined that p is true, q is true, and r is false.

$$p \rightarrow \sim(q \vee r)$$
$$T \rightarrow \sim(T \vee F)$$
$$T \rightarrow \sim(T)$$
$$T \rightarrow F$$
$$F$$

Since the antecedent is true and the consequent is false, the original compound statement is false for this specific case.

Section 2.4 | Exercises

In Exercises 1 – 12, create a truth table for the given statement.

1. $\sim(p \to \sim q)$
2. $(p \vee q) \to \sim p$
3. $(p \wedge \sim q) \to \sim r$
4. $\sim p \to (\sim q \vee r)$
5. $(r \vee \sim p) \to (\sim q \vee p)$
6. $(\sim r \wedge \sim p) \to q$
7. $\sim(p \to q) \wedge (\sim q \to \sim p)$
8. $(\sim p \vee r) \to \sim(p \wedge q)$
9. $[(p \vee q) \wedge \sim q] \to \sim p$
10. $[\sim(p \vee q) \wedge r] \to p$
11. $[(p \vee q) \to (\sim q \vee r)] \to (\sim p \wedge r)$
12. $[\, p \wedge (\sim q \to r)] \to \sim(p \vee \sim r)$

In Exercises 13 – 24, create a truth table for the given statement.

13. $\sim(p \leftrightarrow \sim q)$
14. $(p \vee q) \leftrightarrow \sim q$
15. $\sim r \leftrightarrow (\sim p \vee q)$
16. $\sim q \to (p \leftrightarrow r)$
17. $(\sim p \to \sim q) \leftrightarrow \sim(q \vee p)$
18. $(q \wedge \sim p) \leftrightarrow (\sim r \to p)$
19. $[r \wedge (q \vee \sim p)] \leftrightarrow \sim r$
20. $\sim(q \vee \sim r) \leftrightarrow (p \wedge r)$
21. $\sim p \wedge [(q \leftrightarrow r) \to (\sim p \leftrightarrow q)]$
22. $(q \to r) \leftrightarrow (\sim p \to q)$
23. $(p \to \sim q) \leftrightarrow (\sim p \wedge r)$
24. $[(p \vee q) \to (p \wedge q)] \leftrightarrow \sim r$

In Exercises 25 – 30, create a truth table for the given statement. Insert grouping symbols as appropriate according to the dominance of connectives.

25. $r \vee p \to q \wedge \sim r$
26. $\sim p \to q \vee \sim r$

27. $p \wedge \sim q \rightarrow p \rightarrow \sim q$
28. $\sim r \leftrightarrow p \rightarrow q \vee p$
29. $p \leftrightarrow q \vee \sim r \rightarrow \sim p$
30. $q \vee p \leftrightarrow r \rightarrow \sim q$

In Exercises 31 – 34, create a truth table and then find the truth value for each specific case.

31. a. Create a truth table for the statement "A team makes the tournament if and only if they have won the majority of their games."

    b. Determine the overall truth value of the statement in the case that:
    - A team does not make the tournament.
    - A team has won the majority of their games.

32. a. Create a truth table for the statement "If Neil does not like the Green Bay Packers, then Neil likes the Chicago Bears."

    b. Determine the overall truth value of the statement in the case that:
    - Neil likes the Green Bay Packers.
    - Neil likes the Chicago Bears.

33. a. Create a truth table for the statement "If a bill is rejected by the U.S. Senate, then it does not become a law."

    b. Determine the overall truth value of the statement in the case that:
    - A bill is not rejected by the U.S. Senate.
    - A bill does not become a law.

34. a. Create a truth table for the statement "Curtis Granderson was inducted in the Chicago Sports Hall of Fame if and only if he was drafted by the Detroit Tigers in 2002."

    b. Determine the overall truth value of the statement in the case that:
    - Curtis Granderson was drafted by the Detroit Tigers in 2002.
    - In 2021, Curtis Granderson was inducted in the Chicago Sports Hall of Fame.

In Exercises 35 – 40, create a truth table and then find the truth value for each specific case.

35. a. Create a truth table for the statement "She will attend the concert, if and only if the show is on a Saturday night and tickets are not expensive."

    b. Determine the overall truth value of the statement in the case that:
    - She attends the concert.
    - The show is on a Friday night.
    - Tickets are not expensive.

36. a. Create a truth table for the statement "I will not go to the university, if and only if it is not true that I get a scholarship or a grant."

    b. Determine the overall truth value of the statement in the case that:
    - I go to the university.
    - I do not get a scholarship.
    - I get a grant.

37. a. Create a truth table for the statement "If he doesn't take public transportation to work, then it is not true that it's raining and his bicycle is in the shop for repairs."

    b. Determine the overall truth value of the statement in the case that:
    - He takes public transportation to work.
    - It's not raining.
    - His bicycle is not in the shop for repairs.

38. a. Create a truth table for the statement "The teacher will call home if and only if Shayla studies or she earns an A on her first quiz."

    b. Determine the overall truth value of the statement in the case that:
    - The teacher did not call home.
    - Shayla did not study.
    - Shayla earned a B on her first quiz.

39. a. Create a truth table for the statement "I will buy the dress and I will buy the shoes if and only if I am invited to the party."

    b. Determine the overall truth value of the statement in the case that:
    - I will not buy the dress.
    - I will buy the shoes.
    - I am not invited to the party.

40. a. Create a truth table for the statement "If Carl does not carry an umbrella and does not wear a raincoat, then it is not raining."

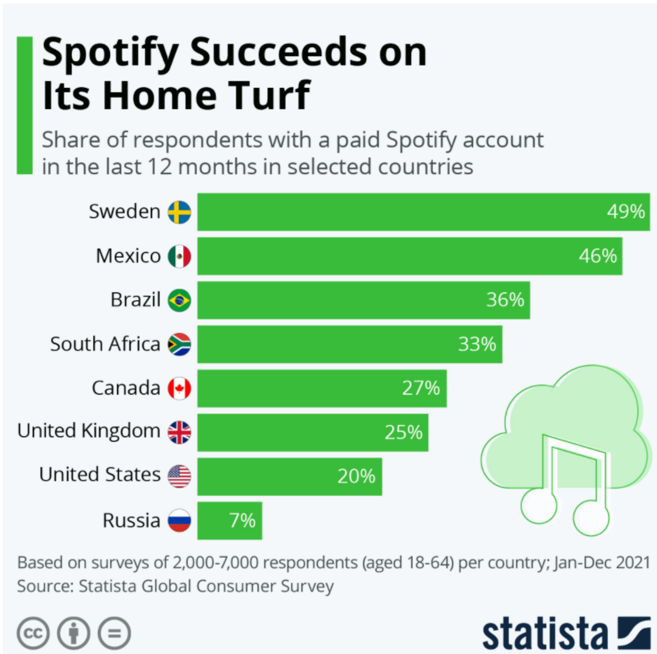
    b. Determine the overall truth value of the statement in the case that:
    - Carl carries an umbrella.
    - Carl does not wear a raincoat.
    - It is raining.

In Exercises 41 – 46, determine the truth value of each statement for the specific case given.

41. $\sim(p \vee q) \leftrightarrow \sim q;\ \ p{:}\,true,\ \ q{:}\,true$
42. $(p \rightarrow \sim q) \leftrightarrow (q \rightarrow \sim p);\ \ p{:}\,false,\ \ q{:}\,false$
43. $(p \wedge \sim q) \rightarrow (r \leftrightarrow \sim q);\ \ p{:}\,true,\ \ q{:}\,false,\ \ r{:}\,true$
44. $(p \wedge \sim r) \leftrightarrow (q \rightarrow \sim r);\ \ p{:}\,false,\ \ q{:}\,true,\ \ r{:}\,false$
45. $[(\sim r \rightarrow q) \wedge \sim p] \rightarrow r;\ \ p{:}\,true,\ \ q{:}\,false,\ \ r{:}\,true$
46. $[(p \vee \sim q) \rightarrow (q \rightarrow r)] \leftrightarrow \sim q;\ \ p{:}\,false,\ \ q{:}\,false,\ \ r{:}\,true$

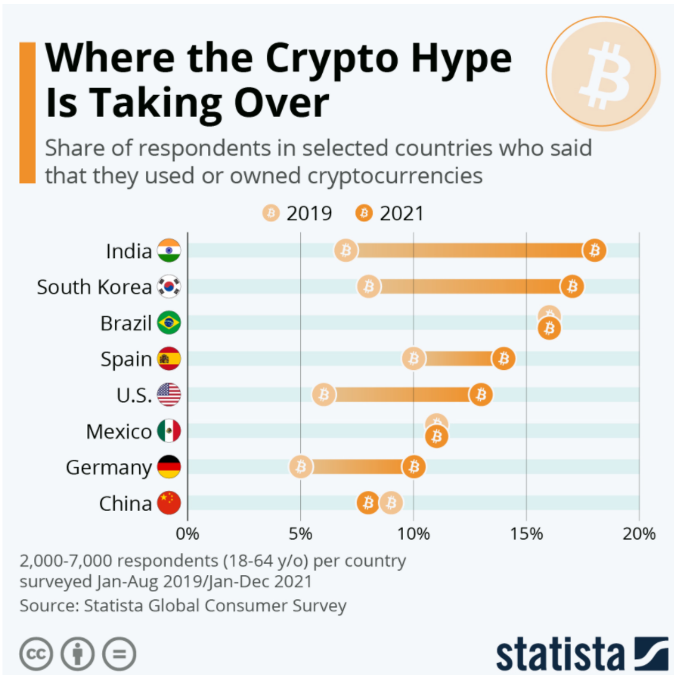
Applications

47. According to a Statista Global Consumer Survey, respondents in selected countries were asked if had a paid Spotify account. Results from Jan-Dec 2021 are provided in the following infographic. Use the infographic to determine the truth value of each given statement.



a. It is not the case that in 2021 a larger percentage of respondents had a paid Spotify account in Sweden than Mexico and less than 20% of respondents in the U.S. had a paid Spotify account.
b. If 36% of respondents had a paid Spotify account in Brazil in 2021, then the percentage of respondents with paid accounts in Mexico was not 19% percent larger than that of Canada in 2021.

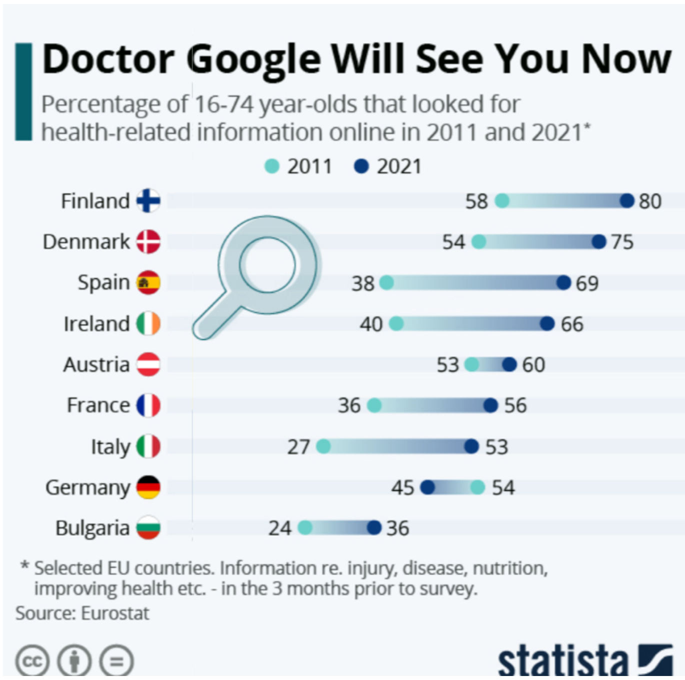
48. According to a Statista Global Consumer Survey, respondents in selected countries were asked if they used or owned cryptocurrencies. Results for 2019 and 2021 are provided in the following infographic. Use the infographic to determine the truth value of each given statement.



a. More than 15% of respondents in India and less than 15% of respondents in the U.S. said that they used or owned cryptocurrencies in 2021, if and only if the number of respondents in Germany who said that they used or owned cryptocurrencies increased by 10% from 2019 to 2021.

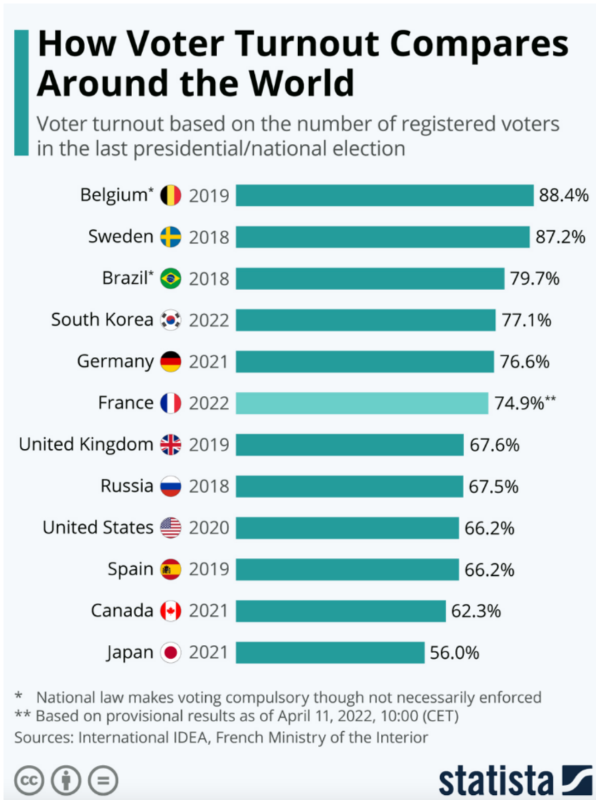
b. If the percentage of respondents in Brazil who said that they used or owned cryptocurrencies was the same in 2019 and 2021, then more than 10% of respondents in China or Mexico said that they used or owned cryptocurrencies in 2019.

49. According to a Statista Global Consumer Survey, respondents in selected countries were asked if they looked for health-related information online. Results for 2011 and 2021 are provided in the following infographic. Use the infographic to determine the truth value of each given statement.



a. If the percentage of respondents who looked for health-related information online increased by at least 10% from 2011 to 2021 in Denmark and Austria, then the percentage increased by at least 20% in Ireland.
b. The percent of respondents who looked for health-related information online nearly doubled in Italy if and only if Bulgaria has the lowest percentage of the countries reported in 2021.

50. According to a Statista Global Consumer Survey, respondents in selected countries were asked if they voted in the most recent presidential/national election. Results are provided in the following infographic. Use the infographic to determine the truth value of each given statement.



a. Belgium reported the highest voter turnout and a national law makes voting compulsory in Belgium. (Hint: use the * note at the bottom of the graphic.)
b. If voter turnout is higher in Germany and Sweden than in the United States, then voter turnout for Germany and Sweden are both more than 20% higher than the United States.

Concept Review

51. When is a conditional statement false?
52. When is a biconditional statement true?
53. If the antecedent is false, determine the truth value needed for the consequent to make a conditional statement true.

Section 2.4 Exercise Solutions

1.

p	q	$\sim q$	$p \rightarrow \sim q$	$\sim(p \rightarrow \sim q)$
T	T	F	F	T
T	F	T	T	F
F	T	F	T	F
F	F	T	T	F

2.

p	q	$p \vee q$	$\sim p$	$(p \vee q) \rightarrow \sim p$
T	T	T	F	F
T	F	T	F	F
F	T	T	T	T
F	F	F	T	T

3.

p	q	r	$\sim q$	$p \wedge \sim q$	$\sim r$	$(p \wedge \sim q) \rightarrow \sim r$
T	T	T	F	F	F	T
T	T	F	F	F	T	T
T	F	T	T	T	F	F
T	F	F	T	T	T	T
F	T	T	F	F	F	T
F	T	F	F	F	T	T
F	F	T	T	F	F	T
F	F	F	T	F	T	T

4.

p	q	r	$\sim p$	$\sim q$	$\sim q \vee r$	$\sim p \rightarrow (\sim q \vee r)$
T	T	T	F	F	T	T
T	T	F	F	F	F	T
T	F	T	F	T	T	T
T	F	F	F	T	T	T
F	T	T	T	F	T	T
F	T	F	T	F	F	F
F	F	T	T	T	T	T
F	F	F	T	T	T	T

5.

p	q	r	$\sim p$	$\sim q$	$r \vee \sim p$	$\sim q \vee p$	$(r \vee \sim p) \rightarrow (\sim q \vee p)$
T	T	T	F	F	T	T	T
T	T	F	F	F	F	T	T
T	F	T	F	T	T	T	T
T	F	F	F	T	F	T	T
F	T	T	T	F	T	F	F
F	T	F	T	F	T	F	F
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

6.

p	q	r	$\sim p$	$\sim r$	$\sim r \wedge \sim p$	$(\sim r \wedge \sim p) \rightarrow q$
T	T	T	F	F	F	T
T	T	F	F	T	F	T
T	F	T	F	F	F	T
T	F	F	F	T	F	T
F	T	T	T	F	F	T
F	T	F	T	T	T	T
F	F	T	T	F	F	T
F	F	F	T	T	T	F

7.

p	q	$p \to q$	$\sim(p \to q)$	$\sim q$	$\sim p$	$\sim q \to \sim p$	$\sim(p \to q) \wedge (\sim q \to \sim p)$
T	T	T	F	F	F	T	F
T	F	F	T	T	F	F	F
F	T	T	F	F	T	T	F
F	F	T	F	T	T	T	F

8.

p	q	r	$\sim p$	$\sim p \vee r$	$p \wedge q$	$\sim(p \wedge q)$	$(\sim p \vee r) \to \sim(p \wedge q)$
T	T	T	F	T	T	F	F
T	T	F	F	F	T	F	T
T	F	T	F	T	F	T	T
T	F	F	F	F	F	T	T
F	T	T	T	T	F	T	T
F	T	F	T	T	F	T	T
F	F	T	T	T	F	T	T
F	F	F	T	T	F	T	T

9.

p	q	$p \vee q$	$\sim q$	$(p \vee q) \wedge \sim q$	$\sim p$	$[(p \vee q) \wedge \sim q] \to \sim p$
T	T	T	F	F	F	T
T	F	T	T	T	F	F
F	T	T	F	F	T	T
F	F	F	T	F	T	T

10.

p	q	r	$p \vee q$	$\sim(p \vee q)$	$\sim(p \vee q) \wedge r$	$[\sim(p \vee q) \wedge r] \to p$
T	T	T	T	F	F	T
T	T	F	T	F	F	T
T	F	T	T	F	F	T
T	F	F	T	F	F	T
F	T	T	T	F	F	T
F	T	F	T	F	F	T
F	F	T	F	T	T	F
F	F	F	F	T	F	T

11.

p	q	r	$\sim p$	$\sim q$	$p \vee q$	$\sim q \vee r$	$(p \vee q) \rightarrow (\sim q \vee r)$	$\sim p \wedge r$	$[(p \vee q) \rightarrow (\sim q \vee r)] \rightarrow (\sim p \wedge r)$
T	T	T	F	F	T	T	T	F	F
T	T	F	F	F	T	F	F	F	T
T	F	T	F	T	T	T	T	F	F
T	F	F	F	T	T	T	T	F	F
F	T	T	T	F	T	T	T	T	T
F	T	F	T	F	T	F	F	F	T
F	F	T	T	T	F	T	T	T	T
F	F	F	T	T	F	T	T	F	F

12.

p	q	r	$\sim q$	$\sim r$	$\sim q \rightarrow r$	$p \wedge (\sim q \rightarrow r)$	$p \vee \sim r$	$\sim(p \vee \sim r)$	$[p \wedge (\sim q \rightarrow r)] \rightarrow \sim(p \vee \sim r)$
T	T	T	F	F	T	T	T	F	F
T	T	F	F	T	T	T	T	F	F
T	F	T	T	F	T	T	T	F	F
T	F	F	T	T	F	F	T	F	T
F	T	T	F	F	T	F	F	T	T
F	T	F	F	T	T	F	T	F	T
F	F	T	T	F	T	F	F	T	T
F	F	F	T	T	F	F	T	F	T

13.

p	q	$\sim q$	$p \leftrightarrow \sim q$	$\sim(p \leftrightarrow \sim q)$
T	T	F	F	T
T	F	T	T	F
F	T	F	T	F
F	F	T	F	T

14.

p	q	$\sim q$	$p \vee q$	$(p \vee q) \leftrightarrow \sim q$
T	T	F	T	F
T	F	T	T	T
F	T	F	T	F
F	F	T	F	F

15.

p	q	r	$\sim p$	$\sim r$	$\sim p \vee q$	$\sim r \leftrightarrow (\sim p \vee q)$
T	T	T	F	F	T	F
T	T	F	F	T	T	T
T	F	T	F	F	F	T
T	F	F	F	T	F	F
F	T	T	T	F	T	F
F	T	F	T	T	T	T
F	F	T	T	F	T	F
F	F	F	T	T	T	T

16.

p	q	r	$\sim q$	$p \leftrightarrow r$	$\sim q \rightarrow (p \leftrightarrow r)$
T	T	T	F	T	T
T	T	F	F	F	T
T	F	T	T	T	T
T	F	F	T	F	F
F	T	T	F	F	T
F	T	F	F	T	T
F	F	T	T	F	F
F	F	F	T	T	T

17.

p	q	$\sim p$	$\sim q$	$\sim p \rightarrow \sim q$	$q \vee p$	$\sim(q \vee p)$	$(\sim p \rightarrow \sim q) \leftrightarrow \sim(q \vee p)$
T	T	F	F	T	T	F	F
T	F	F	T	T	T	F	F
F	T	T	F	F	T	F	T
F	F	T	T	T	F	T	T

18.

p	q	r	$\sim p$	$\sim r$	$q \wedge \sim p$	$\sim r \to p$	$(q \wedge \sim p) \leftrightarrow (\sim r \to p)$
T	T	T	F	F	F	T	F
T	T	F	F	T	F	T	F
T	F	T	F	F	F	T	F
T	F	F	F	T	F	T	F
F	T	T	T	F	T	T	T
F	T	F	T	T	T	F	F
F	F	T	T	F	F	T	F
F	F	F	T	T	F	F	T

19.

p	q	r	$\sim p$	$\sim r$	$q \vee \sim p$	$r \wedge (q \vee \sim p)$	$[r \wedge (q \vee \sim p)] \leftrightarrow \sim r$
T	T	T	F	F	T	T	F
T	T	F	F	T	T	F	F
T	F	T	F	F	F	F	T
T	F	F	F	T	F	F	F
F	T	T	T	F	T	T	F
F	T	F	T	T	T	F	F
F	F	T	T	F	T	T	F
F	F	F	T	T	T	F	F

20.

p	q	r	$\sim r$	$q \vee \sim r$	$\sim(q \vee \sim r)$	$p \wedge r$	$\sim(q \vee \sim r) \leftrightarrow (p \wedge r)$
T	T	T	F	T	F	T	F
T	T	F	T	T	F	F	T
T	F	T	F	F	T	T	T
T	F	F	T	T	F	F	T
F	T	T	F	T	F	F	T
F	T	F	T	T	F	F	T
F	F	T	F	F	T	F	F
F	F	F	T	T	F	F	T

21.

p	q	r	$\sim p$	$q \leftrightarrow r$	$\sim p \leftrightarrow q$	$(q \leftrightarrow r) \rightarrow (\sim p \leftrightarrow q)$	$\sim p \wedge [(q \leftrightarrow r) \rightarrow (\sim p \leftrightarrow q)]$
T	T	T	F	T	F	F	F
T	T	F	F	F	F	T	F
T	F	T	F	F	T	T	F
T	F	F	F	T	T	T	F
F	T	T	T	T	T	T	T
F	T	F	T	F	T	T	T
F	F	T	T	F	F	T	T
F	F	F	T	T	F	F	F

22.

p	q	r	$\sim p$	$q \rightarrow r$	$\sim p \rightarrow q$	$(q \rightarrow r) \leftrightarrow (\sim p \rightarrow q)$
T	T	T	F	T	T	T
T	T	F	F	F	T	F
T	F	T	F	T	T	T
T	F	F	F	T	T	T
F	T	T	T	T	T	T
F	T	F	T	F	T	F
F	F	T	T	T	F	F
F	F	F	T	T	F	F

23.

p	q	r	$\sim p$	$\sim q$	$p \rightarrow \sim q$	$\sim p \wedge r$	$(p \rightarrow \sim q) \leftrightarrow (\sim p \wedge r)$
T	T	T	F	F	F	F	T
T	T	F	F	F	F	F	T
T	F	T	F	T	T	F	F
T	F	F	F	T	T	F	F
F	T	T	T	F	T	T	T
F	T	F	T	F	T	F	F
F	F	T	T	T	T	T	T
F	F	F	T	T	T	F	F

24.

p	q	r	$\sim r$	$p \vee q$	$p \wedge q$	$(p \vee q) \to (p \wedge q)$	$[(p \vee q) \to (p \wedge q)] \leftrightarrow \sim r$
T	T	T	F	T	T	T	F
T	T	F	T	T	T	T	T
T	F	T	F	T	F	F	T
T	F	F	T	T	F	F	F
F	T	T	F	T	F	F	T
F	T	F	T	T	F	F	F
F	F	T	F	F	F	T	F
F	F	F	T	F	F	T	T

25. $(r \vee p) \to (q \wedge \sim r)$

p	q	r	$\sim r$	$r \vee p$	$q \wedge \sim r$	$(r \vee p) \to (q \wedge \sim r)$
T	T	T	F	T	F	F
T	T	F	T	T	T	T
T	F	T	F	T	F	F
T	F	F	T	T	F	F
F	T	T	F	T	F	F
F	T	F	T	F	T	T
F	F	T	F	T	F	F
F	F	F	T	F	F	T

26. $\sim p \to (q \vee \sim r)$

p	q	r	$\sim p$	$\sim r$	$q \vee \sim r$	$\sim p \to (q \vee \sim r)$
T	T	T	F	F	T	T
T	T	F	F	T	T	T
T	F	T	F	F	F	T
T	F	F	F	T	T	T
F	T	T	T	F	T	T
F	T	F	T	T	T	T
F	F	T	T	F	F	F
F	F	F	T	T	T	T

27. $(p \wedge \sim q) \rightarrow (p \rightarrow \sim q)$

p	q	$\sim q$	$p \wedge \sim q$	$p \rightarrow \sim q$	$(p \wedge \sim q) \rightarrow (p \rightarrow \sim q)$
T	T	F	F	F	T
T	F	T	T	T	T
F	T	F	F	T	T
F	F	T	F	T	T

28. $\sim r \leftrightarrow [p \rightarrow (q \vee p)]$

p	q	r	$\sim r$	$q \vee p$	$p \rightarrow (q \vee p)$	$\sim r \leftrightarrow [p \rightarrow (q \vee p)]$
T	T	T	F	T	T	F
T	T	F	T	T	T	T
T	F	T	F	T	T	F
T	F	F	T	T	T	T
F	T	T	F	T	T	F
F	T	F	T	T	T	T
F	F	T	F	F	T	F
F	F	F	T	F	T	T

29. $p \leftrightarrow [(q \vee \sim r) \rightarrow \sim p]$

p	q	r	$\sim r$	$q \vee \sim r$	$\sim p$	$(q \vee \sim r) \rightarrow \sim p$	$p \leftrightarrow [(q \vee \sim r) \rightarrow \sim p]$
T	T	T	F	T	F	F	F
T	T	F	T	T	F	F	F
T	F	T	F	F	F	T	T
T	F	F	T	T	F	F	F
F	T	T	F	T	T	T	F
F	T	F	T	T	T	T	F
F	F	T	F	F	T	T	F
F	F	F	T	T	T	T	F

30. $(q \vee p) \leftrightarrow (r \rightarrow {\sim}q)$

p	q	r	${\sim}q$	$q \vee p$	$r \rightarrow {\sim}q$	$(q \vee p) \leftrightarrow (r \rightarrow {\sim}q)$
T	T	T	F	T	F	F
T	T	F	F	T	T	T
T	F	T	T	T	T	T
T	F	F	T	T	T	T
F	T	T	F	T	F	F
F	T	F	F	T	T	T
F	F	T	T	F	T	F
F	F	F	T	F	T	F

For exercises 31 – 40, answers can vary but one solution is shown.

31. Answers may vary. One way to assign statements p and q is provided.
    a. Let p stand for "A team makes the tournament" and q stand for "A team won the majority of its games."

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

b. False

32. Answers may vary. One way to assign statements p and q is provided.
    a. Let p stand for "Neil likes the Green Bay Packers" and let q stand for "Neil likes the Chicago Bears."

p	q	${\sim}p$	${\sim}p \rightarrow q$
T	T	F	T
T	F	F	T
F	T	T	T
F	F	T	F

b. True

33. Answers may vary. One way to assign statements p and q is provided.
a. Let p stand for "A bill is rejected by the U.S. Senate" and q stand for "A bill becomes a law."

p	q	$\sim q$	$p \to \sim q$
T	T	F	F
T	F	T	T
F	T	F	T
F	F	T	T

b. True

34. Answers may vary. One way to assign statements p and q is provided.
a. Let p stand for "Curtis Granderson was inducted in the Chicago Sports Hall of Fame" and q stand for "Curtis Granderson was drafted by the Detroit Tigers."

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

b. True

35. Answers may vary. One way to assign statements p, q and r is provided.
a. Let p stand for "She attends the concert," q stand for "The show is on a Saturday night," and r stand for "The tickets are expensive."

p	q	r	$\sim r$	$q \wedge \sim r$	$p \leftrightarrow (q \wedge \sim r)$
T	T	T	F	F	F
T	T	F	T	T	T
T	F	T	F	F	F
T	F	F	T	F	F
F	T	T	F	F	T
F	T	F	T	T	F
F	F	T	F	F	T
F	F	F	T	F	T

b. False

36. Answers may vary. One way to assign statements p, q and r is provided.

a. Let p stand for "I go to university," q stand for "I get a scholarship," and r stand for "I get a grant."

$\boldsymbol{p}$	$\boldsymbol{q}$	$\boldsymbol{r}$	$\sim\boldsymbol{p}$	$\boldsymbol{q} \vee \boldsymbol{r}$	$\sim(\boldsymbol{q} \vee \boldsymbol{r})$	$\sim\boldsymbol{p} \leftrightarrow \sim(\boldsymbol{q} \vee \boldsymbol{r})$
T	T	T	F	T	F	T
T	T	F	F	T	F	T
T	F	T	F	T	F	T
T	F	F	F	F	T	F
F	T	T	T	T	F	F
F	T	F	T	T	F	F
F	F	T	T	T	F	F
F	F	F	T	F	T	T

b. True

37. Answers may vary. One way to assign statements p, q and r is provided.

a. Let p stand for "He takes public transportation to work," q stand for "It's raining," and r stand for "His bicycle is in the shop for repairs."

$\boldsymbol{p}$	$\boldsymbol{q}$	$\boldsymbol{r}$	$\sim\boldsymbol{p}$	$\boldsymbol{q} \wedge \boldsymbol{r}$	$\sim(\boldsymbol{q} \wedge \boldsymbol{r})$	$\sim\boldsymbol{p} \rightarrow \sim(\boldsymbol{q} \wedge \boldsymbol{r})$
T	T	T	F	T	F	T
T	T	F	F	F	T	T
T	F	T	F	F	T	T
T	F	F	F	F	T	T
F	T	T	T	T	F	F
F	T	F	T	F	T	T
F	F	T	T	F	T	T
F	F	F	T	F	T	T

b. True

38. Answers may vary. One way to assign statements p, q and r is provided.
a. Let p stand for "The teacher will call home," q stand for "Shayla studies," and r stand for "Shayla earns an A on her first quiz."

$\boldsymbol{p}$	$\boldsymbol{q}$	$\boldsymbol{r}$	$\boldsymbol{q \vee r}$	$\boldsymbol{p \leftrightarrow (q \vee r)}$
T	T	T	T	T
T	T	F	T	T
T	F	T	T	T
T	F	F	F	F
F	T	T	T	F
F	T	F	T	F
F	F	T	T	F
F	F	F	F	T

b. True

39. Answers may vary. One way to assign statements p, q and r is provided.
a. Let p: I will buy the dress. Let q: I will buy the shoes. Let r: I am invited to the party.

$\boldsymbol{p}$	$\boldsymbol{q}$	$\boldsymbol{r}$	$\boldsymbol{p \wedge q}$	$\boldsymbol{(p \wedge q) \leftrightarrow r}$
T	T	T	T	T
T	T	F	T	F
T	F	T	F	F
T	F	F	F	T
F	T	T	F	F
F	T	F	F	T
F	F	T	F	F
F	F	F	F	T

b. True

40. Answers may vary. One way to assign statements p, q and r is provided.
    a. Let p stand for "Carl does not carry an umbrella," q stand for "Carl does not wear a raincoat," and r stand for "It is not raining."

$\boldsymbol{p}$	$\boldsymbol{q}$	$\boldsymbol{r}$	$\boldsymbol{p \wedge q}$	$\boldsymbol{(p \wedge q) \to r}$
T	T	T	T	T
T	T	F	T	F
T	F	T	F	T
T	F	F	F	T
F	T	T	F	T
F	T	F	F	T
F	F	T	F	T
F	F	F	F	T

    b. True

41. True
42. True
43. True
44. False
45. True
46. True
47. a. True
    b. False
48. a. False
    b. True
49. a. True
    b. True
50. a. True
    b. False
51. A conditional statement is false when the antecedent is true and the consequent is false.
52. A biconditional statement is true when both statements being connected have the same truth value.
53. The truth value of the consequent cannot be determined because $F \to T$ is true and $F \to F$ is also true.

Section 2.5 | Equivalent Statements & DeMorgan's Laws

Objectives

- Determine if statements are logically equivalent
- Apply DeMorgan's Laws

Suppose students are told that it is not true that tuition will be raised or student fees will be raised. What are the students really being told? Does this mean that tuition will not be raised **OR** student fees will not be raised? Does it mean that tuition will not be raised **AND** student fees will not be raised? Or does it mean something else? In this section, truth tables will be used to determine if statements that are worded differently express the same meaning. Furthermore, truth tables will be used to develop a basic set of rules, called DeMorgan's Laws, involving statements that are worded differently but have the same meaning.

Equivalent Statements

DEFINITION: Statements are **logically equivalent** if they have the same truth value for every possible case. If statements p and q are logically equivalent, this can be denoted by $p \equiv q$.

Truth tables can be used to determine if statements are logically equivalent. To do this, a truth table is constructed for each statement in question, and then the final truth values of each compound statement are compared for every possible case. If the truth values match for every possible case, then the statements are logically equivalent. If a case exists for which the truth values of each statement do not match, then the statements are not logically equivalent.

➢ **EXAMPLE 2.5.1:** Use truth tables to determine if the statements $(\sim p \wedge q) \leftrightarrow \sim p$ and $(p \vee \sim q) \rightarrow p$ are logically equivalent.

SOLUTION: Both compound statements can be shown in the same truth table.

p	q	$\sim p$	$\sim q$	$\sim p \wedge q$	$(\sim p \wedge q) \leftrightarrow \sim p$	$p \vee \sim q$	$(p \vee \sim q) \rightarrow p$
T	T	F	F	F	T	T	T
T	F	F	T	F	T	T	T
F	T	T	F	T	T	F	T
F	F	T	T	F	F	T	F

The statements are equivalent since the truth values are the same in every possible case. Here the circled columns are compared since they represent the compound statements in question. Symbolically, $(\sim p \wedge q) \leftrightarrow \sim p \ \equiv\ (p \vee \sim q) \rightarrow p$.

➢ **EXAMPLE 2.5.2:** Use truth tables to determine if the statements $(p \wedge \sim r) \rightarrow \sim(p \rightarrow q)$ and $(\sim p \vee \sim q) \rightarrow (q \wedge \sim r)$ are logically equivalent.

SOLUTION: Due to the length of the truth tables, two separate truth tables will be created. In this case, it is important to list the truth values for the p, q, and r columns in the same order so that the final truth values can be compared.

p	q	r	$\sim r$	$p \wedge \sim r$	$p \rightarrow q$	$\sim(p \rightarrow q)$	$(p \wedge \sim r) \rightarrow \sim(p \rightarrow q)$
T	T	T	F	F	T	F	T
T	T	F	T	T	T	F	F
T	F	T	F	F	F	T	T
T	F	F	T	T	F	T	T
F	T	T	F	F	T	F	T
F	T	F	T	F	T	F	T
F	F	T	F	F	T	F	T
F	F	F	T	F	T	F	T

p	q	r	$\sim p$	$\sim q$	$\sim r$	$\sim p \vee \sim q$	$q \wedge \sim r$	$(\sim p \vee \sim q) \to (q \wedge \sim r)$
T	T	T	F	F	F	F	F	T
T	T	F	F	F	T	F	T	T
T	F	T	F	T	F	T	F	F
T	F	F	F	T	T	T	F	F
F	T	T	T	F	F	T	F	F
F	T	F	T	F	T	T	T	T
F	F	T	T	T	F	T	F	F
F	F	F	T	T	T	T	F	F

These statements are **not** equivalent since the truth values are **not** the same in every possible case. Here the circled columns of each table are compared since they represent the compound statements in question. Note that although *some* of the truth values match, for example both compound statements are true when p, q, and r are all true (first entry in each circled column), not *all* truth values match in those columns.

❖ **YOU TRY IT 2.5.A:** Use truth tables to determine if the following statements in each pair are logically equivalent.

a. $(p \to q) \leftrightarrow \sim(\sim p \wedge q)$ and $(q \vee \sim p) \to (\sim p \to q)$
b. $\sim p \to (q \vee \sim r)$ and $(r \wedge \sim q) \to p$

➢ **EXAMPLE 2.5.3:** Use truth tables to determine if the statements "I don't have homework whenever I play basketball after school" and "It is not true that I play basketball after school and have homework" are logically equivalent.

SOLUTION: Assign letters to each of the simple statements given.

$\boldsymbol{p}$: I have homework.
$\boldsymbol{q}$: I play basketball after school.

Recall that "I don't have homework whenever I play basketball after school" can be rewritten as "If I play basketball after school, then I don't have homework." Thus, the first compound statement can be represented symbolically as $q \to \sim p$. The second compound statement can be represented symbolically as $\sim(q \wedge p)$.

p	q	$\sim p$	$q \rightarrow \sim p$	$q \wedge p$	$\sim(q \wedge p)$
T	T	F	F	T	F
T	F	F	T	F	T
F	T	T	T	F	T
F	F	T	T	F	T

The statements are equivalent since the truth values are the same in every possible case. Here the circled columns are compared since they represent the compound statements in question.

❖ **YOU TRY IT 2.5.B:** Use truth tables to determine if the statements "If it is 8:05, then I am late to class" and "It is not 8:05 am or I am late to class" are logically equivalent.

➢ **EXAMPLE 2.5.4:** Use truth tables to determine if any of the following statements are equivalent.

STMT 1: I don't complete the marathon training, or it is raining and I go for a run.
STMT 2: If it is not raining or I don't go for a run, then I don't complete the marathon training.
STMT 3: I go for a run and I complete the marathon training, if and only if it is not raining.

SOLUTION: Assign letters to each of the simple statements given.

p: It is raining.
q: I go for a run.
r: I complete the marathon training.

The three compound statements can be represented symbolically as follows. Note that parentheses are used as grouping symbols in accordance with the placement of the comma in each statement.

STMT 1: $\sim r \vee (p \wedge q)$
STMT 2: $(\sim p \vee \sim q) \rightarrow \sim r$
STMT 3: $(q \wedge r) \leftrightarrow \sim p$

p	q	r	$\sim p$	$\sim q$	$\sim r$	$p \wedge q$	STMT 1	$\sim p \vee \sim q$	STMT 2	$q \wedge r$	STMT 3
T	T	T	F	F	F	T	T	F	T	T	F
T	T	F	F	F	T	T	T	F	T	F	T
T	F	T	F	T	F	F	F	T	F	F	T
T	F	F	F	T	T	F	T	T	T	F	T
F	T	T	T	F	F	F	F	T	F	T	T
F	T	F	T	F	T	F	T	T	T	F	F
F	F	T	T	T	F	F	F	T	F	F	F
F	F	F	T	T	T	F	T	T	T	F	F

STMT 1: $\sim r \vee (p \wedge q)$
STMT 2: $(\sim p \vee \sim q) \rightarrow \sim r$
STMT 3: $(q \wedge r) \leftrightarrow \sim p$

By comparing the circled truth values for each compound statement it can be determined that Statements 1 and 2 are equivalent, but that Statement 3 is **not** equivalent to Statements 1 or 2. Thus, the statements "I don't complete the marathon training, or it is raining and I go for a run" and "If it is not raining or I don't go for a run, then I don't complete the marathon training" are logically equivalent.

❖ **YOU TRY IT 2.5.C:** Use truth tables to determine if any of the following statements are equivalent.

STMT 1: If she visits Europe, then she will see the Eiffel Tower and will not see the giant redwoods.
STMT 2: She will not visit Europe and will not see the Eiffel Tower, or she will see the giant redwoods.
STMT 3: If she will not see the Eiffel Tower or will see the giant redwoods, then she will not visit Europe.

DeMorgan's Laws

Suppose students are told that it is not true that tuition will be raised or student fees will be raised. What are students really being told? Are they being told that tuition will not be raised *or* student fees will not be raised? Or are they being told that tuition will not be raised *and* student fees will not be raised?

Assign letters to each of the simple statements given.

$\boldsymbol{p}$: Tuition will be raised.
$\boldsymbol{q}$: Student fees will be raised.

The original statement "It is not true that tuition will be raised or student fees will be raised" can be represented symbolically as $\sim(p \vee q)$.

The statement "Tuition will not be raised or student fees will not be raised" can be represented symbolically as $\sim p \vee \sim q$. Furthermore, the statement "Tuition will not be raised and student fees will not be raised" can be represented symbolically as $\sim p \wedge \sim q$. Truth tables can be used to determine if either of these statements is logically equivalent to the original statement above.

$\boldsymbol{p}$	$\boldsymbol{q}$	$\boldsymbol{\sim p}$	$\boldsymbol{\sim q}$	$\boldsymbol{p \vee q}$	$\boldsymbol{\sim(p \vee q)}$	$\boldsymbol{\sim p \vee \sim q}$	$\boldsymbol{\sim p \wedge \sim q}$
T	T	F	F	T	F	F	F
T	F	F	T	T	F	T	F
F	T	T	F	T	F	T	F
F	F	T	T	F	T	T	T

By comparing the truth values in the circled columns it can be determined that the statements $\sim(p \vee q)$ and $\sim p \wedge \sim q$ are logically equivalent. However, the statement $\sim p \vee \sim q$ is **not** logically equivalent to $\sim(p \vee q)$ or $\sim p \wedge \sim q$.

Thus, when students are told that it is not the case that tuition will be raised *or* student fees will be raised, they are being told that tuition will not be raised *and* student fees will not be raised.

This example demonstrates one part of a set of common rules in logic called **DeMorgan's Laws**, named after British mathematician and logician Augustus De Morgan (1806-1871).

Recall from set theory in Section 1.4, DeMorgan's Laws states that for any two sets A and B, $(A \cup B)' = A' \cap B'$ and $(A \cap B)' = A' \cup B'$. Similar rules also apply when working with logical statements! Set theory and logic are connected in many ways. Here, the idea of a complement in set theory is like the idea of a negation in logic. Further, intersections and unions in set theory are like conjunctions and disjunctions in logic, respectively.

DeMorgan's Laws

For any two statements p and q,

$$\sim(p \vee q) \text{ is equivalent to } \sim p \wedge \sim q$$
$$\sim(p \wedge q) \text{ is equivalent to } \sim p \vee \sim q$$

NOTE: When using DeMorgan's Laws to write a logically equivalent statement, the connective changes (from conjunction to disjunction or vice versa). It is a common mistake to use the same connective, but in doing so a logically equivalent statement is not produced. As shown in the truth table on the previous page, $\sim(p \vee q)$ is not equivalent to $\sim p \vee \sim q$. It can also be shown that $\sim(p \wedge q)$ is not equivalent to $\sim p \wedge \sim q$.

- **EXAMPLE 2.5.5:** Use DeMorgan's Laws to write a logically equivalent statement in symbolic form.

  d. $\sim(s \wedge \sim t)$
  e. $\sim[p \wedge (\sim q \vee r)]$

  SOLUTION:

  a. According to DeMorgan's Laws, $\sim(s \wedge \sim t)$ is equivalent to $\sim s \vee \sim(\sim t)$. Each simple statement is negated, and the conjunction becomes a disjunction. Since $\sim(\sim t)$ is the same as t, this can be written as $\sim s \vee t$.

  b. According to DeMorgan's Laws, $\sim[p \wedge (\sim q \vee r)]$ is equivalent to $\sim p \vee \sim(\sim q \vee r)$. Each statement is negated, and the conjunction becomes a disjunction. DeMorgan's Laws can also be applied to rewrite $\sim(\sim q \vee r)$ as $\sim(\sim q) \wedge \sim r$. Here, each simple statement is negated, and the disjunction becomes a conjunction. Since $\sim(\sim q)$ is the same as q, this can be written as $q \wedge \sim r$. Thus, $\sim[p \wedge (\sim q \vee r)]$ can be written as $\sim p \vee (q \wedge \sim r)$.

➢ **EXAMPLE 2.5.6:** Use DeMorgan's Laws to write a logically equivalent statement in words.

a. It is not true that Florida is an island or Chicago is on a coast.
b. The first calculation of pi was not discovered by DeMorgan or the first calculation of pi was not discovered by Archimedes.

SOLUTION:

a. This can be written symbolically as $\sim(p \vee q)$, where p represents "Florida is an island" and q represents "Chicago is on a coast." According to DeMorgan's Laws, an equivalent statement in symbolic form is $\sim p \wedge \sim q$. In words this is written "Florida is not an island and Chicago is not on a coast."

b. This can be written symbolically as $\sim p \vee \sim q$, where p represents "The first calculation of pi was discovered by DeMorgan" and q represents "The first calculation of pi was discovered by Archimedes." According to DeMorgan's Laws, an equivalent statement in symbolic form is $\sim(p \wedge q)$. In words this is written "It is not true that the first calculation of pi was discovered by DeMorgan and Archimedes."

❖ **YOU TRY IT 2.5.D:** Use DeMorgan's Laws to write a logically equivalent statement in words.

a. It is not true that the Great Wall of China is the longest man-made structure in the world and the Empire State Building is the tallest building in New York City.
b. Manchester City F.C. is not a basketball team and the Chicago Bulls are not a soccer club.

Quick Review

➢ Statements are **logically equivalent** if they have the same truth value for every possible case.
➢ According to **DeMorgan's Laws**, for any statements p and q:

$$\sim(p \vee q) \text{ is equivalent to } \sim p \wedge \sim q$$
$$\sim(p \wedge q) \text{ is equivalent to } \sim p \vee \sim q$$

YOU TRY IT 2.5.A SOLUTION:

a. The statements are not equivalent, because the truth values are not the same in every possible case.

p	q	$\sim p$	$p \to q$	$\sim p \wedge q$	$\sim(\sim p \wedge q)$	$(p \to q) \leftrightarrow \sim(\sim p \wedge q)$
T	T	F	T	F	T	T
T	F	F	F	F	T	F
F	T	T	T	T	F	F
F	F	T	T	F	T	T

p	q	$\sim p$	$q \vee \sim p$	$\sim p \to q$	$(q \vee \sim p) \to (\sim p \to q)$
T	T	F	T	T	T
T	F	F	F	T	T
F	T	T	T	T	T
F	F	T	T	F	F

b. The statements are equivalent because the truth values are the same in every possible case.

p	q	r	$\sim p$	$\sim q$	$\sim r$	$q \vee \sim r$	$\sim p \to (q \vee \sim r)$	$r \wedge \sim q$	$(r \wedge \sim q) \to p$
T	T	T	F	F	F	T	T	F	T
T	T	F	F	F	T	T	T	F	T
T	F	T	F	T	F	F	T	T	T
T	F	F	F	T	T	T	T	F	T
F	T	T	T	F	F	T	T	F	T
F	T	F	T	F	T	T	T	F	T
F	F	T	T	T	F	F	F	T	F
F	F	F	T	T	T	T	T	F	T

YOU TRY IT 2.5.B SOLUTION: Assign letters to each of the simple statements given.

p: It is 8:05 am.
q: I am late to class.

The first compound statement can be represented symbolically as $p \rightarrow q$, and the second compound statement can be represented symbolically as $\sim p \vee q$. The statements are equivalent since the truth values are the same in every possible case.

p	q	$\sim p$	$p \rightarrow q$	$\sim p \vee q$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

YOU TRY IT 2.5.C SOLUTION: Assign letters to each of the simple statements given.

p: She visits Europe.
q: She sees the Eiffel tower.
r: She sees the giant redwoods.

The three compound statements can be represented symbolically as follows.

STMT 1: $p \rightarrow (q \wedge \sim r)$
STMT 2: $(\sim p \wedge \sim q) \vee r$
STMT 3: $(\sim q \vee r) \rightarrow \sim p$

Comparing the circled truth values for each compound statement shows that the statements $p \rightarrow (q \wedge \sim r)$and $(\sim q \vee r) \rightarrow \sim p$ are equivalent, but that the statement $(\sim p \wedge \sim q) \vee r$ is **not** equivalent to the other two statements. Thus, the statements "If she visits Europe, then she will see the Eiffel Tower and will not see the giant redwoods" and "If she will not see the Eiffel Tower or will see the giant redwoods, then she will not visit Europe" are logically equivalent.

p	q	r	$\sim p$	$\sim q$	$\sim r$	$q \wedge \sim r$	STMT 1	$\sim p \wedge \sim q$	STMT 2	$\sim q \vee r$	STMT 3
T	T	T	F	F	F	F	F	F	T	T	F
T	T	F	F	F	T	T	T	F	F	F	T
T	F	T	F	T	F	F	F	F	T	T	F
T	F	F	F	T	T	F	F	F	F	T	F
F	T	T	T	F	F	F	T	F	T	T	T
F	T	F	T	F	T	T	T	F	F	F	T
F	F	T	T	T	F	F	T	T	T	T	T
F	F	F	T	T	T	F	T	T	T	T	T

YOU TRY IT 2.5.D SOLUTION:

c. This can be written symbolically as $\sim(p \wedge q)$, where p represents "The Great Wall of China is the longest man-made structure in the world" and q represents "The Empire State Building is the tallest building in New York City." According to DeMorgan's Laws, an equivalent statement in symbolic form is $\sim p \vee \sim q$. In words this is written "The Great Wall of China is not the longest man-made structure in the world or the Empire State Building is not the tallest building in New York City."

d. This can be written symbolically as $\sim p \wedge \sim q$, where p represents "Manchester City F.C. is a basketball team" and q represents "The Chicago Bulls are a soccer club." According to DeMorgan's Laws, an equivalent statement in symbolic form is $\sim(p \vee q)$. In words this is written "It is not true that Manchester City F.C. is a basketball team or the Chicago Bulls are a soccer club."

Section 2.5 | Exercises

In Exercises 1 – 8, use truth tables to determine whether the statements are equivalent.

1. $\sim p \rightarrow q,\ \sim q \rightarrow p$
2. $\sim p \rightarrow q,\ \ p \rightarrow \sim q$
3. $(p \wedge q) \leftrightarrow \sim p,\ \ p \rightarrow \sim(p \vee q)$
4. $p \rightarrow (q \wedge \sim p),\ q \rightarrow (p \vee \sim q)$
5. $(\sim q \rightarrow p) \wedge (p \rightarrow \sim q),\ \ \sim q \leftrightarrow p$
6. $(\sim q \leftrightarrow p) \wedge q,\ \sim[q \rightarrow (p \vee \sim q)]$
7. $(p \vee q) \rightarrow (p \wedge q),\ (p \wedge q) \rightarrow (p \vee q)$
8. $\sim p \rightarrow (\sim p \wedge q),\ (\sim p \wedge q) \vee p$

In Exercises 9 – 16, use truth tables to determine whether the statements are equivalent.

9. $(p \vee r) \wedge q,\ \ p \vee (r \wedge q)$
10. $r \leftrightarrow (p \vee q \vee r),\ (\sim r \vee q) \leftrightarrow \sim(p \wedge q)$
11. $r \rightarrow \sim(p \vee q),\ (\sim p \wedge \sim q) \rightarrow r$
12. $r \rightarrow (\sim p \wedge \sim q),\ (p \vee q) \rightarrow \sim r$
13. $(p \rightarrow q) \vee (p \rightarrow r),\ p \rightarrow (q \vee r)$
14. $[(\sim p \vee r) \rightarrow (\sim q \wedge r)],\ \ \sim p \leftrightarrow (\sim q \wedge r)$
15. $\sim(\sim p \rightarrow q) \vee r,\ (\sim p \wedge \sim q) \vee r$
16. $(\sim p \wedge \sim q) \rightarrow \sim r,\ r \rightarrow [\sim(p \wedge q)]$

In Exercises 17 – 20, use truth tables to determine whether the statements are equivalent.

17. It is not the case that Haley likes to watch movies or go out to eat.
    Haley does not like to watch movies or Haley does not like to go out to eat.
18. It is not true that if I ditch class, then I will not pass the course.
    I ditch class and I pass the course.
19. If it is Saturday, then I will order pizza and play video games.
    It is not Saturday, or I will order pizza and play video games.
20. I will do well on my test if and only if I slept well the night before and ate a good breakfast.
    I don't do well on my test, or I slept well the night before and I ate a good breakfast.

In Exercises 21 – 24, use truth tables to determine which statements are equivalent, if any.

21. (a) If it is St. Patrick's Day, then Sharon wears green.
    (b) Sharon wears green or it is not St. Patrick's Day.
    (c) If it is not St. Patrick's Day, then Sharon does not wear green.
22. (a) It is not true that I go to the coffee shop or I drink coffee.
    (b) It is not true that if I go to the coffee shop, then I drink coffee.
    (c) I go to the coffee shop and I do not drink coffee.
23. (a) If a child passes a swim test, then they can ride the large slide and they do not require parent supervision in the water.
    (b) A child cannot ride the large slide, if and only if they don't pass the swim test and require parent supervision in the water.
    (c) If a child can't ride the large slide or requires parent supervision in the water, then they did not pass the swim test.
24. (a) A student can return to campus five days after a positive Covid test if they are asymptomatic or have been without a temperature of more than 100.3 degrees for 24 hours. (Hint: rewrite this statement in ***if … then*** form.)
    (b) If a student can return to campus five days after a positive Covid test, then they are asymptomatic or have been without a temperature of more than 100.3 degrees for 24 hours.
    (c) If a student cannot return to campus five days after a positive Covid test, then they are not asymptomatic and have had a temperature of more than 100.3 degrees in the past 24 hours.

In Exercises 25 – 28, use DeMorgan's Laws to write an equivalent statement in symbolic form.

25. $\sim(\sim p \vee q)$
26. $\sim[(p \vee q) \wedge (\sim p \vee \sim q)]$
27. $\sim[(p \wedge \sim q) \vee r]$
28. $\sim[\sim(p \wedge \sim q) \vee r]$

In Exercises 29 – 34, use DeMorgan's Laws to write a statement that is equivalent to the given statement.

29. It is not true that Beijing is in France or Paris is in China.
30. Christian Bale did not play Batman or Robert Downey Jr. did not play Spiderman.
31. It is not true that I did not go to the concert and I missed the opening act.
32. Pigs cannot fly and I am not lying.
33. It is not the case that *The Ghostbusters: Afterlife* didn't contain music like the original film or didn't contain some of the actors from the original film.

34. It is not the case that calculus isn't a form of mathematics and chemistry isn't a form of history.

Concept Review

35. What are logically equivalent statements?
36. How can it be shown that statements are logically equivalent?
37. Use words to describe DeMorgan's Laws.

Section 2.5 | Exercise Solutions

1. Equivalent
2. Not equivalent
3. Not equivalent
4. Not equivalent
5. Equivalent
6. Equivalent
7. Not equivalent
8. Equivalent
9. Not equivalent
10. Not equivalent
11. Not equivalent
12. Equivalent
13. Equivalent
14. Not equivalent
15. Equivalent
16. Not equivalent
17. Not equivalent
18. Equivalent
19. Equivalent
20. Not equivalent
21. Statements (a) and (b) are equivalent.
22. Statements (b) and (c) are equivalent.
23. Statements (a) and (c) are equivalent.
24. Statements (a) and (c) are equivalent.
25. $p \wedge \sim q$
26. $(\sim p \wedge \sim q) \vee (p \wedge q)$
27. $\sim(p \wedge \sim q) \wedge \sim r$ or $(\sim p \vee q) \wedge \sim r$
28. $(p \wedge \sim q) \wedge \sim r$
29. Beijing is not in France and Paris is not in China.
30. It is not true that Christian Bale played Batman and Robert Downey Jr. played Spiderman.
31. I went to the concert or I did not miss the opening act.
32. It is not the case that pigs can fly or I am lying.
33. *The Ghostbusters: Afterlife* did contain music like the original film and did contain some of the actors from the original film.
34. Calculus is a form of mathematics or chemistry is a form of history.
35. Statements are logically equivalent if they have the same truth value for every possible case.
36. Truth tables can be used to determine if statements are equivalent. The final truth values of each statement are compared. If the truth values match for every possible case, then the statements are equivalent.
37. When negating a conjunction, negate each statement on either side of the conjunction and switch the conjunction to disjunction. Similarly, when negating a disjunction, negate each statement on either side of the disjunction and switch the disjunction to conjunction.

Section 2.6 | Conditionals: Equivalency & Negation

Objectives

- Define converse, inverse, and contrapositive of a conditional statement
- Define the negation of a conditional statement
- State the equivalence of conditional, converse, inverse, and contrapositive statements
- Use truth tables to find the equivalency and negation of conditional statements

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In this section, conditional statements will be discussed in more depth. Specifically, truth values of complex statements where simple statements are rearranged will be defined. Furthermore, the negation of a conditional statement will be defined.

The truth value of conditional statements is quite an interesting topic. Recall from Section 2.4 the truth table of a conditional statement "If p, then q."

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

Consider the conditional statement: *"If you drink lots of caffeine, then you'll be more alert."*

p**:** You drink lots of caffeine.
q**:** You'll be more alert.

We can consider the variations of that statement:

"If you're more alert, then you've drunk lots of caffeine." This doesn't necessarily mean the same thing as the original conditional statement. There are other ways to be more alert besides lots of caffeine (nerves or stress, for instance).

"If you don't drink lots of caffeine, then you won't be more alert." This also doesn't mean the same as the original conditional statement. With the original conditional statement, nothing was said about what would happen if you didn't drink lots of caffeine.

"If you aren't more alert, then you haven't drunk lots of caffeine." This actually has the same meaning as the original statement.

Mathematicians ask questions such as: what are the truth values of these statements? Are any statements equivalent and, if so, which ones? What does it mean to negate any one of these statements? Those questions will be explored in this section.

Equivalency of the Conditional Statement

DEFINITION: Given a conditional statement "If p, then q," the **converse** is "If q, then p." Symbolically, the converse of $p \rightarrow q$ is $q \rightarrow p$. The following is the truth table for the converse.

Truth Table for the Converse of the Conditional Statement

p	q	$q \rightarrow p$
T	T	T
T	F	T
F	T	F
F	F	T

Referring back to the opening example, the converse of the conditional statement is "If you're more alert, then you've drunk lots of caffeine."

DEFINITION: Given a conditional statement "If p, then q," the **inverse** is "If not p, then not q." Symbolically, the inverse of $p \rightarrow q$ is $\sim p \rightarrow \sim q$. The following is the truth table for the inverse.

Truth Table for the Inverse of the Conditional Statement

p	q	$\sim p$	$\sim q$	$\sim p \rightarrow \sim q$
T	T	F	F	T
T	F	F	T	T
F	T	T	F	F
F	F	T	T	T

Referring back to the opening example, the inverse of the conditional statement is "If you don't drink lots of caffeine, then you won't be more alert."

DEFINITION: Given a conditional statement "If p, then q," the **contrapositive** is "If not q, then not p." Symbolically, the contrapositive of $p \to q$ is $\sim q \to \sim p$. The following is the truth table for the contrapositive.

Truth Table for the Contrapositive of the Conditional Statement

$\boldsymbol{p}$	$\boldsymbol{q}$	$\boldsymbol{\sim q}$	$\boldsymbol{\sim p}$	$\boldsymbol{\sim q \to \sim p}$
T	T	F	F	T
T	F	T	F	F
F	T	F	T	T
F	F	T	T	T

Referring back to the opening example, the contrapositive of the conditional statement is "If you aren't more alert, then you haven't drunk lots of caffeine."

The following truth table shows all four statements: (original) conditional $p \to q$, converse $q \to p$, inverse $\sim p \to \sim q$, and contrapositive $\sim q \to \sim p$.

$\boldsymbol{p}$	$\boldsymbol{q}$	$\boldsymbol{\sim p}$	$\boldsymbol{\sim q}$	**Conditional** $\boldsymbol{p \to q}$	**Converse** $\boldsymbol{q \to p}$	**Inverse** $\boldsymbol{\sim p \to \sim q}$	**Contrapositive** $\boldsymbol{\sim q \to \sim p}$
T	T	F	F	T	T	T	T
T	F	F	T	F	T	T	F
F	T	T	F	T	F	F	T
F	F	T	T	T	T	T	T

QUESTION: Are any of the ***if … then*** statements equivalent?

ANSWER: The conditional statement $p \to q$ is logically equivalent to its contrapositive $\sim q \to \sim p$. Similarly, the inverse statement $q \to p$ is logically equivalent to the converse $\sim p \to \sim q$.

$\boldsymbol{p}$	$\boldsymbol{q}$	$\boldsymbol{\sim p}$	$\boldsymbol{\sim q}$	**Conditional** $\boldsymbol{p \to q}$	**Converse** $\boldsymbol{q \to p}$	**Inverse** $\boldsymbol{\sim p \to \sim q}$	**Contrapositive** $\boldsymbol{\sim q \to \sim p}$
T	T	F	F	T	T	T	T
T	F	F	T	F	T	T	F
F	T	T	F	T	F	F	T
F	F	T	T	T	T	T	T

NOTE: The original conditional statement is NOT equivalent to the converse or inverse statement.

Referring back to the opening example this means the original conditional statement "If you drink lots of caffeine, then you'll be more alert" is logically equivalent to the contrapositive "If you aren't more alert, then you haven't drunk lots of caffeine." Similarly, the converse statement "If you're more alert, then you've drunk lots of caffeine" is logically equivalent to the inverse statement "If you don't drink lots of caffeine, then you won't be more alert."

Equivalences of Conditional Statements

For any two statements p and q, where "If p, then q":

The conditional statement is logically equivalent to its contrapositive.

$$p \to q \equiv {\sim}q \to {\sim}p$$

The converse of the conditional statement is logically equivalent to the inverse of the conditional statement.

$$q \to p \equiv {\sim}p \to {\sim}q$$

- **EXAMPLE 2.6.1:** Write the converse, inverse, and contrapositive in symbols and in words given the simple statements and the conditional "If I live in Chicago, then I live in Illinois."

  $\boldsymbol{p}$**:** I live in Chicago.
  $\boldsymbol{q}$**:** I live in Illinois.

  SOLUTION:
  Converse: $q \to p$ is written as "If I live in Illinois, then I live in Chicago."
  Inverse: ${\sim}p \to {\sim}q$ is written as "If I don't live in Chicago, then I don't live in Illinois."
  Contrapositive: ${\sim}q \to {\sim}p$ is written as "If I don't live in Illinois, then I don't live in Chicago."

- **EXAMPLE 2.6.2:** Write the converse, inverse, and contrapositive in symbols and in words given the simple statements and the conditional "If I go to Miami, then I do not go to Key West."

  $\boldsymbol{s}$**:** I go to Miami.
  $\boldsymbol{t}$**:** I go to Key West.

SOLUTION:
Converse: $\sim t \rightarrow s$ is written as "If I do not go to Key West, then I go to Miami."
Inverse: $\sim s \rightarrow t$ is written as "If I don't go to Miami, then I go to Key West."
Contrapositive: $t \rightarrow \sim s$ is written as "If I go to Key West, then I don't go to Miami."

❖ **YOU TRY IT 2.6.A:** Write the converse, inverse, and contrapositive in symbols and in words given the simple statements and the conditional "If it is snowing, then it is less than 32 degrees outside."

$\boldsymbol{p}$**:** It is snowing.
$\boldsymbol{q}$**:** It is less than 32 degrees outside.

➢ **EXAMPLE 2.6.3:** Write the converse, inverse, and contrapositive of the conditional statement: *"If I do not eat healthy foods, then I gain weight."*

SOLUTION:
Converse: "If I gain weight, then I do not eat healthy foods."
Inverse: "If I eat healthy foods, then I do not gain weight."
Contrapositive: "If I do not gain weight, then I eat healthy foods."

➢ **EXAMPLE 2.6.4:** Create a truth table for the converse, inverse, and contrapositive of the conditional statement: *"If I do not eat healthy foods, then I gain weight."*

SOLUTION:

$\boldsymbol{p}$: I eat healthy foods.
$\boldsymbol{q}$: I gain weight.

Conditional: $\sim p \rightarrow q$
Converse: $q \rightarrow \sim p$
Inverse: $p \rightarrow \sim q$
Contrapositive: $\sim q \rightarrow p$

NOTE: The simple statement selected for p is phrased without using the word ***not***. If a simple statement for p includes the word not, then some steps of the truth table will be different; however, the final truth values will be the same. The conditional and contrapositive will remain logically equivalent. The converse and inverse will also remain logically equivalent.

p	q	$\sim p$	$\sim q$	Conditional $\sim p \rightarrow q$	Converse $q \rightarrow \sim p$	Inverse $p \rightarrow \sim q$	Contrapositive $\sim q \rightarrow p$
T	T	F	F	T	F	F	T
T	F	F	T	T	T	T	T
F	T	T	F	T	T	T	T
F	F	T	T	F	T	T	F

❖ **YOU TRY IT 2.6.B:** Write the converse, inverse, and contrapositive of the conditional statement: *"If I do not stay out late, then I am not tired the next day."* Then create a truth table for each statement: conditional, converse, inverse, and contrapositive.

➢ **EXAMPLE 2.6.5:** Given the complex statement $(p \vee q) \rightarrow r$, symbolically write the converse, inverse, and contrapositive.

SOLUTION:
Converse: $r \rightarrow (p \vee q)$
Inverse: $\sim(p \vee q) \rightarrow \sim r$ which can be written as $(\sim p \wedge \sim q) \rightarrow \sim r$ after applying DeMorgan's Law
Contrapositive: $\sim r \rightarrow \sim(p \vee q)$ which can be written as $\sim r \rightarrow (\sim p \wedge \sim q)$ after applying DeMorgan's Law

❖ **YOU TRY IT 2.6.C:** Given the complex statement $(p \wedge q) \rightarrow r$, symbolically write the converse, inverse, and contrapositive.

Negation of the Conditional Statement

Recall that given the conditional statement "If p, then q," the only case where the conditional fails is when the antecedent is true (statement p is true) and the consequent is false (q is false). Therefore, to negate a conditional statement, use the information p and $\sim q$ are both confirmed which is written symbolically as $p \wedge \sim q$.

Consider the conditional statement: *"If Billy cleans his room, then he can have ice cream."*

$\boldsymbol{p}$**:** Billy cleans his room.
$\boldsymbol{q}$**:** Billy gets ice cream.

The negation of the statement is Billy cleaned his room **AND** he did not get ice cream. Simply put, the original conditional statement would be false if Billy cleaned his room and he did not get ice cream. So, again, the negation of "*If* p*, then* q" is "p ***and*** $\sim q$."

DEFINITION: Given a conditional statement "If p, then q," the **negation** is p and $\sim q$. So, symbolically, the negation of a conditional statement, $\sim(p \rightarrow q)$, is $p \wedge \sim q$. The following is the truth table for the negation of a conditional where truth table that shows that $\sim(p \rightarrow q)$ is logically equivalent to $p \wedge \sim q$.

$\boldsymbol{p}$	$\boldsymbol{q}$	$\boldsymbol{\sim q}$	$\boldsymbol{p \rightarrow q}$	$\boldsymbol{\sim(p \rightarrow q)}$	$\boldsymbol{p \wedge \sim q}$
T	T	F	T	F	F
T	F	T	F	T	T
F	T	F	T	F	F
F	F	T	T	F	F

Negation of the Conditional Statement

For any two statements p and q, the negation of "If p, then q," is "p **and** $\sim q$."

$$\sim(p \rightarrow q) \equiv p \wedge \sim q$$

- **EXAMPLE 2.6.6:** Write the negation of the conditional statement: "If I buy the car, then I don't have to take the train."

  SOLUTION: "I buy the car and I have to take the train."

- **EXAMPLE 2.6.7:** Given the statement "If I buy the car, then I do not have to take the train" let

  $\boldsymbol{p}$**:** I buy the car.
  $\boldsymbol{q}$**:** I have to take the train.

  Create a truth table for following:

  a. The conditional statement: $p \rightarrow \sim q$
  b. The negation of the conditional statement: $\sim(p \rightarrow \sim q)$
  c. The logically equivalent negation of the conditional statement: $p \wedge q$

SOLUTION:

p	q	$\sim q$	$p \to \sim q$	$\sim(p \to \sim q)$	$p \wedge q$
T	T	F	F	T	T
T	F	T	T	F	F
F	T	F	T	F	F
F	F	T	T	F	F

NOTE: The negation of the conditional statement $\sim(p \to \sim q)$ is logically equivalent to the conjunction $p \wedge q$.

➢ **EXAMPLE 2.6.8:** Write the negation of the conditional statement: "If I finish my homework and clean my dorm room, then I can meet you for coffee."

SOLUTION: I finish my homework and I clean my dorm room, and I can't meet you for coffee.

➢ **EXAMPLE 2.6.9:** Given the statement "If I finish my homework and clean my dorm room, then I can meet you for coffee" create a truth table for following:

a. The conditional statement
b. The negation of the conditional statement
c. The logically equivalent negation of the conditional statement

SOLUTION:

$\boldsymbol{p}$**:** I finish my homework.
$\boldsymbol{q}$**:** I clean my dorm room.
$\boldsymbol{r}$**:** I meet you for coffee.

The original conditional statement can be represented symbolically as $(p \wedge q) \to r$. The negation of $(p \wedge q) \to r$ can be written as $\sim[(p \wedge q) \to r]$ which is logically equivalent to $(p \wedge q) \wedge \sim r$.

p	q	r	$\sim r$	$(p \wedge q)$	$(p \wedge q) \rightarrow r$	$\sim[(p \wedge q) \rightarrow r]$	$(p \wedge q) \wedge \sim r$
T	T	T	F	T	T	F	F
T	T	F	T	T	F	T	T
T	F	T	F	F	T	F	F
T	F	F	T	F	T	F	F
F	T	T	F	F	T	F	F
F	T	F	T	F	T	F	F
F	F	T	F	F	T	F	F
F	F	F	T	F	T	F	F

❖ **YOU TRY IT 2.6.D:** Given the statement "If I graduate college and attain a nice job, then I will not be as stressed out" write the negation of the conditional statement. Then create a truth table for the following:

a. The conditional statement
b. The negation of the conditional statement
c. The logically equivalent negation of the conditional statement

Quick Review

- Conditional: $p \rightarrow q$
- Converse: $q \rightarrow p$
- Inverse: $\sim p \rightarrow \sim q$
- Contrapositive: $\sim q \rightarrow \sim p$
- The conditional statement is logically equivalent to the contrapositive of the conditional statement: $p \rightarrow q \equiv \sim q \rightarrow \sim p$
- The converse of the conditional statement is logically equivalent to the inverse of the conditional statement: $q \rightarrow p \equiv \sim p \rightarrow \sim q$
- Negation of the conditional statement: $\sim(p \rightarrow q)$ is equivalent to $p \wedge \sim q$

YOU TRY IT 2.6.A SOLUTION:

Converse: $q \rightarrow p$ is written as "If it is less than 32 degrees outside, then it is snowing."
Inverse: $\sim p \rightarrow \sim q$ is written as "If it is not snowing, then it is not less than 32 degrees outside."
Contrapositive: $\sim q \rightarrow \sim p$ is written as "If it is not less than 32 degrees outside, then it is not snowing."

YOU TRY IT 2.6.B SOLUTION:

Let $\boldsymbol{p}$: I stay out late. $\boldsymbol{q}$: I am tired the next day.
Converse: "If I am not tired the next day, then I did not stay out late."
Inverse: "If I stay out late, then I am tired the next day."
Contrapositive: "If I am tired the next day, then I stayed out late."

$\boldsymbol{p}$	$\boldsymbol{q}$	$\boldsymbol{\sim p}$	$\boldsymbol{\sim q}$	**Conditional** $\boldsymbol{\sim p \rightarrow \sim q}$	**Converse** $\boldsymbol{\sim q \rightarrow \sim p}$	**Inverse** $\boldsymbol{p \rightarrow q}$	**Contrapositive** $\boldsymbol{q \rightarrow p}$
T	T	F	F	T	T	T	T
T	F	F	T	T	F	F	T
F	T	T	F	F	T	T	F
F	F	T	T	T	T	T	T

YOU TRY IT 2.6.C SOLUTION:

Converse: $r \rightarrow (p \wedge q)$
Inverse: $\sim(p \wedge q) \rightarrow \sim r$ which can be written as $(\sim p \vee \sim q) \rightarrow \sim r$ after applying DeMorgan's Law.
Contrapositive: $\sim r \rightarrow \sim(p \wedge q)$ which can be written as $\sim r \rightarrow (\sim p \vee \sim q)$ after applying DeMorgan's Law.

YOU TRY IT 2.6.D SOLUTION:

Negation: "I graduate college and attain a nice job, and I am stressed out."

$\boldsymbol{p}$: I graduate college.
$\boldsymbol{q}$: I attain a nice job.
$\boldsymbol{r}$: I am stressed out.

p	q	r	$\sim r$	$(p \wedge q)$	$(p \wedge q) \rightarrow \sim r$	$\sim[(p \wedge q) \rightarrow \sim r]$	$(p \wedge q) \wedge r$
T	T	T	F	T	F	T	T
T	T	F	T	T	T	F	F
T	F	T	F	F	T	F	F
T	F	F	T	F	T	F	F
F	T	T	F	F	T	F	F
F	T	F	T	F	T	F	F
F	F	T	F	F	T	F	F
F	F	F	T	F	T	F	F

Section 2.6 | Exercises

For exercises 1 – 8, given the simple statements and the provided conditional, write the converse, inverse, and contrapositive in symbols and in words.

1. $\boldsymbol{p}$: I go skydiving.
   $\boldsymbol{q}$: I go bungee jumping.
   If I do not go skydiving, then I go bungee jumping.

2. $\boldsymbol{p}$: I have a pet cat.
   $\boldsymbol{q}$: I have a mouse problem.
   If I have a pet cat, then I don't have a mouse problem.

3. $\boldsymbol{r}$: I use a laptop.
   $\boldsymbol{s}$: I use a tablet.
   If I do not use a laptop, then I do not use a tablet.

4. $\boldsymbol{r}$: Zac Brown Band is a county music band artist.
   $\boldsymbol{s}$: Zac Brown Band is playing at Wrigley Field this summer.
   If Zac Brown Band is a country music artist, then Zac Brown Band is playing at Wrigley Field this summer.

5. $\boldsymbol{t}$: I wear contacts.
   $\boldsymbol{u}$: I wear glasses.
   If I wear contacts, then I do not wear glasses.

6. $\boldsymbol{t}$: I go to bed early.
   $\boldsymbol{u}$: I need 8 hours of sleep each night.
   If I don't go to bed early, then I don't need 8 hours of sleep each night.

7. $\boldsymbol{a}$: It is Friday.
   $\boldsymbol{b}$: My friends and I order pizza.
   If it is Friday, then my friends and I order pizza.

8. $\boldsymbol{a}$: I do not like parties.
   $\boldsymbol{b}$: I like staying home.
   If I do not like parties, then I like staying home.

For exercises 9 – 17, write the converse, inverse, and contrapositive of the conditional statement given.

9. If I will go camping, then I will make s'mores.
10. If I don't study every day, then I won't pass this class.
11. If I do not go out, then I will watch a movie.
12. If I walk my dog, then she will leave me alone the rest of the day.
13. If I do not bake cookies, then I do not use my oven.
14. If I want to become an engineer, then I need to pass Calculus III.
15. If it is St. Patty's Day, then I drink green beer.
16. If it is Patriot's Day in Massachusetts, then runners compete in the Boston Marathon.
17. If it is 2 am, then I am not sleeping.

For exercises 18 – 28, given the complex conditional statement, symbolically write the converse, inverse, and contrapositive.

18. $\sim r \rightarrow (p \vee \sim q)$
19. $p \rightarrow (\sim p \wedge q)$
20. $\sim(q \wedge \sim p) \rightarrow p$
21. $\sim(p \vee \sim q) \rightarrow q$
22. $\sim(\sim q \wedge r) \rightarrow \sim p$
23. $\sim p \rightarrow \sim(q \vee r)$
24. $q \rightarrow (\sim p \wedge \sim q)$
25. $(\sim p \vee r) \rightarrow (\sim q \wedge r)$
26. $(r \vee p) \rightarrow \sim(q \vee \sim p)$
27. $\sim(\sim p \vee r) \rightarrow \sim(q \vee r)$
28. $\sim(p \wedge \sim r) \rightarrow \sim(\sim q \vee p)$

For exercises 29 – 37, write the negation of the conditional statement. Then create a truth table for the following:

a. The conditional statement
b. The negation of the conditional statement
c. The logically equivalent negation of the conditional statement

29. If I do not smoke, then I will be happy.
30. If you earn a bachelor's degree, then your income potential increases.
31. If I do not watch basketball, then I do not eat nachos.
32. If I do not travel to Iceland, then I will travel to Tahiti.
33. If I go to Las Vegas, then I will not play poker.
34. If pigs can fly, then dogs can talk.
35. If it is 5:30 pm, then I will watch the news.

36. If I build a rocket, then I will not fly to Mars.
37. If I am not in California, then I am not in Napa Valley.

For exercises 38 – 46, write the negation of the statement. Then create a truth table for following:

a. The conditional statement
b. The negation of the conditional statement
c. The logically equivalent negation of the conditional statement b

38. If I do not go to the doctor, then I will get sick and prolong a diagnosis.
39. If a toddler gets his first haircut, then he doesn't look quite so young and his hair won't be in his eyes.
40. If I am not stressed, then I went to yoga or I meditated.
41. If I get a promotion, then I cook dinner and invite friends over.
42. If I have a headache, then I did not drink enough water and I need to take medicine.
43. If you aren't wearing a shirt and aren't wearing shoes, then you can't enter stores.
44. If you don't stop smoking or you don't start exercising, then you won't be considered healthy.
45. If I go on a small boat, then I have to take medicine and I will not be seasick.
46. If Caedon travels to Japan, then he will visit Kyoto and he will not climb to the top of Mount Fuji.

Concept Review

47. The truth value of a conditional statement $p \rightarrow q$ is true while the truth value of its converse is false. Determine the truth values of the simple statements p and q.
48. The truth value of a conditional statement $p \rightarrow q$ is true while the truth value of its inverse is false. Determine the truth values of the simple statements p and q.
49. The truth value of a biconditional statement $p \leftrightarrow q$ is false while the truth value of a conditional $p \rightarrow q$ containing the same statements is false. Determine the truth values of the simple statements p and q.
50. The truth value of a biconditional statement $p \leftrightarrow q$ is true while the truth value of a conjunction $p \wedge q$ containing the same statements is false. Determine the truth values of the simple statements p and q.

Section 2.6 | Exercises Solutions

1. Conditional $\sim p \rightarrow q$: If I do not go skydiving, then I go bungee jumping.
   Converse $q \rightarrow \sim p$: If I go bungee jumping, then I do not go skydiving.
   Inverse $p \rightarrow \sim q$: If I go skydiving, then I do not go bungee jumping.
   Contrapositive $\sim q \rightarrow p$: If I do not go bungee jumping, then I go skydiving.

2. Conditional $p \rightarrow \sim q$: If I have a pet cat, then I don't have a mouse problem.
   Converse $\sim q \rightarrow p$: If I don't have a mouse problem, then I have a pet cat.
   Inverse $\sim p \rightarrow q$: If I don't have a pet cat, then I do have a mouse problem.
   Contrapositive $q \rightarrow \sim p$: If I have a mouse problem, then I don't have a pet cat.

3. Conditional $\sim r \rightarrow \sim s$: If I do not use a laptop, then I do not use a tablet.
   Converse $\sim s \rightarrow \sim r$: If I do not use a tablet, then I do not use a laptop.
   Inverse $r \rightarrow s$: If I use a laptop, then I use a tablet.
   Contrapositive $s \rightarrow r$: If I use a tablet, then I use a laptop.

4. Conditional $r \rightarrow s$: If Zac Brown Band (ZBB) is a country music band, then they are playing at Wrigley Field this summer.
   Converse $s \rightarrow r$: If ZBB is playing at Wrigley Field this summer, then they are a country music band.
   Inverse $\sim r \rightarrow \sim s$: If ZBB is not a country music band, then they are not playing at Wrigley Field this summer.
   Contrapositive $\sim s \rightarrow \sim r$: If ZBB is not playing at Wrigley Field this summer, then they are not a country music band.

5. Conditional $t \rightarrow \sim u$: If I wear contacts, then I do not wear glasses.
   Converse $\sim u \rightarrow t$: If I do not wear glasses, then I wear contacts.
   Inverse $\sim t \rightarrow u$: If I do not wear contacts, then I wear glasses.
   Contrapositive $u \rightarrow \sim t$: If I wear glasses, then I do not wear contacts.

6. Conditional $\sim t \rightarrow \sim u$: If I do not go to bed early, then I do not need 8 hours of sleep each night.
   Converse $\sim u \rightarrow \sim t$: If I do not need 8 hours of sleep each night, then I do not go to bed early.
   Inverse $t \rightarrow u$: If I go to bed early, then I need 8 hours of sleep each night.
   Contrapositive $u \rightarrow t$: If need 8 hours of sleep each night, then I go to bed early.

7. Conditional $a \rightarrow b$: If it is Friday, then my friends and I order pizza.
   Converse $b \rightarrow a$: If my friends and I order pizza, then it is Friday.
   Inverse $\sim a \rightarrow \sim b$: If it is not Friday, then my friends and I do not order pizza.
   Contrapositive $\sim b \rightarrow \sim a$: If my friends and I do not order pizza, then it is not Friday.

8. Conditional $\sim a \rightarrow b$: If I do not like parties, then I like staying home.
   Converse $b \rightarrow \sim a$: If I like staying home, then I do not like parties.
   Inverse $a \rightarrow \sim b$: If I like parties, then I do not like staying home.
   Contrapositive $\sim b \rightarrow a$: If I do not like staying home, then I like parties.

9. Converse: If I will make s'mores, then I will go camping.
   Inverse: If I will not go camping, then I will not make s'mores.
   Contrapositive: If will not make s'mores, then I will not go camping.

10. Converse: If I don't pass this class, then I don't study every day.
    Inverse: If I do study every day, then I will pass this class.
    Contrapositive: If I do pass this class, then I do study every day.

11. Converse: If I watch a movie, then I did not go out.
    Inverse: If I go out, then I will not watch a movie.
    Contrapositive: If I will not watch a movie, then I did go out.

12. Converse: If my dog leaves me alone the rest of the day, then I walk her.
    Inverse: If I don't walk my dog, then she won't leave me alone for the rest of the day.
    Contrapositive: If my dog doesn't leave me alone for the rest of the day, then I didn't walk her.

13. Converse: If I do not use my oven, then I do not bake cookies.
    Inverse: If I bake cookies, then I use my oven.
    Contrapositive: If I use my oven, then I bake cookies.

14. Converse: If I need to pass Calculus III, then I want to become an engineer.
    Inverse: If I do not want to become an engineer, then I do not need to pass Calculus III.
    Contrapositive: If I do not need to pass Calculus III, then I do not want to become an engineer.

15. Converse: If I drink green beer, then it is St. Patty's Day.
    Inverse: If it is not St. Patty's Day, then I did not drink green beer.
    Contrapositive: If I did not drink green beer, then it is not St. Patty's Day.

16. Converse: If runners compete in the Boston Marathon, then it is Patriot's Day in Massachusetts.
Inverse: If it is not Patriot's Day in Massachusetts, then runners do not compete in the Boston Marathon.
Contrapositive: If runners do not compete in the Boston Marathon, then it is not Patriot's Day in Massachusetts.

17. Converse: If I am not sleeping, then it is 2 am.
Inverse: If it is not 2 am, then I am sleeping.
Contrapositive: If I am sleeping, then it is not 2 am.

18. Converse: $(p \vee {\sim}q) \rightarrow {\sim}r$
Inverse: $r \rightarrow {\sim}(p \vee {\sim}q)$ or after applying DeMorgan's Law $r \rightarrow ({\sim}p \wedge q)$
Contrapositive: ${\sim}(p \vee {\sim}q) \rightarrow r$ or after applying DeMorgan's Law $({\sim}p \wedge q) \rightarrow r$

19. Converse: $({\sim}p \wedge q) \rightarrow p$
Inverse: ${\sim}p \rightarrow {\sim}({\sim}p \wedge q)$ or after applying DeMorgan's Law ${\sim}p \rightarrow (p \vee {\sim}q)$
Contrapositive: ${\sim}({\sim}p \wedge q) \rightarrow {\sim}p$ or after applying DeMorgan's Law $(p \vee {\sim}q) \rightarrow {\sim}p$

20. Converse: $p \rightarrow {\sim}(q \wedge {\sim}p)$ or after applying DeMorgan's Law $p \rightarrow ({\sim}q \vee p)$
Inverse: $(q \wedge {\sim}p) \rightarrow {\sim}p$
Contrapositive: ${\sim}p \rightarrow (q \wedge {\sim}p)$

21. Converse: $q \rightarrow {\sim}(p \vee {\sim}q)$ or after applying DeMorgan's Law $q \rightarrow ({\sim}p \wedge q)$
Inverse: $(p \vee {\sim}q) \rightarrow {\sim}q$
Contrapositive: ${\sim}q \rightarrow (p \vee {\sim}q)$

22. Converse: ${\sim}p \rightarrow {\sim}({\sim}q \wedge r)$ or after applying DeMorgan's Law ${\sim}p \rightarrow (q \vee {\sim}r)$
Inverse: $({\sim}q \wedge r) \rightarrow p$
Contrapositive: $p \rightarrow ({\sim}q \wedge r)$

23. Converse: ${\sim}(q \vee r) \rightarrow {\sim}p$ or after applying DeMorgan's Law $({\sim}q \wedge {\sim}r) \rightarrow {\sim}p$
Inverse: $p \rightarrow (q \vee r)$
Contrapositive: $(q \vee r) \rightarrow p$

24. Converse: $({\sim}p \wedge {\sim}q) \rightarrow q$
Inverse: ${\sim}q \rightarrow {\sim}({\sim}p \wedge {\sim}q)$ or after applying DeMorgan's Law ${\sim}q \rightarrow (p \vee q)$
Contrapositive: ${\sim}({\sim}p \wedge {\sim}q) \rightarrow {\sim}q$ or after applying DeMorgan's Law $(p \vee q) \rightarrow {\sim}q$

25. Converse: $(\sim q \wedge r) \rightarrow (\sim p \vee r)$
Inverse: $\sim(\sim p \vee r) \rightarrow \sim(\sim q \wedge r)$ or after applying DeMorgan's Law
$(p \wedge \sim r) \rightarrow (q \vee \sim r)$
Contrapositive: $\sim(\sim q \wedge r) \rightarrow \sim(\sim p \vee r)$ or after applying DeMorgan's Law
$(q \vee \sim r) \rightarrow (p \wedge \sim r)$

26. Converse: $\sim(q \vee \sim p) \rightarrow (r \vee p)$ or after applying DeMorgan's Law $(\sim q \wedge p) \rightarrow (r \vee p)$
Inverse: $\sim(r \vee p) \rightarrow (q \vee \sim p)$ or after applying DeMorgan's Law $(\sim r \wedge \sim p) \rightarrow (q \vee \sim p)$
Contrapositive: $(q \vee \sim p) \rightarrow \sim(r \vee p)$ or after applying DeMorgan's Law
$(q \vee \sim p) \rightarrow (\sim r \wedge \sim p)$

27. Converse: $\sim(q \vee r) \rightarrow \sim(\sim p \vee r)$ or after applying DeMorgan's Law
$(\sim q \wedge \sim r) \rightarrow (p \wedge \sim r)$
Inverse: $(\sim p \vee r) \rightarrow (q \vee r)$
Contrapositive: $(q \vee r) \rightarrow (\sim p \vee r)$

28. Converse: $\sim(\sim q \vee p) \rightarrow \sim(p \wedge \sim r)$ or after applying DeMorgan's Law
$(q \wedge \sim p) \rightarrow (\sim p \vee r)$
Inverse: $(p \wedge \sim r) \rightarrow (\sim q \vee p)$
Contrapositive: $(\sim q \vee p) \rightarrow (p \wedge \sim r)$

For exercises 29 – 46, answers can vary but one solution is shown.

29. Answers can vary. One way to assign statements p and q is provided.
Negation: I do not smoke and I am not happy.
$\boldsymbol{p}$: I smoke. $\boldsymbol{q}$: I am happy.

$\boldsymbol{p}$	$\boldsymbol{q}$	$\sim\boldsymbol{p}$	$\sim\boldsymbol{q}$	**a.** $\sim\boldsymbol{p} \rightarrow \boldsymbol{q}$	**b.** $\sim(\sim\boldsymbol{p} \rightarrow \boldsymbol{q})$	**c.** $\sim\boldsymbol{p} \wedge \sim\boldsymbol{q}$
T	T	F	F	T	F	F
T	F	F	T	T	F	F
F	T	T	F	T	F	F
F	F	T	T	F	T	T

30. Answers can vary. One way to assign statements p and q is provided.
Negation: You earn a bachelor's degree and your income potential does not increase.
$\boldsymbol{p}$: You earn a bachelor's degree. $\boldsymbol{q}$: Your income potential increase.

$\boldsymbol{p}$	$\boldsymbol{q}$	$\sim\boldsymbol{q}$	**a.** $\boldsymbol{p \to q}$	**b.** $\sim(\boldsymbol{p \to q})$	**c.** $\boldsymbol{p \wedge \sim q}$
T	T	F	T	F	F
T	F	T	F	T	T
F	T	F	T	F	F
F	F	T	T	F	F

31. Answers can vary. One way to assign statements p and q is provided.
Negation: I do not watch basketball and I eat nachos.
$\boldsymbol{p}$: I watch basketball. $\boldsymbol{q}$: I eat nachos.

$\boldsymbol{p}$	$\boldsymbol{q}$	$\sim\boldsymbol{p}$	$\sim\boldsymbol{q}$	**a.** $\sim\boldsymbol{p} \to \sim\boldsymbol{q}$	**b.** $\sim(\sim\boldsymbol{p} \to \sim\boldsymbol{q})$	**c.** $\sim\boldsymbol{p} \wedge \boldsymbol{q}$
T	T	F	F	T	F	F
T	F	F	T	T	F	F
F	T	T	F	F	T	T
F	F	T	T	T	F	F

32. Answers can vary. One way to assign statements p and q is provided.
Negation: I do not travel to Iceland and I do not travel to Tahiti.
$\boldsymbol{p}$: I travel to Iceland. $\boldsymbol{q}$: I travel to Tahiti.

$\boldsymbol{p}$	$\boldsymbol{q}$	$\sim\boldsymbol{p}$	$\sim\boldsymbol{q}$	**a.** $\sim\boldsymbol{p} \to \boldsymbol{q}$	**b.** $\sim(\sim\boldsymbol{p} \to \boldsymbol{q})$	**c.** $\sim\boldsymbol{p} \wedge \sim\boldsymbol{q}$
T	T	F	F	T	F	F
T	F	F	T	T	F	F
F	T	T	F	T	F	F
F	F	T	T	F	T	T

33. Answers can vary. One way to assign statements p and q is provided.
Negation: I go to Las Vegas and I play poker.
$\boldsymbol{p}$: I go to Las Vegas. $\boldsymbol{q}$: I play poker.

$\boldsymbol{p}$	$\boldsymbol{q}$	$\sim\boldsymbol{q}$	**a.** $\boldsymbol{p \to \sim q}$	**b.** $\boldsymbol{\sim(p \to \sim q)}$	**c.** $\boldsymbol{p \wedge q}$
T	T	F	F	T	T
T	F	T	T	F	F
F	T	F	T	F	F
F	F	T	T	F	F

34. Answers can vary. One way to assign statements p and q is provided.
Negation: Pigs can fly and dogs can't talk.
$\boldsymbol{p}$: Pigs can fly. $\boldsymbol{q}$: Dogs can talk.

$\boldsymbol{p}$	$\boldsymbol{q}$	$\sim\boldsymbol{q}$	**a.** $\boldsymbol{p \to q}$	**b.** $\boldsymbol{\sim(p \to q)}$	**c.** $\boldsymbol{p \wedge \sim q}$
T	T	F	T	F	F
T	F	T	F	T	T
F	T	F	T	F	F
F	F	T	T	F	F

35. Answers can vary. One way to assign statements p and q is provided.
Negation: It is 5:30 pm and I will not watch the news.
$\boldsymbol{p}$: It is 5:30 pm. $\boldsymbol{q}$: I will watch the news.

$\boldsymbol{p}$	$\boldsymbol{q}$	$\sim\boldsymbol{q}$	**a.** $\boldsymbol{p \to q}$	**b.** $\boldsymbol{\sim(p \to q)}$	**c.** $\boldsymbol{p \wedge \sim q}$
T	T	F	T	F	F
T	F	T	F	T	T
F	T	F	T	F	F
F	F	T	T	F	F

36. Answers can vary. One way to assign statements p and q is provided.
Negation: I build a rocket and I fly to Mars.
$\boldsymbol{p}$: I build a rocket. $\boldsymbol{q}$: I fly to Mars.

p	q	$\sim q$	a. $p \to \sim q$	b. $\sim(p \to \sim q)$	c. $p \land q$
T	T	F	F	T	T
T	F	T	T	F	F
F	T	F	T	F	F
F	F	T	T	F	F

37. Answers can vary. One way to assign statements p and q is provided.
Negation: I am not in California and I am in Napa Valley.
$\boldsymbol{p}$: I am in California. $\boldsymbol{q}$: I am in Napa Valley.

p	q	$\sim p$	$\sim q$	a. $\sim p \to \sim q$	b. $\sim(\sim p \to \sim q)$	c. $\sim p \land q$
T	T	F	F	T	F	F
T	F	F	T	T	F	F
F	T	T	F	F	T	T
F	F	T	T	T	F	F

38. Answers can vary. One way to assign statements p, q and r is provided.
Negation: I do not go to the doctor, and it is not true that I am sick and prolong a diagnosis.
$\boldsymbol{p}$: I go to the doctor. $\boldsymbol{q}$: I will get sick. $\boldsymbol{r}$: I prolong a diagnosis.

p	q	r	$\sim p$	$q \land r$	$\sim(q \land r)$	a. $\sim p \to (q \land r)$	b. $\sim(\sim p \to (q \land r))$	c. $\sim p \land \sim(q \land r)$
T	T	T	F	T	F	T	F	F
T	T	F	F	F	T	T	F	F
T	F	T	F	F	T	T	F	F
T	F	F	F	F	T	T	F	F
F	T	T	T	T	F	T	F	F
F	T	F	T	F	T	F	T	T
F	F	T	T	F	T	F	T	T
F	F	F	T	F	T	F	T	T

39. Answers can vary. One way to assign statements p, q and r is provided.
Negation: A toddler got his first haircut, and it is not true that he doesn't look quite so young and his hair won't be in his eyes.
$\boldsymbol{p}$: A toddler gets his first haircut. $\boldsymbol{q}$: He does look quite so young. $\boldsymbol{r}$: His hair is in his eyes.

$\boldsymbol{p}$	$\boldsymbol{q}$	$\boldsymbol{r}$	$\sim\boldsymbol{q}$	$\sim\boldsymbol{r}$	$\sim\boldsymbol{q} \wedge \sim\boldsymbol{r}$	$\sim(\sim\boldsymbol{q} \wedge \sim\boldsymbol{r})$	**a.** $\boldsymbol{p} \rightarrow (\sim\boldsymbol{q} \wedge \sim\boldsymbol{r})$	**b.** $\sim(\boldsymbol{p} \rightarrow (\sim\boldsymbol{q} \wedge \sim\boldsymbol{r}))$	**c.** $\boldsymbol{p} \wedge \sim(\sim\boldsymbol{q} \wedge \boldsymbol{r})$
T	T	T	F	F	F	T	F	T	T
T	T	F	F	T	F	T	F	T	T
T	F	T	T	F	F	T	F	T	T
T	F	F	T	T	T	F	T	F	F
F	T	T	F	F	F	T	T	F	F
F	T	F	F	T	F	T	T	F	F
F	F	T	T	F	F	T	T	F	F
F	F	F	T	T	T	F	T	F	F

40. Answers can vary. One way to assign statements p, q and r is provided.
Negation: I am not stressed, and it is not true that I went to yoga or I meditated.
$\boldsymbol{p}$: I am stressed. $\boldsymbol{q}$: I went to yoga. $\boldsymbol{r}$: I meditated.

$\boldsymbol{p}$	$\boldsymbol{q}$	$\boldsymbol{r}$	$\sim\boldsymbol{p}$	$\boldsymbol{q} \vee \boldsymbol{r}$	$\sim(\boldsymbol{q} \vee \boldsymbol{r})$	**a.** $\sim\boldsymbol{p} \rightarrow (\boldsymbol{q} \vee \boldsymbol{r})$	**b.** $\sim(\sim\boldsymbol{p} \rightarrow (\boldsymbol{q} \vee \boldsymbol{r}))$	**c.** $\sim\boldsymbol{p} \wedge \sim(\boldsymbol{q} \vee \boldsymbol{r})$
T	T	T	F	T	F	T	F	F
T	T	F	F	T	F	T	F	F
T	F	T	F	T	F	T	F	F
T	F	F	F	F	T	T	F	F
F	T	T	T	T	F	T	F	F
F	T	F	T	T	F	T	F	F
F	F	T	T	T	F	T	F	F
F	F	F	T	F	T	F	T	T

41. Answers can vary. One way to assign statements p, q and r is provided.
Negation: I get a promotion, and it is not true that I cook dinner and invite friends over.
$\boldsymbol{p}$: I get a promotion. $\boldsymbol{q}$: I cook dinner. $\boldsymbol{r}$: I invite friends over.

$\boldsymbol{p}$	$\boldsymbol{q}$	$\boldsymbol{r}$	$\sim\boldsymbol{p}$	$\boldsymbol{q} \wedge \boldsymbol{r}$	$\sim(\boldsymbol{q} \wedge \boldsymbol{r})$	**a.** $\boldsymbol{p} \rightarrow (\boldsymbol{q} \wedge \boldsymbol{r})$	**b.** $\sim(\boldsymbol{p} \rightarrow (\boldsymbol{q} \wedge \boldsymbol{r}))$	**c.** $\boldsymbol{p} \wedge \sim(\boldsymbol{q} \wedge \boldsymbol{r})$
T	T	T	F	T	F	T	F	F
T	T	F	F	F	T	F	T	T
T	F	T	F	F	T	F	T	T
T	F	F	F	F	T	F	T	T
F	T	T	T	T	F	T	F	F
F	T	F	T	F	T	T	F	F
F	F	T	T	F	T	T	F	F
F	F	F	T	F	T	T	F	F

42. Answers can vary. One way to assign statements p, q and r is provided.
Negation: I have a headache, and it is not true that I did not drink enough water and I need to take medicine.
$\boldsymbol{p}$: I have a headache. $\boldsymbol{q}$: I drink enough water. $\boldsymbol{r}$: I need to take medicine.

$\boldsymbol{p}$	$\boldsymbol{q}$	$\boldsymbol{r}$	$\sim\boldsymbol{p}$	$\sim\boldsymbol{q}$	$\sim\boldsymbol{q} \wedge \boldsymbol{r}$	$\sim(\sim\boldsymbol{q} \wedge \boldsymbol{r})$	**a.** $\boldsymbol{p} \rightarrow (\sim\boldsymbol{q} \wedge \boldsymbol{r})$	**b.** $\sim(\boldsymbol{p} \rightarrow (\sim\boldsymbol{q} \wedge \boldsymbol{r}))$	**c.** $\boldsymbol{p} \wedge \sim(\sim\boldsymbol{q} \wedge \boldsymbol{r})$
T	T	T	F	F	F	T	F	T	T
T	T	F	F	F	F	T	F	T	T
T	F	T	F	T	T	F	T	F	F
T	F	F	F	T	F	T	F	T	T
F	T	T	T	F	F	T	T	F	F
F	T	F	T	F	F	T	T	F	F
F	F	T	T	T	T	F	T	F	F
F	F	F	T	T	F	T	T	F	F

43. Answers can vary. One way to assign statements p, q and r is provided.
Negation: You aren't wearing a shirt and aren't wearing shoes and you can enter stores.
$\boldsymbol{p}$: You are wearing a shirt. $\boldsymbol{q}$: You are wearing shoes. $\boldsymbol{r}$: You can enter stores.

$\boldsymbol{p}$	$\boldsymbol{q}$	$\boldsymbol{r}$	$\sim\boldsymbol{p}$	$\sim\boldsymbol{q}$	$\sim\boldsymbol{r}$	$\sim\boldsymbol{p} \wedge \sim\boldsymbol{q}$	**a.** $(\sim\boldsymbol{p} \wedge \sim\boldsymbol{q}) \rightarrow \sim\boldsymbol{r}$	**b.** $\sim[(\sim\boldsymbol{p} \wedge \sim\boldsymbol{q}) \rightarrow \sim\boldsymbol{r}]$	**c.** $(\sim\boldsymbol{p} \wedge \sim\boldsymbol{q}) \wedge \boldsymbol{r}$
T	T	T	F	F	F	F	T	F	F
T	T	F	F	F	T	F	T	F	F
T	F	T	F	T	F	F	T	F	F
T	F	F	F	T	T	F	T	F	F
F	T	T	T	F	F	F	T	F	F
F	T	F	T	F	T	F	T	F	F
F	F	T	T	T	F	T	F	T	T
F	F	F	T	T	T	T	T	F	F

44. Answers can vary. One way to assign statements p, q and r is provided.
Negation: You don't stop smoking or you don't start exercising, and you will be considered healthy.
$\boldsymbol{p}$: You stop smoking. $\boldsymbol{q}$: You start exercising. $\boldsymbol{r}$: You are considered healthy.

$\boldsymbol{p}$	$\boldsymbol{q}$	$\boldsymbol{r}$	$\sim\boldsymbol{p}$	$\sim\boldsymbol{q}$	$\sim\boldsymbol{r}$	$\sim\boldsymbol{p} \vee \sim\boldsymbol{q}$	**a.** $(\sim\boldsymbol{p} \vee \sim\boldsymbol{q}) \rightarrow \sim\boldsymbol{r}$	**b.** $\sim[(\sim\boldsymbol{p} \vee \sim\boldsymbol{q}) \rightarrow \sim\boldsymbol{r}]$	**c.** $(\sim\boldsymbol{p} \vee \sim\boldsymbol{q}) \wedge \boldsymbol{r}$
T	T	T	F	F	F	F	T	F	F
T	T	F	F	F	T	F	T	F	F
T	F	T	F	T	F	T	F	T	T
T	F	F	F	T	T	T	T	F	F
F	T	T	T	F	F	T	F	T	T
F	T	F	T	F	T	T	T	F	F
F	F	T	T	T	F	T	F	T	T
F	F	F	T	T	T	T	T	F	F

45. Answers can vary. One way to assign statements p, q and r is provided.
Negation: I go on a small boat, and it is not true that I have to take medicine and I will not be seasick.
$\boldsymbol{p}$: I go on a small boat. $\boldsymbol{q}$: I have to take medicine. $\boldsymbol{r}$: I will be seasick.

$\boldsymbol{p}$	$\boldsymbol{q}$	$\boldsymbol{r}$	$\sim\boldsymbol{r}$	$\boldsymbol{q} \wedge \sim\boldsymbol{r}$	$\sim(\boldsymbol{q} \wedge \sim\boldsymbol{r})$	**a.** $\boldsymbol{p} \rightarrow (\boldsymbol{q} \wedge \sim\boldsymbol{r})$	**b.** $\sim(\boldsymbol{p} \rightarrow (\boldsymbol{q} \wedge \sim\boldsymbol{r}))$	**c.** $\boldsymbol{p} \wedge \sim(\boldsymbol{q} \wedge \sim\boldsymbol{r})$
T	T	T	F	F	T	F	T	T
T	T	F	T	T	F	T	F	F
T	F	T	F	F	T	F	T	T
T	F	F	T	F	T	F	T	T
F	T	T	F	F	T	T	F	F
F	T	F	T	T	F	T	F	F
F	F	T	F	F	T	T	F	F
F	F	F	T	F	T	T	F	F

46. Answers can vary. One way to assign statements p, q and r is provided.
Negation: Caedon travels to Japan, and it is not true that Caedon visits Kyoto and he does not climb to the top of Mount Fuji.
$\boldsymbol{p}$: Caedon travels to Japan. $\boldsymbol{q}$: Caedon visits Kyoto. $\boldsymbol{r}$: Caedon climbs to the top of Mount Fuji.

$\boldsymbol{p}$	$\boldsymbol{q}$	$\boldsymbol{r}$	$\sim\boldsymbol{r}$	$\boldsymbol{q} \wedge \sim\boldsymbol{r}$	$\sim(\boldsymbol{q} \wedge \sim\boldsymbol{r})$	**a.** $\boldsymbol{p} \rightarrow (\boldsymbol{q} \wedge \sim\boldsymbol{r})$	**b.** $\sim(\boldsymbol{p} \rightarrow (\boldsymbol{q} \wedge \sim\boldsymbol{r}))$	**c.** $\boldsymbol{p} \wedge \sim(\boldsymbol{q} \wedge \sim\boldsymbol{r})$
T	T	T	F	F	T	F	T	T
T	T	F	T	T	F	T	F	F
T	F	T	F	F	T	F	T	T
T	F	F	T	F	T	F	T	T
F	T	T	T	T	F	T	F	F
F	T	F	T	T	F	T	F	F
F	F	T	T	F	T	T	F	F
F	F	F	T	F	T	T	F	F

47. p is false and q is true
48. p is false and q is true
49. p is true and q is false
50. p is false and q is false

Section 2.7 | Verifying Logic Arguments & Common Arguments & Fallacies

Objectives

- Use truth tables to determine the validity of arguments
- Identify some common forms of valid arguments
- Identify some common forms of fallacies

An argument is not always an exchange of differing opinions. In logic, an argument is a series of statements (called premises) that lead to a conclusion. The premises are used to determine the truth value of a conclusion. The mathematics of logic theory can determine whether an argument is valid or invalid.

Logic in Alice's Adventures in Wonderland

Lewis Carroll was a famous mathematician and wrote children's literature. His knowledge of mathematical logic can be found in his famous novel *Alice's Adventures in Wonderland* (1865, 72-73). For example, in Chapter 5 Alice has a conversation with a pigeon. The pigeon claims Alice is a serpent who is out to eat his eggs while Alice tries to counter the pigeon's logic.

> "But I'm not a serpent," said Alice. "I—I'm a little girl," said Alice, rather doubtfully, as she remembered the number of changes she had gone through, that day.
>
> "A likely story indeed!" said the Pigeon, "I've seen a good many little girls in my time, but never one with such a neck as that! No, no! You're a serpent, and there's no use denying it. I suppose you'll be telling me next that you never tasted an egg!"
>
> "I *have* tasted eggs, certainly," said Alice, who was a very truthful child, "but little girls eat eggs quite as much as serpents do, you know."
>
> "I don't believe it," said the Pigeon; "but if they do, why, then they're a kind of serpent; that's all I can say."

The Pigeon's argument could be summarized as follows:

> If you have a long neck, then you are a serpent.
> Girls don't have long necks and you have a long neck, so then you are a serpent.
> If you eat eggs, then you are a serpent.

In the Pigeon's logical thinking, Alice has a long neck and eats eggs; therefore, she is a serpent. Alice is, in fact, not a serpent but is having a difficult time proving the Pigeon wrong.

DEFINITION: In logic theory, an **argument** is a series of statements (called **premises**) that lead to a **conclusion**.

The premises of the argument are used to determine the truth value of the conclusion. For example: Given the conditional "If it is Tuesday, then Nola eats tacos" and the statement "Today is Tuesday," then the conclusion "Nola eats tacos" is also true. This is an example of a direct argument. Direct arguments as well as several other types of arguments will be discussed in this section.

Using Truth Tables to Determine the Validity of Arguments

Truth tables can be used to determine whether an argument is valid. For an argument to be **valid**, the argument must be true for every possible case. More specifically, an argument is valid whenever all of the premise statements are true and the conclusion is also true.

Writing the argument similar to a vertical addition problem can be helpful. The premise statements are placed above the horizontal line while the conclusion statement is placed below the line.

For example, the argument mentioned previously about eating tacos could also be written in the form:

$$\left.\begin{array}{l}\text{If it is Tuesday, then Nola eats tacos.}\\ \text{It is Tuesday.}\\ \hline\end{array}\right\}\ \text{Premise statements}$$

$$\left.\begin{array}{l}\text{Therefore, Nola eats tacos.}\end{array}\right\}\ \text{Conclusion statement}$$

Symbolic notation could also be used to represent the premise statements and conclusion statement. First, define the simple statements and then write them using the appropriate logic notation.

> $\boldsymbol{p}$: It is Tuesday.
> $\boldsymbol{q}$: Nola eats tacos.

The argument could then be written symbolically as

$p \to q$
p
$\therefore q$

NOTE: Mathematicians use the symbol ∴ to represent "therefore" or "the conclusion is."

A truth table can be used to determine whether the argument is valid, or true for every possible case. To use a truth table, connect all of the premise statements to the conclusion statement using a conditional ***if … then*** connector.

For example, "If premise 1 is true AND premise 2 is true, then the conclusion is true."
Using symbolic form for the connectors, this becomes
(premise 1 ∧ premise 2) → conclusion.

For the argument above,

IF	premise 1	$p \to q$	is given to be true,
AND	premise 2	p	is also given to be true,
THEN	the conclusion	q	is also true.

There are two ways in which to prove an argument valid. Both will be described below.

METHOD 1: Convert this argument into symbolic form: (premise 1 ∧ premise 2) → conclusion
Then, complete a truth table using the symbolic form of the argument. The argument is valid if all of the final argument truth values are true.

(premise 1 ∧ premise 2) → conclusion
$[(p \to q) \;\wedge\; p\,] \quad \to \quad q$

premise 2 $\boldsymbol{p}$	conclusion $\boldsymbol{q}$	premise 1 $\boldsymbol{p \to q}$	premise 1 AND 2 $\boldsymbol{(p \to q) \wedge p}$	argument $\boldsymbol{[(p \to q) \wedge p\,] \to q}$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

Since the argument is true for all possible cases (a tautology), the argument is valid.

METHOD 2: First, complete a truth table using the symbolic form for each premise and conclusion statements. The argument is valid if the premise statements are true and the conclusion is also true.

p	q	premise 1 $p \to q$	premise 2 p	conclusion q
T	T	**T**	**T**	**T**
T	F	F	T	F
F	T	T	F	T
F	F	T	F	F

The first row of the truth table is the only case where the premise statements are true. Because the conclusion is also true, the argument is valid.

➢ **EXAMPLE 2.7.1:** Given the argument, use a truth table to determine whether the argument is valid or invalid.

If there is lightening, then there is electricity in the air.

There is electricity in the air.

Therefore, there is lightening.

SOLUTION:

Convert the argument into symbolic form.

$\boldsymbol{p}$: There is lightening.
$\boldsymbol{q}$: There is electricity in the air.

$$\begin{array}{l} p \to q \\ q \\ \hline \therefore p \end{array}$$

METHOD 1: Convert this argument into symbolic form. Then, complete a truth table using the symbolic form of the argument. The argument is valid if all of the final argument truth values are true.

(premise 1 ∧ premise 2) → conclusion

$[(p \to q) \land \quad q\,] \quad \to p$

conclusion p	premise 2 q	premise 1 $p \to q$	premise 1 AND 2 $(p \to q) \wedge q$	argument $[(p \to q) \wedge q] \to p$
T	T	T	T	T
T	F	F	F	T
F	T	T	T	F
F	F	T	F	T

Not all of the possible cases result in a truth value that is true. Therefore, the argument is invalid.

METHOD 2: First, complete a truth table using the symbolic form for each premise and conclusion statements. The argument is valid if the premise statements are true and the conclusion is also true.

p	q	premise 1 $p \to q$	premise 2 q	conclusion p
T	T	T	T	T
T	F	F	F	T
F	T	T	T	F
F	F	T	F	F

Find the row(s) in which all of the premise statements are true (highlighted). To be a valid argument, the conclusion must also be true. Since row three shows the premise statements are true while the conclusion is not true, the argument is invalid.

➢ **EXAMPLE 2.7.2:** Given the argument, use a truth table to determine whether the argument is valid or invalid.

If Sam practices, then his running skills improve.

Sam practices and does not eat junk food.

Therefore, if Sam eats junk food, then his running skills do not improve.

SOLUTION:
Convert the argument into symbolic form.

$\boldsymbol{p}$: Sam practices.
$\boldsymbol{q}$: His running skills improve.
$\boldsymbol{r}$: Sam eats junk food.

$$\begin{array}{l} p \to q \\ p \wedge \sim r \\ \hline \therefore r \to \sim q \end{array}$$

METHOD 1: Convert this argument into symbolic form. Then, complete a truth table using the symbolic form of the argument. The argument is valid if all of the final argument truth values are true.

(premise 1 ∧ premise 2) → conclusion
$[\,(p \to q) \;\wedge\; (p \wedge \sim r\,)\,] \;\to (r \to \sim q)$

$\boldsymbol{p}$	$\boldsymbol{q}$	$\boldsymbol{r}$	$\boldsymbol{\sim q}$	$\boldsymbol{\sim r}$	premise 1 $\boldsymbol{p \to q}$	premise 2 $\boldsymbol{p \wedge \sim r}$	*	conclusion $(\boldsymbol{r \to \sim q})$	argument **
T	T	T	F	F	T	F	F	F	T
T	T	F	F	T	T	T	T	T	T
T	F	T	T	F	F	F	F	T	T
T	F	F	T	T	F	T	F	T	T
F	T	T	F	F	T	F	F	F	T
F	T	F	F	T	T	F	F	T	T
F	F	T	T	F	T	F	F	T	T
F	F	F	T	T	T	F	F	T	T

* premise 1 AND premise 2: $(p \to q) \;\wedge\; (p \wedge \sim r)$
** argument: $[\,(p \to q) \;\wedge\; (p \wedge \sim r\,)\,] \;\to (r \to \sim q)$

The argument is true for all possible cases. Therefore, the argument is valid.

METHOD 2: First, complete a truth table using the symbolic form for each premise and conclusion statements. The argument is valid if the premise statements are true and the conclusion is also true.

p	q	r	$\sim q$	$\sim r$	premise 1 $p \rightarrow q$	premise 2 $p \wedge \sim r$	conclusion $(r \rightarrow \sim q)$
T	T	T	F	F	T	F	F
T	T	F	F	T	T	T	T
T	F	T	T	F	F	F	T
T	F	F	T	T	F	T	T
F	T	T	F	F	T	F	F
F	T	F	F	T	T	F	T
F	F	T	T	F	T	F	T
F	F	F	T	T	T	F	T

Find the row(s) in which all of the premise statements are true (highlighted). Because the conclusion is also true, the argument is valid.

❖ **YOU TRY IT 2.7.A:** Use a truth table to determine whether the argument is valid or invalid.

If it is raining, then I will carry an umbrella.

I did not carry an umbrella.

Therefore, it is not raining.

❖ **YOU TRY IT 2.7.B:** Use a truth table to determine whether the argument is valid or invalid.

If Ondro studies history, then he will pass the test.

Ondro studies history or plays video games.

Therefore, Ondro passes the test.

Common Valid Argument Forms

At the beginning of this section, an argument about Nola and tacos was presented and proven valid. This argument is an example of a direct argument.

If it is Tuesday, then Nola eats tacos.	$p \rightarrow q$
It is Tuesday.	p
Therefore, Nola eats tacos.	$\therefore q$

As shown previously using a truth table, this is a valid argument. Other statements that fit into this same pattern would also be valid. For example,

If the animal is a pig, then it can fly.	$p \rightarrow q$
The animal is a pig.	p
Therefore, it can fly.	$\therefore q$

NOTE: The premise statements are given information and assumed true for that specific argument. In the example above, the first premise states "If the animal is a pig, then it can fly." In the real world, this is not a true statement. However, because the information provided in the premise statements are assumed to be true, pigs can fly for this specific argument.

There are several common forms of valid arguments. Figure 2.7.1 provides a summary of each symbolic form with an explanation of why the conclusion is always true, making it a valid argument.

FIGURE 2.7.1

Common Form	Symbolic Form	Example	Explanation
Direct Reasoning	$p \rightarrow q$ p $\therefore q$	If it is Tuesday, then Nola eats tacos. It is Tuesday. Therefore, Nola eats tacos.	The first premise says, "If it is Tuesday, then Nola eats tacos." The second premise confirms that for this specific situation, it is Tuesday. Therefore, the conclusion of Nola eats Tacos will always be true.
Contrapositive Reasoning	$p \rightarrow q$ $\sim q$ $\therefore \sim p$	If it is raining, then there are clouds. There are no clouds. Therefore, it is not raining.	From Section 2.6, the original conditional and the contrapositive will always have the same truth value. If given $p \rightarrow q$ and $\sim q$, then the conclusion of $\sim p$ will always be true.
Transitive Reasoning	$p \rightarrow q$ $q \rightarrow r$ $\therefore p \rightarrow r$ $p \rightarrow q$ $q \rightarrow r$ $\therefore \sim r \rightarrow \sim p$	If I drive fast, then I will get a speeding ticket. If I get a speeding ticket, then I will have to pay a fine. Therefore, if I drive fast, then I will have to pay a fine.	In this case, statement q is acting as a transition from statement p to statement r. $p \rightarrow q$ and $q \rightarrow r$, then $p \rightarrow q \rightarrow r$ **NOTE:** Using the contrapositive of $p \rightarrow r$, it is known that $\sim r \rightarrow \sim p$ is also a valid conclusion.
Disjunctive Reasoning	$p \vee q$ $\sim p$ $\therefore q$ $p \vee q$ $\sim q$ $\therefore p$	Izzy went bowling or to the movies. Izzy did not go bowling. Therefore, Izzy went to the movies. Izzy went bowling or to the movies. Izzy did not go to the movies. Therefore, Izzy went bowling.	The first premise is a disjunction, which is where the form gets its name. The first premise $p \vee q$ is true, so it is known that at least one statement p or q is true. The second premise states that $\sim p$ is given true, which means that p is false. By default, statement q must be true. In the second example, premise 2 states that $\sim q$ is given true. By default, statement p must be true.

- **EXAMPLE 2.7.3:** Identify the common form of the argument. Also determine whether the argument is valid or invalid.

  If the dog's name is Snoopy, then he drives a red airplane.

  If he drives a red airplane, then he lives with Charlie Brown.

  Therefore, if the dog's name is Snoopy, then he lives with Charlie Brown.

  SOLUTION:

  Convert the argument into symbolic form.

  $\boldsymbol{p}$: The dog's name is Snoopy.
  $\boldsymbol{q}$: He drives a red airplane.
  $\boldsymbol{r}$: He lives with Charlie Brown.

  $$\begin{array}{l} p \rightarrow q \\ q \rightarrow r \\ \hline \therefore p \rightarrow r \end{array}$$

  This argument fits the form transitive reasoning.
  The argument is valid.

- **EXAMPLE 2.7.4:** Identify the common form of the argument. Also determine whether the argument is valid or invalid.

  Beckham enjoys soccer or hang gliding.

  Beckham does not enjoy hang gliding.

  Therefore, Beckham enjoys soccer.

  SOLUTION:

  Convert the argument into symbolic form.

  $\boldsymbol{p}$: Beckham enjoys soccer.
  $\boldsymbol{q}$: Beckham enjoys hang gliding.

  $$\begin{array}{l} p \vee q \\ \sim q \\ \hline \therefore p \end{array}$$

  This argument fits the form disjunctive reasoning.
  The argument is valid.

❖ **YOU TRY IT 2.7.C:** Identify the common form of the argument. Also determine whether the argument is valid or invalid.

James will take AP Calculus or AP Statistics.

James will not take AP Calculus.

Therefore, James will take AP Statistics.

Common Fallacy Forms

DEFINITION: A **fallacy** is an argument with faulty reasoning that results in an invalid conclusion. A fallacy can seem like a strong argument. However, upon inspection there is a hole or problem with the logic used that makes the conclusion false.

For example,

If it is 9 pm, then Bill watches the news.

It is not 9 pm.

Therefore, Bill is not watching the news.

In the example above, the argument follows the common fallacy form called Fallacy of the Inverse. The initial premise states, "If it is 9 pm, then Bill watches the news." The second premise states, "It is not 9 pm." However, the conclusion "Bill is not watching the news" is not necessarily true since Bill could watch the news at 7 am or 11 pm. Just because it is not 9 pm does not imply that Bill is not watching the news. The argument can be shown invalid using a truth table.

$\boldsymbol{p}$: It is 9 pm.
$\boldsymbol{q}$: Bill watches the news.

$$\begin{array}{l} p \to q \\ \sim p \\ \hline \therefore \sim q \end{array}$$

METHOD 1: The argument is valid if all of the final truth values for the argument are true.

(premise 1 ∧ premise 2) → conclusion
$[(p \to q) \quad \wedge \quad \sim p\,] \quad \to \quad \sim q$

p	q	premise 2 $\sim p$	conclusion $\sim q$	premise 1 $p \to q$	premise 1 AND 2 $(p \to q) \wedge \sim p$	argument $[(p \to q) \wedge \sim p\,] \to \sim q$
T	T	F	F	T	F	T
T	F	F	T	F	F	T
F	T	T	F	T	T	F
F	F	T	T	T	T	T

Not all of the possible cases result in a truth value that is true. Therefore, the argument is invalid.

METHOD 2: The argument is valid whenever all of the premise statements are true and the conclusion is also true.

p	q	premise 1 $p \to q$	premise 2 $\sim p$	conclusion $\sim q$
T	T	T	F	F
T	F	F	F	T
F	T	T	T	F
F	F	T	T	T

Find the rows in which all of the premise statements are true (highlighted). To be a valid argument, the conclusion must also be true. Since row three shows that the premise statements are true but the conclusion is not true, the argument is invalid.

There are several common forms of invalid arguments. Provided in Figure 2.7.2 is a summary of each symbolic form with explanations describing why these arguments are invalid.

FIGURE 2.7.2

Common Form	Symbolic Form	Example	Explanation
Fallacy of the Converse	$p \to q$ q ——— $\therefore p$	If it is raining, then there are clouds. There are clouds. ——— Therefore, it is raining.	From Section 2.6, the original conditional $p \to q$ and the converse $q \to p$ do not have the same meaning. An argument using this logic will NOT always be true, making the argument INVALID.
Fallacy of the Inverse	$p \to q$ $\sim p$ ——— $\therefore \sim q$	If it is Friday, then Carl eats pizza. It is not Friday. ——— Therefore, Carl does not eat pizza.	From Section 2.6, the original conditional $p \to q$ and the inverse $\sim p \to \sim q$ do not have the same meaning. An argument using this logic will NOT always be true, making the argument INVALID.
Misuse of Transitive Reasoning	$p \to q$ $q \to r$ ——— $\therefore r \to p$ $p \to q$ $q \to r$ ——— $\therefore \sim p \to \sim r$	If it is Friday, then we go bowling. If we go bowling, then we eat pizza. ——— Therefore, if we eat pizza then it is Friday. If it is Friday, then we go bowling. If we go bowling, then we eat pizza. ——— Therefore, if it is not Friday, then we don't eat pizza.	Statement q is acting as a transition from statement p to statement r. $p \to q$ and $q \to r$, then $p \to r$. However, the direction of the arrow is important. From Section 2.6, the conditional $p \to r$ does not always have the same truth value as its converse $r \to p$ or as its inverse $\sim p \to \sim r$.
Misuse of Disjunctive Reasoning	$p \vee q$ p ——— $\therefore \sim q$ $p \vee q$ q ——— $\therefore \sim p$	Ryan likes the Ford Raptor or the Nissan GTR. Ryan likes the Ford Raptor. ——— Therefore, Ryan does not like the Nissan GTR. Ryan likes the Ford Raptor or the Nissan GTR. Ryan likes the Nissan GTR. ——— Therefore, Ryan does not like the Ford Raptor.	The first premise $p \vee q$ is true, so it is known that statement p or q or both are true. The second premise states that p is given true. Since it is possible for both statements to be true, there is no guaranteed truth value for statement q. In the second example, premise 2 states that q is given true. Since it is possible for both statements to be true, there is no guaranteed truth value for statement p.

➢ **EXAMPLE 2.7.5:** Identify the form of the common argument or fallacy. Also determine whether the argument is valid or invalid.

If it is morning, then Boden brushes his teeth.

Boden is brushing his teeth.

Therefore, it is morning.

SOLUTION:

Convert the argument into symbolic form.

$\boldsymbol{p}$: It is morning.
$\boldsymbol{q}$: Boden brushes his teeth.

$$\begin{array}{l} p \to q \\ q \\ \hline \therefore p \end{array}$$

This argument fits the form of fallacy of the converse.
The argument is invalid.

➢ **EXAMPLE 2.7.6:** Identify the form of the common argument or fallacy. Also determine whether the argument is valid or invalid.

Maheen will go to dinner or play laser tag with friends.

Maheen went to dinner with friends.

Therefore, Maheen did not play laser tag.

SOLUTION:

Convert the argument into symbolic form.

$\boldsymbol{p}$: Maheen will go to dinner with friends.
$\boldsymbol{q}$: Maheen will play laser tag with friends.

$$\begin{array}{l} p \vee q \\ p \\ \hline \therefore \sim q \end{array}$$

This argument fits the form misuse of disjunctive reasoning.
The argument is invalid.

❖ **YOU TRY IT 2.7.D:** Identify the common form. Also determine whether the argument is valid or invalid.

If the animal is a shark, then it is a fish.

The animal is not a shark.

Therefore, the animal is not a fish.

Quick Review

- An **argument** is a series of statements (called **premises**) that lead to a **conclusion**.
- An argument is **valid** if it is true for every possible case (Method 1).
- An argument is **valid** whenever all of the premise statements are true and the conclusion is also true (Method 2).
- A **fallacy** is an argument with faulty reasoning that results in an invalid argument.

Common Forms of Valid Arguments				
Direct Argument	**Contrapositive Reasoning**	**Transitive Reasoning**	**Disjunctive Reasoning**	
$p \to q$ p $\therefore q$	$p \to q$ $\sim q$ $\therefore \sim p$	$p \to q$ $q \to r$ $\therefore p \to r$	$p \vee q$ $\sim p$ $\therefore q$	$p \vee q$ $\sim q$ $\therefore p$

Common Forms of Fallacies				
Fallacy of the Converse	**Fallacy of the Inverse**	**Misuse of Transitive Reasoning**	**Misuse of Disjunctive Reasoning**	
$p \to q$ q $\therefore p$	$p \to q$ $\sim p$ $\therefore \sim q$	$p \to q$ $q \to r$ $\therefore r \to p$	$p \vee q$ p $\therefore \sim q$	$p \vee q$ q $\therefore \sim p$

YOU TRY IT 2.7.A SOLUTION:

Convert the argument into symbolic form.

$\boldsymbol{p}$: It is raining.
$\boldsymbol{q}$: I will carry an umbrella.

$$\begin{array}{l} p \rightarrow q \\ \sim q \\ \hline \therefore \sim p \end{array}$$

METHOD 1: (Method 2 not shown here.) Convert this argument into symbolic form. Then, complete a truth table using the symbolic form of the argument. The argument is valid if all of the final argument truth values are true.

(premise 1 ∧ premise 2) → conclusion
$[(p \rightarrow q) \;\wedge\; \sim q\,] \quad \rightarrow \quad \sim p$

$\boldsymbol{p}$	$\boldsymbol{q}$	conclusion $\boldsymbol{\sim p}$	premise 2 $\boldsymbol{\sim q}$	premise 1 $\boldsymbol{p \rightarrow q}$	premise 1 AND 2 $\boldsymbol{(p \rightarrow q) \wedge \sim q}$	argument $\boldsymbol{[(p \rightarrow q) \wedge \sim q\,] \rightarrow \sim p}$
T	T	F	F	T	F	T
T	F	F	T	F	F	T
F	T	T	F	T	F	T
F	F	T	T	T	T	T

The argument is valid.

YOU TRY IT 2.7.B SOLUTION:

Convert the argument into symbolic form.

$\boldsymbol{p}$: Ondro studies history.
$\boldsymbol{q}$: He passes the test.
$\boldsymbol{r}$: Ondro plays video games.

$$\begin{array}{l} p \rightarrow q \\ p \vee r \\ \hline \therefore q \end{array}$$

METHOD 2: (Method 1 solution not shown.) First, complete a truth table using the symbolic form for each premise and conclusion statements. The argument is valid whenever all of the premise statements are true and the conclusion is also true.

p	q	r	premise 1 $p \to q$	premise 2 $p \vee r$	conclusion q
T	T	T	T	T	T
T	T	F	T	T	T
T	F	T	F	T	F
T	F	F	F	T	F
F	T	T	T	T	T
F	T	F	T	F	T
F	F	T	T	T	F
F	F	F	T	F	F

Find the row(s) in which all of the premise statements are true (highlighted). Because the conclusion is NOT ALWAYS true for the highlighted rows, the argument is invalid.

YOU TRY IT 2.7.C SOLUTION:

Convert the argument into symbolic form.

$\boldsymbol{p}$: James will take AP Calculus.
$\boldsymbol{q}$: James will take AP Statistics.

The argument could then be written symbolically as

$$\begin{array}{l} p \vee q \\ \sim p \\ \hline \therefore q \end{array}$$

This argument fits the form of disjunctive reasoning.
The argument is valid.

YOU TRY IT 2.7.D SOLUTION:

Convert the argument into symbolic form.

$\boldsymbol{p}$: The animal is a shark.
$\boldsymbol{q}$: It is a fish.

The argument could then be written symbolically as

$$\begin{array}{l} p \rightarrow q \\ \underline{\sim p \quad\quad} \\ \therefore \sim q \end{array}$$

This argument fits the form of fallacy of the inverse.
The argument is invalid.

Section 2.7 | Exercises

In Exercises 1 – 8,
(a) Use **Method 1** (example 2.7.2) to convert the argument into symbolic form.
(b) Complete a truth table using the symbolic form of the argument.
(c) Determine whether the argument is valid or invalid. The argument is valid if all of the final argument truth values are true.

1. $$\begin{array}{l} p \to q \\ \underline{q} \\ \therefore p \end{array}$$

2. $$\begin{array}{l} p \to q \\ \underline{q \to r} \\ \therefore p \to r \end{array}$$

3. $$\begin{array}{l} p \to {\sim}q \\ \underline{p \vee q} \\ \therefore {\sim}q \end{array}$$

4. $$\begin{array}{l} (p \vee q) \to r \\ \underline{p} \\ \therefore r \end{array}$$

5. $$\begin{array}{l} (p \wedge q) \to {\sim}r \\ \underline{{\sim}r} \\ \therefore {\sim}(p \wedge q) \end{array}$$

6. $$\begin{array}{l} p \leftrightarrow q \\ \underline{{\sim}q} \\ \therefore {\sim}p \end{array}$$

7. $$\begin{array}{l} p \to r \\ \underline{p \wedge q} \\ \therefore r \end{array}$$

8. $(p \to q) \to r$
$\underline{p \wedge q}$
$\therefore r$

In Exercises 9 – 16, repeat exercises 1 – 8 above using method 2.
(a) Use **Method 2** (example 2.7.2) to complete a truth table using the symbolic form for the premise and conclusion statements.
(b) Determine whether the argument is valid or invalid. The argument is valid if the premise statements are true and the conclusion is also true.

9. $p \to q$
$\underline{q}$
$\therefore p$

10. $p \to q$
$\underline{q \to r}$
$\therefore p \to r$

11. $p \to \sim q$
$\underline{p \vee q}$
$\therefore \sim q$

12. $(p \vee q) \to r$
$\underline{p}$
$\therefore r$

13. $(p \wedge q) \to \sim r$
$\underline{\sim r}$
$\therefore \sim (p \wedge q)$

14. $p \leftrightarrow q$
$\underline{\sim q}$
$\therefore \sim p$

15. $p \to r$
$\underline{p \wedge q}$
$\therefore r$

16. $(p \to q) \to r$
$\underline{p \wedge q}$
$\therefore r$

In Exercises 17 – 26,
(a) Use **<u>Method 1</u>** (example 2.7.2) to convert the argument into symbolic form.
(b) Complete a truth table using the symbolic form of the argument.
(c) Determine whether the argument is valid or invalid. The argument is valid if all of the final argument truth values are true.

17. If Sharon is hungry, she will eat a snack.
    Sharon eats a snack.
    ___
    Therefore, Sharon is hungry.

18. If it is 9 pm, then Bill watches the news.
    Bill is not watching the news.
    ___
    Therefore, it is not 9 pm.

19. If it is January, then it snows.
    If it snows, then Kat makes a snowman.
    ___
    Therefore, it snows and Kat makes a snowman.

20. If Kadin cooks pumpkin muffins, then he earns an A in cooking class.
    Kadin cooks pumpkin muffins or blueberry pancakes.
    ___
    Therefore, Kadin does not earn an A in cooking class.

21. If Shayla does not pass biology class, then she must take summer school.
    Shayla passed her biology class and passed her math class.
    ___
    Therefore, Shayla does not take summer school.

22. Carson cleans his room or does not earn his allowance.
    Carson does not earn his allowance and helps his brother with math homework.
    ___
    Therefore, Carson cleans his room.

23. If William doesn't get a snack and doesn't get a nap, then William will be cranky.
    William is cranky.
    ___
    Therefore, William didn't get a snack and didn't get a nap.

24. If I go to Oahu and Maui, then I do not go to Alaska.
    I do not go to Alaska.
    ___
    Therefore, I do not go to Oahu or I do not go to Maui.

25. I will retire early if and only if I have enough savings and I'm bored with my job.
    I'm not bored with my job.
    ___
    Therefore, I don't retire early.

26. I eat salads for lunch for a month and I lose ten pounds.
    I lose ten pounds.
    ___
    Therefore, I ate salads for lunch for a month.

In Exercises 27 – 36, repeat exercises 17 – 26 above using method 2.
(a) Use **Method 2** (example 2.7.2) to complete a truth table using the symbolic form for the premise and conclusion statements.
(b) Determine whether the argument is valid or invalid. The argument is valid if the premise statements are true and the conclusion is also true.

27. If Sharon is hungry, she will eat a snack.
    Sharon eats a snack.
    ___
    Therefore, Sharon is hungry.

28. If it is 9 pm, then Bill watches the news.
    Bill is not watching the news.
    ___
    Therefore, it is not 9 pm.

29. If it is January, then it snows.
    If it snows, then Kat makes a snowman.
    ___
    Therefore, it snows and Kat makes a snowman.

30. If Kadin cooks pumpkin muffins, then he earns an A in cooking class.
    Kadin cooks pumpkin muffins or blueberry pancakes.
    ___
    Therefore, Kadin does not earn an A in cooking class.

31. If Shayla does not pass biology class, then she must take summer school.
    Shayla passed her biology class and passed her math class.
    ___
    Therefore, Shayla does not take summer school.

32. Carson cleans his room or does not earn his allowance.
    Carson does not earn his allowance and helps his brother with math homework.
    ___
    Therefore, Carson cleans his room.

33. If William doesn’t get a snack and doesn’t get a nap, then William will be cranky.
    William is cranky.

    Therefore, William didn’t get a snack and didn’t get a nap.

34. If I go to Oahu and Maui, then I do not go to Alaska.
    I do not go to Alaska.

    Therefore, I do not go to Oahu or I do not go to Maui.

35. I will retire early if and only if I have enough savings and I’m bored with my job.
    I’m not bored with my job.

    Therefore, I don’t retire early.

36. I eat salads for lunch for a month and I lose ten pounds.
    I lose ten pounds.

    Therefore, I ate salads for lunch for a month.

In Exercises 37 – 46, identify the common form of the argument or fallacy. Also determine whether the argument or fallacy is valid or invalid.

37. If Aisha enjoys music class, then she will take guitar lessons.
    Aisha did not enjoy music class.

    Therefore, she did not take guitar lessons.

38. Elsa will let it go or she makes it snow.
    Elsa did not let it go.

    Therefore, she makes it snow.

39. Batman will defeat the Joker or he did not see the Bat signal.
    Batman defeated the Joker.

    Therefore, Batman did see the Bat signal.

40. If Ferris Bueller skips school, then Cameron will borrow his dad’s Ferrari.
    If Cameron borrows his dad’s Ferrari, then Mr. Rooney will not catch Ferris.

    If Ferris Bueller skips school, then Mr. Rooney will not catch him.

41. If Eliud Kipchoge runs the 2018 Berlin marathon, then he will set a new marathon world record.
    Eliud Kipchoge ran the 2018 Berlin marathon.

    Therefore, he set a new marathon world record.

42. If Boden wakes early, then he will go to class.
    If Boden goes to class, then he will pass the final exam.

    Therefore, if Boden passed the final exam, then we woke early.

43. If I am social, then I like to party.
    I do not like to party.

    Therefore, I am not social.

44. If I vote, then my candidate will win.
    My candidate won.

    Therefore, I voted.

45. My neighbor stops tap dancing at night or I am not able to sleep at night.
    I am able to sleep at night.

    Therefore, my neighbor did stop tap dancing.

46. If it rains a lot in April, then in May lots of flowers will bloom.
    It doesn't rain a lot in April.

    Therefore, lots of flowers aren't blooming in May.

In Exercises 47 – 52, supply a conclusion that will make the argument valid.

47. If you watch too much television, then you will not complete your homework.
    If you do not complete your homework, then you will not be prepared for the exam.

    Therefore,

48. Drake lifted weights or went jogging.
    Drake did not go jogging.

    Therefore,

49. If Jasmine becomes a journalist, then she completed a lot of writing courses.
Jasmine did not complete a lot of writing courses.

Therefore,

50. If Bella passes this course, then she will graduate from college.
Bella passes this course.

Therefore,

51. I go bungee jumping or I go skydiving.
I did not go bungee jumping.

Therefore,

52. If Harry finds all the Horcruxes, then he can destroy Voldemort.
If Harry destroys Voldemort, then Voldemort's followers will go into hiding.

Therefore,

Section 2.7 Exercise Solutions

1. a. $[\,(p \to q) \wedge q\,] \to p$
   b.

conclusion $\boldsymbol{p}$	premise 2 $\boldsymbol{q}$	premise 1 $\boldsymbol{p \to q}$	premise 1 AND 2 $\boldsymbol{(p \to q) \wedge q}$	argument $\boldsymbol{[(p \to q) \wedge q\,] \to p}$
T	T	T	T	T
T	F	F	F	T
F	T	T	T	F
F	F	T	F	T

   c. Invalid. Not all final argument truth values are true.

2. a. $[\,(p \to q) \wedge (q \to r)\,] \to (p \to r)$
   b.

$\boldsymbol{p}$	$\boldsymbol{q}$	$\boldsymbol{r}$	premise 1 $\boldsymbol{p \to q}$	premise 2 $\boldsymbol{q \to r}$	premise 1 AND 2 $\boldsymbol{(p \to q) \wedge (q \to r)}$	conclusion $\boldsymbol{p \to r}$	argument *
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	F	T	T
T	F	F	F	T	F	F	T
F	T	T	T	T	T	T	T
F	T	F	T	F	F	T	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

   * $[\,(p \to q) \wedge (q \to r)\,] \to (p \to r)$

   c. Valid. All final argument truth values are true.

3. a. $[\,(p \to \sim q) \wedge (p \vee q)\,] \to \sim q$
   b.

$\boldsymbol{p}$	$\boldsymbol{q}$	conclusion $\boldsymbol{\sim q}$	premise 1 $\boldsymbol{p \to \sim q}$	premise 2 $\boldsymbol{p \vee q}$	premise 1 AND 2 $(\boldsymbol{p \to \sim q}) \wedge (\boldsymbol{p \vee q})$	argument *
T	T	F	F	T	F	T
T	F	T	T	T	T	T
F	T	F	T	T	T	F
F	F	T	T	F	F	T

* $[\,(p \to \sim q) \wedge (p \vee q)\,] \to \sim q$

   c. Invalid. Not all final argument truth values are true.

4. a. $[\,((p \vee q) \to r) \wedge p\,] \to r$
   b.

premise 2 $\boldsymbol{p}$	$\boldsymbol{q}$	conclusion $\boldsymbol{r}$	$(\boldsymbol{p \vee q})$	premise 1 $(\boldsymbol{p \vee q}) \to \boldsymbol{r}$	premise 1 AND 2 *	argument **
T	T	T	T	T	T	T
T	T	F	T	F	F	T
T	F	T	T	T	T	T
T	F	F	T	F	F	T
F	T	T	T	T	F	T
F	T	F	T	F	F	T
F	F	T	F	T	F	T
F	F	F	F	T	F	T

$*$ $[\,((p \vee q) \to r) \wedge p\,]$
$**$ $[\,((p \vee q) \to r) \wedge p\,] \to r$

   c. Valid. All final argument truth values are true.

5. a. $\left[[(p \wedge q) \rightarrow \sim r] \wedge \sim r\right] \rightarrow \sim(p \wedge q)$

b.

p	q	r	$p \wedge q$	premise 2 $\sim r$	premise 1 $(p \wedge q) \rightarrow \sim r$	premise 1 AND 2 $[(p \wedge q) \rightarrow \sim r] \wedge \sim r$	conclusion $\sim(p \wedge q)$	argument *
T	T	T	T	F	F	F	F	T
T	T	F	T	T	T	T	F	F
T	F	T	F	F	T	F	T	T
T	F	F	F	T	T	T	T	T
F	T	T	F	F	T	F	T	T
F	T	F	F	T	T	T	T	T
F	F	T	F	F	T	F	T	T
F	F	F	F	T	T	T	T	T

* $\left[[(p \wedge q) \rightarrow \sim r] \wedge \sim r\right] \rightarrow \sim(p \wedge q)$

c. Invalid. Not all final argument truth values are true.

6. a. $[(p \leftrightarrow q) \wedge \sim q] \rightarrow \sim p$

b.

p	q	conclusion $\sim p$	premise 1 $p \leftrightarrow q$	premise 2 $\sim q$	premise 1 AND 2 $(p \leftrightarrow q) \wedge \sim q$	argument *
T	T	F	T	F	F	T
T	F	F	F	T	F	T
F	T	T	F	F	F	T
F	F	T	T	T	T	T

*$[(p \leftrightarrow q) \wedge \sim q] \rightarrow \sim p$

c. Valid. All final argument truth values are true.

7. a. $[(p \to r) \land (p \land q)] \to r$

b.

$\boldsymbol{p}$	$\boldsymbol{q}$	conclusion $\boldsymbol{r}$	premise 1 $\boldsymbol{p \to r}$	premise 2 $\boldsymbol{p \land q}$	premise 1 AND 2 $\boldsymbol{(p \to r) \land (p \land q)}$	argument *
T	T	T	T	T	T	T
T	T	F	F	T	F	T
T	F	T	T	F	F	T
T	F	F	F	F	F	T
F	T	T	T	F	F	T
F	T	F	T	F	F	T
F	F	T	T	F	F	T
F	F	F	T	F	F	T

*$[(p \to r) \land (p \land q)] \to r$

c. Valid. All final argument truth values are true.

8. a. $\big[[(p \to q) \to r] \land (p \land q)\big] \to r$

b.

$\boldsymbol{p}$	$\boldsymbol{q}$	conclusion $\boldsymbol{r}$	$\boldsymbol{p \to q}$	premise 1 $\boldsymbol{(p \to q) \to r}$	premise 2 $\boldsymbol{p \land q}$	*	argument **
T	T	T	T	T	T	T	T
T	T	F	T	F	T	F	T
T	F	T	F	T	F	F	T
T	F	F	F	T	F	F	T
F	T	T	T	T	F	F	T
F	T	F	T	F	F	F	T
F	F	T	T	T	F	F	T
F	F	F	T	F	F	F	T

* premise 1 AND premise 2 $[(p \to q) \to r] \land (p \land q)$

**$\big[[(p \to q) \to r] \land (p \land q)\big] \to r$

c. Valid. All final argument truth values are true.

9. a.

p	premise 2 q	premise 1 $p \rightarrow q$	conclusion p
T	T	T	T
T	F	F	T
F	T	T	F
F	F	T	F

b. Invalid. Find the row(s) in which all of the premise statements are true. If the conclusion(s) are also true, then the argument is valid.

10. a.

p	q	r	premise 1 $p \rightarrow q$	premise 2 $q \rightarrow r$	conclusion $p \rightarrow r$
T	T	T	T	T	T
T	T	F	T	F	F
T	F	T	F	T	T
T	F	F	F	T	F
F	T	T	T	T	T
F	T	F	T	F	T
F	F	T	T	T	T
F	F	F	T	T	T

b. Valid. Find the row(s) in which all of the premise statements are true. If the conclusion(s) are also true, then the argument is valid.

11. a.

p	q	$\sim q$	premise 1 $p \rightarrow \sim q$	premise 2 $p \vee q$	conclusion $\sim q$
T	T	F	F	T	F
T	F	T	T	T	T
F	T	F	T	T	F
F	F	T	T	F	T

b. Invalid. Find the row(s) in which all of the premise statements are true. If the conclusion(s) are also true, then the argument is valid.

12. a.

p	q	r	$(p \vee q)$	premise 1 $(p \vee q) \rightarrow r$	premise 2 p	conclusion r
T	T	T	T	T	T	T
T	T	F	T	F	T	F
T	F	T	T	T	T	T
T	F	F	T	F	T	F
F	T	T	T	T	F	T
F	T	F	T	F	F	F
F	F	T	F	T	F	T
F	F	F	F	T	F	F

b. Valid. Find the row(s) in which all of the premise statements are true. If the conclusion(s) are also true, then the argument is valid.

13. a.

p	q	r	$p \wedge q$	premise 2 $\sim r$	premise 1 $(p \wedge q) \rightarrow \sim r$	conclusion $\sim(p \wedge q)$
T	T	T	T	F	F	F
T	T	F	T	T	T	F
T	F	T	F	F	T	T
T	F	F	F	T	T	T
F	T	T	F	F	T	T
F	T	F	F	T	T	T
F	F	T	F	F	T	T
F	F	F	F	T	T	T

b. Invalid. Find the row(s) in which all of the premise statements are true. If the conclusion(s) are also true, then the argument is valid.

14. a.

p	q	premise 1 $p \leftrightarrow q$	premise 2 $\sim q$	conclusion $\sim p$
T	T	T	F	F
T	F	F	T	F
F	T	F	F	T
F	F	T	T	T

b. Valid. Find the row(s) in which all of the premise statements are true. If the conclusion is also true, then the argument is valid.

15. a.

p	q	r	premise 1 $p \to r$	premise 2 $p \wedge q$	conclusion r
T	T	T	T	T	T
T	T	F	F	T	F
T	F	T	T	F	T
T	F	F	F	F	F
F	T	T	T	F	T
F	T	F	T	F	F
F	F	T	T	F	T
F	F	F	T	F	F

b. Valid. Find the row(s) in which all of the premise statements are true. If the conclusion(s) are also true, then the argument is valid.

16. a.

p	q	r	$p \to q$	premise 1 $(p \to q) \to r$	premise 2 $p \wedge q$	conclusion r
T	T	T	T	T	T	T
T	T	F	T	F	T	F
T	F	T	F	T	F	T
T	F	F	F	T	F	F
F	T	T	T	T	F	T
F	T	F	T	F	F	F
F	F	T	T	T	F	T
F	F	F	T	F	F	F

b. Valid. Find the row(s) in which all the premise statements are true. If the conclusion(s) are also true, then the argument is valid.

17. a. One possible way to assign letters to simple statements is as follows:

$\boldsymbol{p}$: Sharon is hungry.

$\boldsymbol{q}$: She will eat a snack.

$$\begin{array}{l} p \to q \\ \underline{q \quad\quad} \\ \therefore p \end{array}$$

$$* \; [\,(p \to q) \wedge q\,] \to p$$

b.

conclusion p	premise 2 q	premise 1 $p \to q$	premise 1 AND 2 $(p \to q) \wedge q$	argument $[(p \to q) \wedge q] \to p$
T	T	T	T	T
T	F	F	F	T
F	T	T	T	F
F	F	T	F	T

c. Invalid. Not all final argument truth values are true.

18. a. One possible way to assign letters to simple statements is as follows:

p: It is 9 pm.

q: Bill watches the news.

$$\begin{array}{l} p \to q \\ \sim p \\ \hline \therefore \sim q \end{array}$$

* $[(p \to q) \wedge \sim p] \to \sim q$

b.

p	q	premise 2 $\sim p$	conclusion $\sim q$	premise 1 $p \to q$	premise 1 AND 2 $(p \to q) \wedge \sim p$	argument *
T	T	F	F	T	F	T
T	F	F	T	F	F	T
F	T	T	F	T	T	F
F	F	T	T	T	T	T

c. Invalid. Not all final argument truth values are true.

19. a. One possible way to assign letters to simple statements is as follows:

p: It is January.

q: It snows.

r: Kat makes a snowman.

$$\begin{array}{l} p \to q \\ q \to r \\ \hline \therefore q \wedge r \end{array}$$

* $[(p \to q) \wedge (q \to r)] \to (q \wedge r)$

b.

$\boldsymbol{p}$	$\boldsymbol{q}$	$\boldsymbol{r}$	premise 1 $\boldsymbol{p \to q}$	premise 2 $\boldsymbol{q \to r}$	premise 1 AND 2 $\boldsymbol{(p \to q) \wedge (q \to r)}$	conclusion $\boldsymbol{q \wedge r}$	argument *
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	F	F	T
T	F	F	F	T	F	F	T
F	T	T	T	T	T	T	T
F	T	F	T	F	F	F	T
F	F	T	T	T	T	F	F
F	F	F	T	T	T	F	F

c. Invalid. Not all final argument truth values are true.

20. a. One possible way to assign letters to simple statements is as follows:

$\boldsymbol{p}$: Kadin cooks pumpkin muffins.

$\boldsymbol{q}$: He earns an A in cooking class.

$\boldsymbol{r}$: Kadin cooks blueberry pancakes.

$$\begin{array}{l} p \to q \\ p \vee r \\ \hline \therefore \sim q \end{array}$$

* $[\,(p \to q) \wedge (p \vee r)\,] \to \sim q$

b.

$\boldsymbol{p}$	$\boldsymbol{q}$	$\boldsymbol{r}$	premise 1 $\boldsymbol{p \to q}$	premise 2 $\boldsymbol{p \vee r}$	premise 1 AND 2 $\boldsymbol{(p \to q) \wedge (p \vee r)}$	conclusion $\boldsymbol{\sim q}$	argument *
T	T	T	T	T	T	F	F
T	T	F	T	T	T	F	F
T	F	T	F	T	F	T	T
T	F	F	F	T	F	T	T
F	T	T	T	T	T	F	F
F	T	F	T	F	F	F	T
F	F	T	T	T	T	T	T
F	F	F	T	F	F	T	T

c. Invalid. Not all final argument truth values are true.

21. a. One possible way to assign letters to simple statements is as follows:

$\boldsymbol{p}$: Shayla does not pass biology class.
$\boldsymbol{q}$: She must take summer school.
$\boldsymbol{r}$: Shayla passes math.

$$\begin{array}{l} p \to q \\ \underline{\sim p \wedge r} \\ \therefore \sim q \end{array}$$

* $[\,(p \to q) \wedge (\sim p \wedge r)\,] \to \sim q$

b.

$\boldsymbol{p}$	$\boldsymbol{q}$	$\boldsymbol{r}$	$\sim\boldsymbol{p}$	premise 1 $\boldsymbol{p \to q}$	premise 2 $\sim\boldsymbol{p \wedge r}$	premise 1 AND 2 $(\boldsymbol{p \to q}) \wedge (\sim\boldsymbol{p \wedge r})$	conclusion $\sim\boldsymbol{q}$	argument *
T	T	T	F	T	F	F	F	T
T	T	F	F	T	F	F	F	T
T	F	T	F	F	F	F	T	T
T	F	F	F	F	F	F	T	T
F	T	T	T	T	T	T	F	F
F	T	F	T	T	F	F	F	T
F	F	T	T	T	T	T	T	T
F	F	F	T	T	F	F	T	T

c. Invalid. Not all final argument truth values are true.

22. a. One possible way to assign letters to simple statements is as follows:

$\boldsymbol{p}$: Carson cleans his room.
$\boldsymbol{q}$: Carson does not earn his allowance.
$\boldsymbol{r}$: He helps his brother with his math homework.

$$\begin{array}{l} p \vee q \\ \underline{q \wedge r} \\ \therefore p \end{array}$$

$[\,(p \vee q) \wedge (q \wedge r)\,] \to p$

b.

conclusion $\boldsymbol{p}$	$\boldsymbol{q}$	$\boldsymbol{r}$	premise 1 $\boldsymbol{p \vee q}$	premise 2 $\boldsymbol{q \wedge r}$	premise 1 AND 2 $(\boldsymbol{p \vee q}) \wedge (\boldsymbol{q \wedge r})$	argument $[\,(\boldsymbol{p \vee q}) \wedge (\boldsymbol{q \wedge r})\,] \to \boldsymbol{p}$
T	T	T	T	T	T	T
T	T	F	T	F	F	T
T	F	T	T	F	F	T
T	F	F	T	F	F	T
F	T	T	T	T	T	F
F	T	F	T	F	F	T
F	F	T	F	F	F	T
F	F	F	F	F	F	T

c. Invalid. Not all final argument truth values are true.

23. a. One possible way to assign letters to simple statements is as follows:

$\boldsymbol{p}$: William gets a snack.
$\boldsymbol{q}$: William gets a nap.
$\boldsymbol{r}$: William is cranky.

$$\begin{array}{l} (\sim p \wedge \sim q) \rightarrow r \\ \underline{r \qquad\qquad\qquad} \\ \therefore \sim p \wedge \sim q \end{array}$$

* $\left[[(\sim p \wedge \sim q) \rightarrow r] \wedge r\right] \rightarrow (\sim p \wedge \sim q)$

b.

$\boldsymbol{p}$	$\boldsymbol{q}$	premise 2 $\boldsymbol{r}$	$\sim\boldsymbol{p}$	$\sim\boldsymbol{q}$	conclusion $\sim\boldsymbol{p} \wedge \sim\boldsymbol{q}$	premise 1 $(\sim\boldsymbol{p} \wedge \sim\boldsymbol{q}) \rightarrow \boldsymbol{r}$	premise 1 AND 2 $[(\sim\boldsymbol{p} \wedge \sim\boldsymbol{q}) \rightarrow \boldsymbol{r}] \wedge \boldsymbol{r}$	argument *
T	T	T	F	F	F	T	T	F
T	T	F	F	F	F	T	F	T
T	F	T	F	T	F	T	T	F
T	F	F	F	T	F	T	F	T
F	T	T	T	F	F	T	T	F
F	T	F	T	F	F	T	F	T
F	F	T	T	T	T	T	T	T
F	F	F	T	T	T	F	F	T

c. Invalid. Not all final argument truth values are true.

24. a. One possible way to assign letters to simple statements is as follows:

$\boldsymbol{p}$: I go to Oahu.
$\boldsymbol{q}$: I go to Maui.
$\boldsymbol{r}$: I go to Alaska.

$$\begin{array}{l} (p \wedge q) \rightarrow \sim r \\ \underline{\sim r \qquad\qquad\quad} \\ \therefore \sim p \vee \sim q \end{array}$$

* $\left[[(p \wedge q) \rightarrow \sim r] \wedge \sim r\right] \rightarrow (\sim p \vee \sim q)$

b.

$\boldsymbol{p}$	$\boldsymbol{q}$	$\boldsymbol{r}$	$\sim\boldsymbol{p}$	$\sim\boldsymbol{q}$	premise 2 $\sim\boldsymbol{r}$	$\boldsymbol{p} \wedge \boldsymbol{q}$	premise 1 $(\boldsymbol{p} \wedge \boldsymbol{q}) \rightarrow \sim\boldsymbol{r}$	premise 1 AND 2 $[(\boldsymbol{p} \wedge \boldsymbol{q}) \rightarrow \sim\boldsymbol{r}] \wedge \sim\boldsymbol{r}$	conclusion $\sim\boldsymbol{p} \vee \sim\boldsymbol{q}$	argument *
T	T	T	F	F	F	T	F	F	F	T
T	T	F	F	F	T	T	T	T	F	F
T	F	T	F	T	F	F	T	F	T	T
T	F	F	F	T	T	F	T	T	T	T
F	T	T	T	F	F	F	T	F	T	T
F	T	F	T	F	T	F	T	T	T	T
F	F	T	T	T	F	F	T	F	T	T
F	F	F	T	T	T	F	T	T	T	T

c. Invalid. Not all final argument truth values are true.

25. a. One possible way to assign letters to simple statements is as follows:

$\boldsymbol{p}$: I retire early.
$\boldsymbol{q}$: I have enough savings.
$\boldsymbol{r}$: I'm bored with my job.

$$\begin{array}{l} p \leftrightarrow (q \wedge r) \\ \underline{\sim r \qquad\qquad} \\ \therefore \sim p \end{array}$$

$*\big[\big(p \leftrightarrow (q \wedge r)\big) \wedge \sim r\big] \rightarrow \sim p$

b.

$\boldsymbol{p}$	$\boldsymbol{q}$	$\boldsymbol{r}$	conclusion $\sim\boldsymbol{p}$	$\boldsymbol{q \wedge r}$	premise 1 $\boldsymbol{p \leftrightarrow (q \wedge r)}$	premise 2 $\sim\boldsymbol{r}$	premise 1 AND 2 $\boldsymbol{(p \leftrightarrow (q \wedge r)) \wedge \sim r}$	argument *
T	T	T	F	T	T	F	F	T
T	T	F	F	F	F	T	F	T
T	F	T	F	F	F	F	F	T
T	F	F	F	F	F	T	F	T
F	T	T	T	T	F	F	F	T
F	T	F	T	F	T	T	T	T
F	F	T	T	F	T	F	F	T
F	F	F	T	F	T	T	T	T

c. Valid. All final argument truth values are true.

26. a. One possible way to assign letters to simple statements is as follows:

$\boldsymbol{p}$: I eat salads for lunch for a month.
$\boldsymbol{q}$: I lose ten pounds.

$$\begin{array}{l} p \wedge q \\ \underline{q \qquad\qquad} \\ \therefore p \end{array}$$

$[(\boldsymbol{p} \wedge \boldsymbol{q}) \ \wedge \ \boldsymbol{q}\,] \rightarrow \boldsymbol{p}$

b.

conclusion $\boldsymbol{p}$	premise 2 $\boldsymbol{q}$	premise 1 $\boldsymbol{p \wedge q}$	premise 1 AND 2 $\boldsymbol{(p \wedge q) \wedge q}$	argument $\boldsymbol{[(p \wedge q) \wedge q] \rightarrow p}$
T	T	T	T	T
T	F	F	F	T
F	T	F	F	T
F	F	F	F	T

c. Valid. All final argument truth values are true.

27. a. One possible way to assign letters to simple statements is as follows:

$\boldsymbol{p}$: Sharon is hungry.

$\boldsymbol{q}$: She will eat a snack.

$$\begin{array}{l} p \rightarrow q \\ q \\ \hline \therefore p \end{array}$$

$\boldsymbol{p}$	premise 2 $\boldsymbol{q}$	premise 1 $\boldsymbol{p \rightarrow q}$	conclusion $\boldsymbol{p}$
T	T	T	T
T	F	F	T
F	T	T	F
F	F	T	F

b. Invalid. Find the row(s) in which all of the premise statements are true. If the conclusion(s) are also true, then the argument is valid.

28. a. One possible way to assign letters to simple statements is as follows:

$\boldsymbol{p}$: It is 9 pm.

$\boldsymbol{q}$: Bill watches the news.

$$\begin{array}{l} p \rightarrow q \\ \sim p \\ \hline \therefore \sim q \end{array}$$

$\boldsymbol{p}$	$\boldsymbol{q}$	premise 2 $\boldsymbol{\sim p}$	premise 1 $\boldsymbol{p \rightarrow q}$	conclusion $\boldsymbol{\sim q}$
T	T	F	T	F
T	F	F	F	T
F	T	T	T	F
F	F	T	T	T

b. Invalid. Find the row(s) in which all of the premise statements are true. If the conclusion(s) are also true, then the argument is valid.

29. a. One possible way to assign letters to simple statements is as follows:

p: It is January.
q: It snows.
r: Kat makes a snowman.

$$\begin{array}{l} p \rightarrow q \\ q \rightarrow r \\ \hline \therefore q \wedge r \end{array}$$

p	q	r	premise 1 $p \rightarrow q$	premise 2 $q \rightarrow r$	conclusion $q \wedge r$
T	T	T	T	T	T
T	T	F	T	F	F
T	F	T	F	T	F
T	F	F	F	T	F
F	T	T	T	T	T
F	T	F	T	F	F
F	F	T	T	T	F
F	F	F	T	T	F

b. Invalid. Find the row(s) in which all of the premise statements are true. If the conclusion(s) are also true, then the argument is valid.

30. a. One possible way to assign letters to simple statements is as follows:

p: Kadin cooks pumpkin muffins.
q: He earns an A in cooking class.
r: Kadin cooks blueberry pancakes.

$$\begin{array}{l} p \rightarrow q \\ p \vee r \\ \hline \therefore \sim q \end{array}$$

p	q	r	premise 1 $p \rightarrow q$	premise 2 $p \vee r$	conclusion $\sim q$
T	T	T	T	T	F
T	T	F	T	T	F
T	F	T	F	T	T
T	F	F	F	T	T
F	T	T	T	T	F
F	T	F	T	F	F
F	F	T	T	T	T
F	F	F	T	F	T

b. Invalid. Find the row(s) in which all of the premise statements are true. If the conclusion(s) are also true, then the argument is valid.

31. a. One possible way to assign letters to simple statements is as follows:

p: Shayla does not pass biology class.
q: She must take summer school.
r: Shayla passes math.

$$\begin{array}{l} p \rightarrow q \\ \sim p \wedge r \\ \hline \therefore \sim q \end{array}$$

p	q	r	$\sim p$	premise 1 $p \rightarrow q$	premise 2 $\sim p \wedge r$	conclusion $\sim q$
T	T	T	F	T	F	F
T	T	F	F	T	F	F
T	F	T	F	F	F	T
T	F	F	F	F	F	T
F	T	T	T	T	T	F
F	T	F	T	T	F	F
F	F	T	T	T	T	T
F	F	F	T	T	F	T

b. Invalid. Find the row(s) in which all of the premise statements are true. If the conclusion(s) are also true, then the argument is valid.

32. a. One possible way to assign letters to simple statements is as follows:

p: Carson cleans his room.
q: Carson does not earn his allowance.
r: He helps his brother with his math homework.

$$\begin{array}{l} p \vee q \\ q \wedge r \\ \hline \therefore p \end{array}$$

p	q	r	premise 1 $p \vee q$	premise 2 $q \wedge r$	conclusion p
T	T	T	T	T	T
T	T	F	T	F	T
T	F	T	T	F	T
T	F	F	T	F	T
F	T	T	T	T	F
F	T	F	T	F	F
F	F	T	F	F	F
F	F	F	F	F	F

b. Invalid. Find the row(s) in which all of the premise statements are true. If the conclusion(s) are also true, then the argument is valid.

33. a. One possible way to assign letters to simple statements is as follows:

$\boldsymbol{p}$: William gets a snack.
$\boldsymbol{q}$: William gets a nap.
$\boldsymbol{r}$: William is cranky.

$$\begin{array}{l} (\sim p \wedge \sim q) \rightarrow r \\ r \\ \hline \therefore \sim p \wedge \sim q \end{array}$$

$\boldsymbol{p}$	$\boldsymbol{q}$	premise 2 $\boldsymbol{r}$	$\boldsymbol{\sim p}$	$\boldsymbol{\sim q}$	$\boldsymbol{\sim p \wedge \sim q}$	premise 1 $(\boldsymbol{\sim p \wedge \sim q}) \rightarrow \boldsymbol{r}$	conclusion $\boldsymbol{\sim p \wedge \sim q}$
T	T	T	F	F	F	T	F
T	T	F	F	F	F	T	F
T	F	T	F	T	F	T	F
T	F	F	F	T	F	T	F
F	T	T	T	F	F	T	F
F	T	F	T	F	F	T	F
F	F	T	T	T	T	T	T
F	F	F	T	T	T	F	T

b. Invalid. Find the row(s) in which all of the premise statements are true. If the conclusion(s) are also true, then the argument is valid.

34. a. One possible way to assign letters to simple statements is as follows:

$\boldsymbol{p}$: I go to Oahu.
$\boldsymbol{q}$: I go to Maui.
$\boldsymbol{r}$: I go to Alaska.

$$\begin{array}{l} (p \wedge q) \rightarrow \sim r \\ \sim r \\ \hline \therefore \sim p \vee \sim q \end{array}$$

$\boldsymbol{p}$	$\boldsymbol{q}$	$\boldsymbol{r}$	$\boldsymbol{\sim p}$	$\boldsymbol{\sim q}$	premise 2 $\boldsymbol{\sim r}$	$\boldsymbol{p \wedge q}$	premise 1 $(\boldsymbol{p \wedge q}) \rightarrow \boldsymbol{\sim r}$	conclusion $\boldsymbol{\sim p \vee \sim q}$
T	T	T	F	F	F	T	F	F
T	T	F	F	F	T	T	T	F
T	F	T	F	T	F	F	T	T
T	F	F	F	T	T	F	T	T
F	T	T	T	F	F	F	T	T
F	T	F	T	F	T	F	T	T
F	F	T	T	T	F	F	T	T
F	F	F	T	T	T	F	T	T

b. Invalid. Find the row(s) in which all of the premise statements are true. If the conclusion(s) are also true, then the argument is valid.

35. a. One possible way to assign letters to simple statements is as follows:

$\boldsymbol{p}$: I retire early.
$\boldsymbol{q}$: I have enough savings.
$\boldsymbol{r}$: I'm bored with my job.

$$\begin{array}{l} p \leftrightarrow (q \wedge r) \\ \underline{\sim r \qquad\quad} \\ \therefore \sim p \end{array}$$

$\boldsymbol{p}$	$\boldsymbol{q}$	$\boldsymbol{r}$	$\boldsymbol{q \wedge r}$	premise 1 $\boldsymbol{p \leftrightarrow (q \wedge r)}$	premise 2 $\boldsymbol{\sim r}$	conclusion $\boldsymbol{\sim p}$
T	T	T	T	T	F	F
T	T	F	F	F	T	F
T	F	T	F	F	F	F
T	F	F	F	F	T	F
F	T	T	T	F	F	T
F	T	F	F	T	T	T
F	F	T	F	T	F	T
F	F	F	F	T	T	T

b. Valid. Find the row(s) in which all of the premise statements are true. If the conclusion(s) are also true, then the argument is valid.

36. a. One possible way to assign letters to simple statements is as follows:

$\boldsymbol{p}$: I eat salads for lunch for a month.
$\boldsymbol{q}$: I lose ten pounds.

$$\begin{array}{l} p \wedge q \\ \underline{q \qquad\quad} \\ \therefore p \end{array}$$

conclusion $\boldsymbol{p}$	premise 2 $\boldsymbol{q}$	premise 1 $\boldsymbol{p \wedge q}$	conclusion $\boldsymbol{p}$
T	T	T	T
T	F	F	T
F	T	F	F
F	F	F	F

b. Valid. Find the row(s) in which all of the premise statements are true. If the conclusion(s) are also true, then the argument is valid.

37. Fallacy of the inverse. The argument is invalid.

38. Disjunctive reasoning. The argument is valid.

39. Misuse of disjunctive reasoning. The argument is invalid.

40. Transitive reasoning. The argument is valid.
41. Direct reasoning. The argument is valid.
42. Misuse of transitive reasoning. The argument is invalid.
43. Contrapositive reasoning. The argument is valid.
44. Fallacy of the converse. The argument is invalid.
45. Disjunctive reasoning. This argument is valid.
46. Fallacy of the inverse. This argument is invalid.
47. If you watch too much television, then you will not be prepared for the exam.
48. Drake lifted weights.
49. Jasmine did not become a journalist.
50. Bella will graduate from college.
51. I go skydiving.
52. If Harry finds all the Horcruxes, then Voldemort's followers will go into hiding.

Section 2.8 | Syllogisms and Euler Diagrams

Objectives

- Define syllogism
- Draw Euler diagrams to represent a syllogism
- Determine the validity of a syllogism

Syllogisms are like the arguments studied in the previous section. An argument contains multiple statements or conditionals strung together and can either be valid (logical) or invalid (illogical). Syllogisms are similar, but syllogisms don't contain connectives and may contain quantifiers in at least one statement. Recall there are two types of quantifiers: existential and universal. Existential quantifiers state "there exists" where universal quantifiers state that a condition holds "for all" or "for none."

Syllogisms

DEFINITION: A **syllogism** consists of two or more premises followed by a conclusion. Typically, a syllogism contains at least one quantifier.

A syllogism takes the form:

All A (or some A or no A) are B.
There is an object from group A (or not from group A).
Therefore, A is (or is not) B.

An example of a syllogism is:

Some math professors are a little socially awkward.
Prof. Jackie is a math professor.
Therefore, Prof. Jackie is a little socially awkward.

➢ **EXAMPLE 2.1.1:** Are the following strings of statements arguments or syllogisms? Why?

a. All roses are red. I have a rose. Therefore, my rose is red.
b. It is not raining outside. When it rains, my joints swell and ache. Therefore, my joints aren't swollen and achy.
c. "…all men are created equal, that they are endowed by their creator with certain unalienable Rights…That to secure these Rights, Governments are instituted among Men…That when whenever any form of Government becomes destructive of these ends, it is the Right of the People to alter or abolish it, and to institute new Government..."[3]

SOLUTION:

a. This is a syllogism. The first statement contains the quantifier ***all***. Further, these statements follow the pattern of a syllogism.

b. This is not a syllogism. This is an argument. There is no quantifier, and the statements don't follow the pattern of a syllogism. (This happens to be an invalid argument.)

c. This piece of the Declaration of Independence is an argument. Even though the first statement contains the quantifier ***all***, the rest of the statements do not follow the pattern of a syllogism. (This happens to be a valid argument.)

Like an argument, a syllogism can be valid or invalid. To show a syllogism is valid:

- Assume all the premises are true.
- Show that the conclusion must be true.

To show a syllogism is invalid:

- Assume all the premises are true.
- Show that the conclusion can be false.

However, unlike an argument, a truth table cannot be used to prove a syllogism valid or invalid.

Instead, Euler diagrams (as seen in Section 2.1) are used to prove the validity of syllogisms.

---

[3] From the Declaration of Independence. This is the logic the founding fathers used to justify their separation from England.

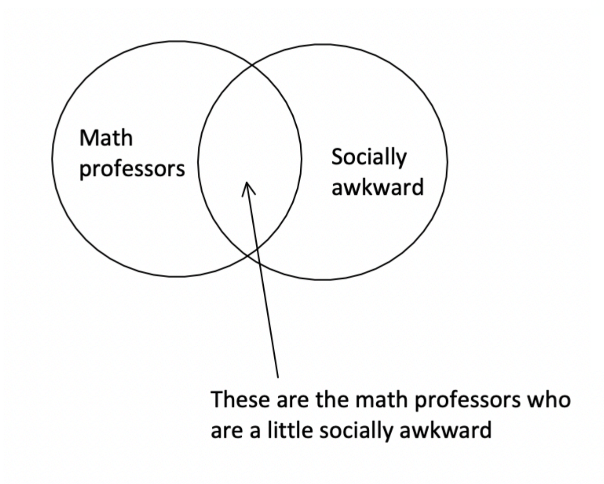
Euler Diagrams to Represent Statements

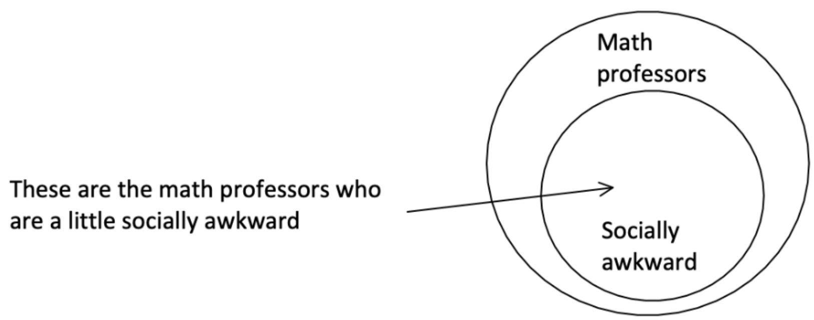
From Section 2.1, recall that an **Euler diagram** is a visual representation used to see the relationship between objects and certain characteristics. An Euler diagram is like a Venn diagram.

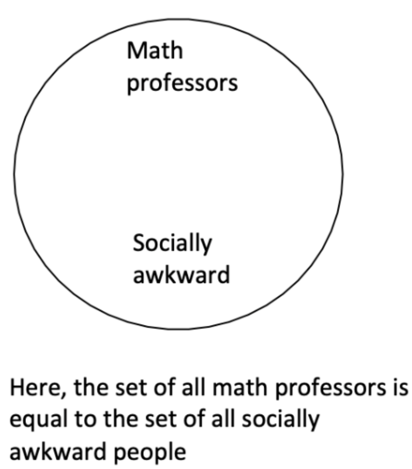
Sometimes, one Euler diagram will represent all possible scenarios of a statement. Other times, multiple Euler diagrams are required to show all scenarios.

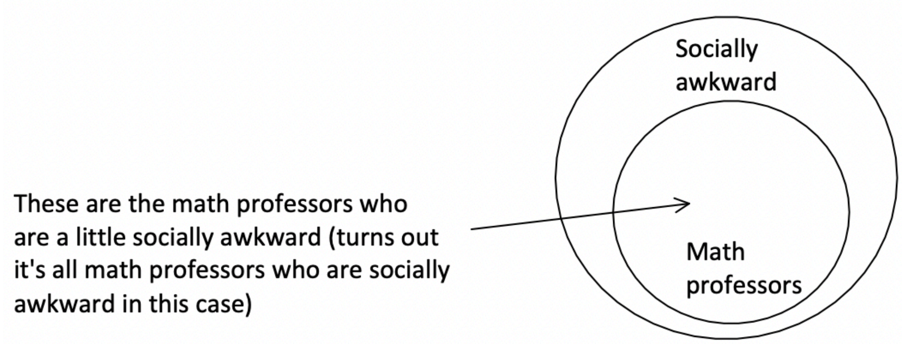
For example, consider the statement "Some math professors are a little socially awkward."

As an Euler diagram, that statement could look like:





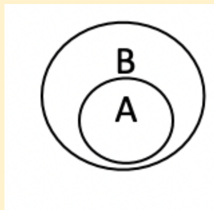
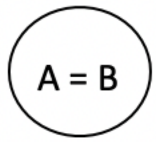
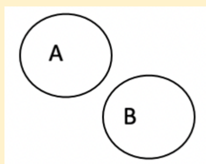
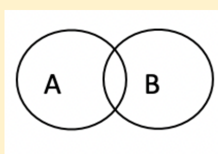
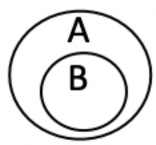
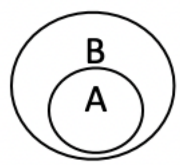
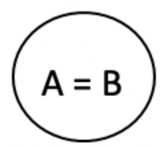
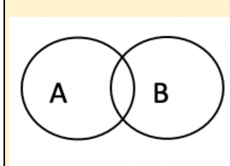
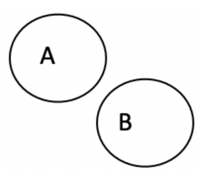
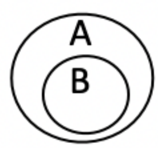




The statement "Some math professors are a little socially awkward" takes the form "Some A are B." Thus, any time an Euler diagram is needed for a "Some A are B" statement all four diagrams could potentially represent that statement.

Statements such as "All A are B" and "No A are B" and "Some A are not B" will have their own set of Euler diagrams. See Figure 2.8.1.

FIGURE 2.8.1

Statement	Possible Euler Diagrams			
All A are B				
No A are B				
Some A are B				
Some A are not B				

For the purposes of this textbook, only the first Euler diagram (highlighted) in each row will be used to represent all the possible Euler diagrams for the quantified statement in that row. This is possible because the other diagrams are special cases that can be represented using the highlighted Euler diagram. Here the regions not necessary would be empty.

For example, for the quantified statement "All A are B" if set A equals set B, then all elements in set A would be in set B. In the special case where $A = B$, there would be no elements outside in set $B - A$. All elements would be in set A.

If given statements about 3 qualities, such as "Some A are B" and "All C are B," then there are many more possible Euler diagrams to consider. The above table can be used as a guide when drawing Euler diagrams for 3 or more qualities.

❖ **THINK ABOUT IT:** Does the statement "All A are B" imply that "Some A are B?" Does the statement "Some A are B" imply that "All A are B?" Explain.

ANSWER: The statement "All A are B" implies that "Some A are B." This can be seen when considering the possible cases as Euler diagrams; the two possible cases of "All A are B" are also cases of "Some A are B." Consider the statement "The sun rises every day." This implies that the sun rises on some days.

"Some A are B" does not imply "All A are B." Consider the statement "Some days in January it snows." This does not imply that "All days in January it snows."

That is, it is correct to draw an Euler diagram for "All A are B" when requested to draw an Euler diagram for "Some A are B." However, the converse is not true; it is not correct to draw an Euler diagram for "Some A are B" when asked for an Euler diagram for "All A are B."

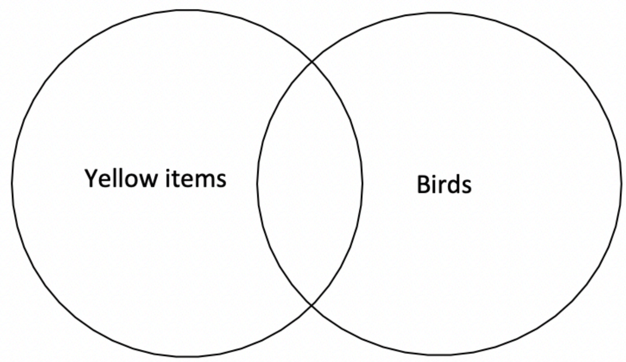
➢ **EXAMPLE 2.8.1:** Draw all possible Euler diagrams to represent the statement. (Remember, the first column of figure 2.8.1 will be used to represent all possible Euler diagrams.)

a. Some birds are yellow.
b. All presidents of the United States have publicly identified as men.
c. No presidents of the United States have publicly identified as women (thus far).
d. All mathematicians like patterns. Some historians like patterns.

SOLUTION:

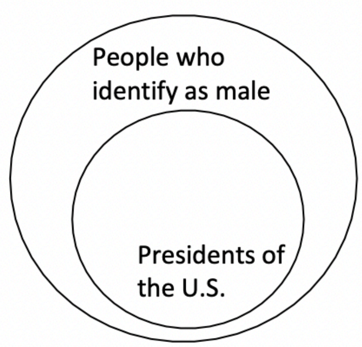
a. Some birds are yellow.

This statement takes the form "Some A are B." In this case, set A is the set of all yellow birds. Set B is the set of yellow items.

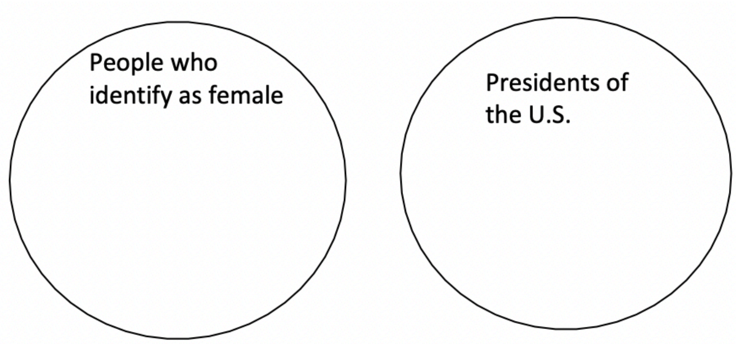


b. All presidents of the United States have publicly identified as male.

This statement takes the form “All A are B.” Set A in this case is the set of all presidents of the U.S. Set B is the set of people who publicly identify as male.



c. No presidents of the United States have publicly identified as women (thus far).



d. All mathematicians like patterns. Some historians like patterns.

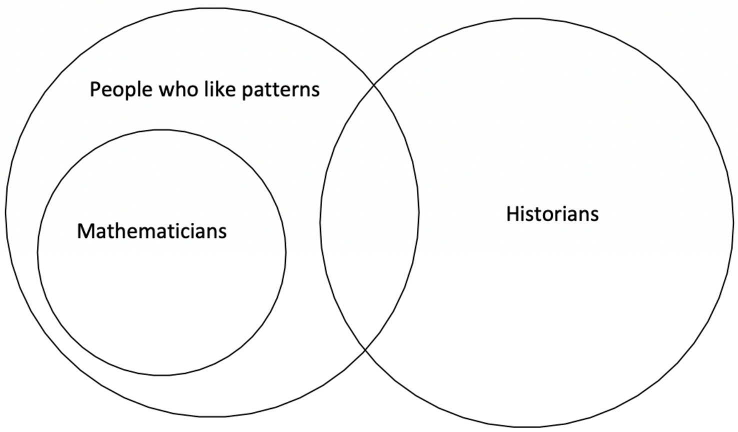


Diagram 1

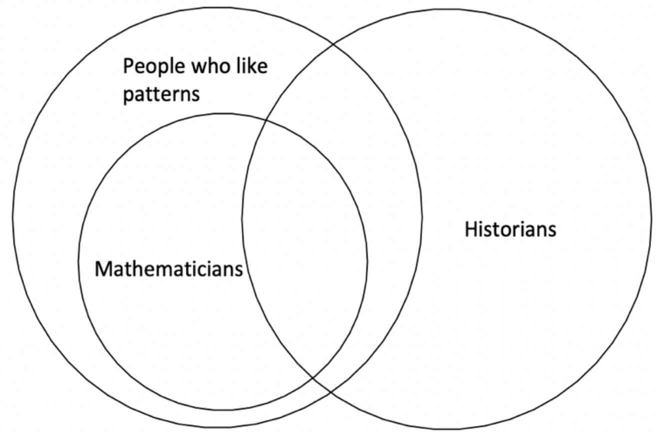


Diagram 2

It is important to point out the difference between Diagram 1 and Diagram 2. In Diagram 1, there is no overlap between historians and mathematicians. That is, no historians are mathematicians. In Diagram 2, there is overlap between mathematicians and historians. That is, some historians are mathematicians.

Proving a Syllogism is Valid or Invalid

Before discussing how to prove a syllogism valid or invalid, one more definition is needed.

DEFINITION: A **counterexample** is a case that shows the conclusion is not true for the given premises.

To prove a syllogism valid or invalid, start by assuming all premises are true and see if the conclusion must be true or if the conclusion could be false.

To prove a syllogism valid:
- Draw all possible Euler diagrams for the premises.
- Show that the conclusion is true for all possible Euler diagrams.

To prove a syllogism invalid:

- Draw an Euler diagram for the premises.
- Show that the conclusion is false for at least one Euler diagram. Such a diagram is called a counterexample.

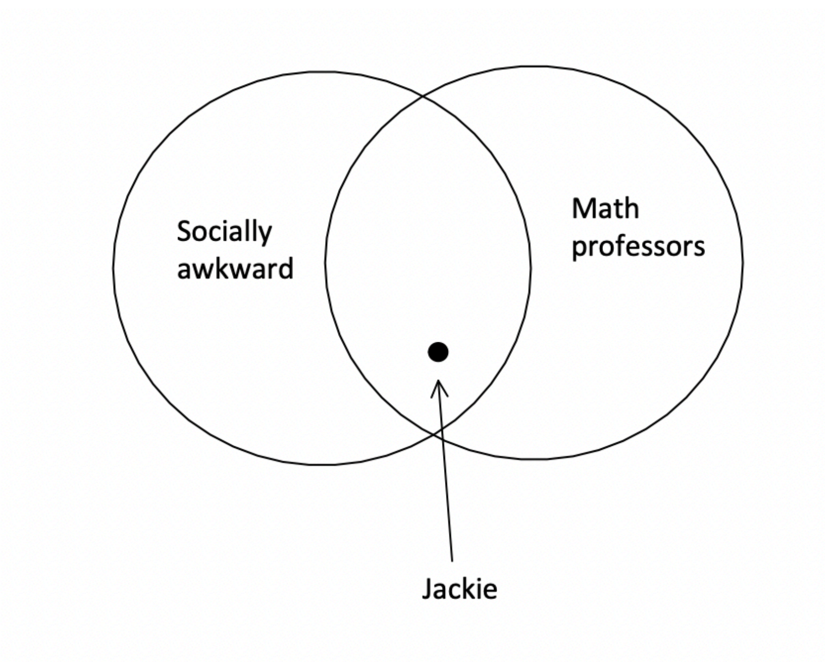
➢ **EXAMPLE 2.8.3:** Prove the following syllogisms are valid or invalid.

a. Some math professors are a little socially awkward. Jackie is a math professor. Therefore, Jackie is a little socially awkward.
b. No presidents of the U.S. have identified as female (thus far). Hillary Clinton identifies as female. Therefore, Hillary Clinton has not been president of the U.S.
c. At least one dog does not have four legs. My animal Zoey has three legs. Therefore, Zoey is a dog.
d. All mathematicians like patterns. Some historians like patterns. Joe is a mathematician. Therefore, Joe is a historian.
e. All mathematicians like patterns. Some mathematicians are historians. Therefore, some historians like patterns.

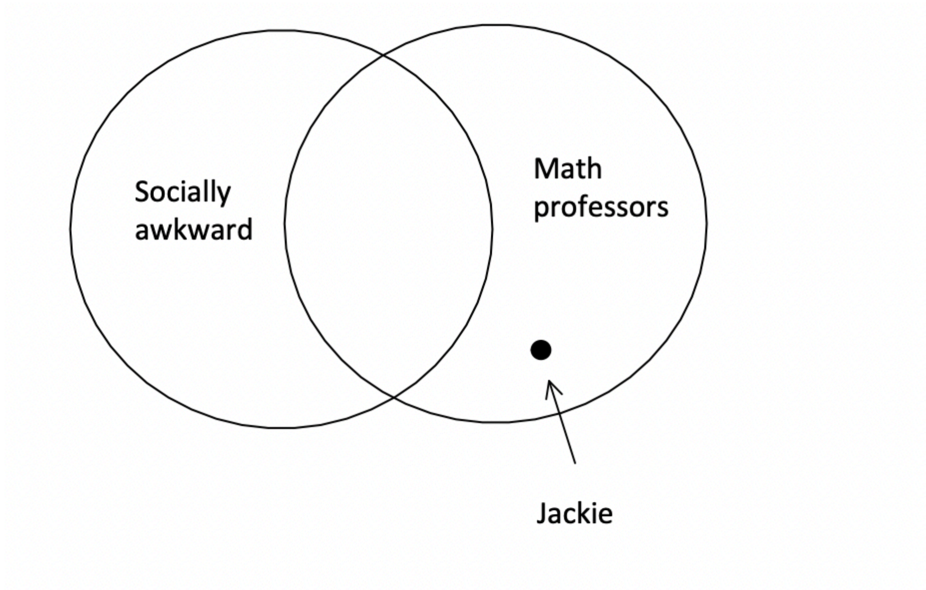
SOLUTION:

a. Some math professors are a little socially awkward. Jackie is a math professor. Therefore, Jackie is a little socially awkward.

Here, it is important that all possible regions in which to place Jackie are considered. Consider the following Euler diagrams.



In the Euler diagram above, Jackie is socially awkward and a math professor. In this diagram, both the premises and the conclusion are true. However, all possible regions in which to place Jackie must be considered too.

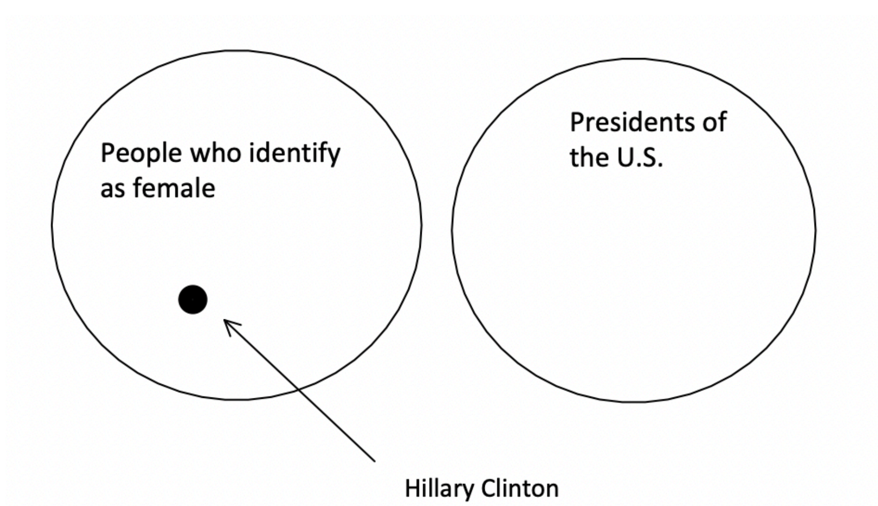


The above Euler diagram shows that the premises can be true and the conclusion false. That is, Jackie can be a math professor who is not socially awkward. Therefore, the conclusion of the syllogism is false, and the syllogism is invalid.

NOTE: A syllogism is always invalid if the conclusion is false for at least one case, even if the conclusion is true for one or more cases.

b. No president of the U.S. has identified as female (thus far). Hillary Clinton identifies as female. Therefore, Hillary Clinton has not been president of the U.S.

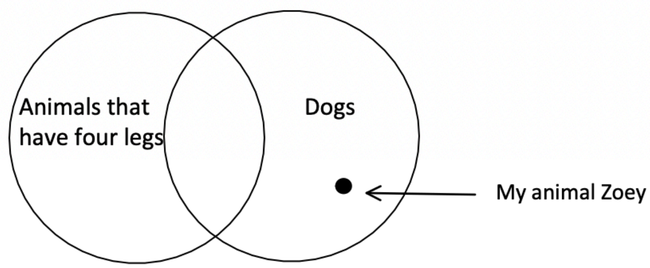
Since Hillary Clinton identifies as female, there is only one location to place her name.



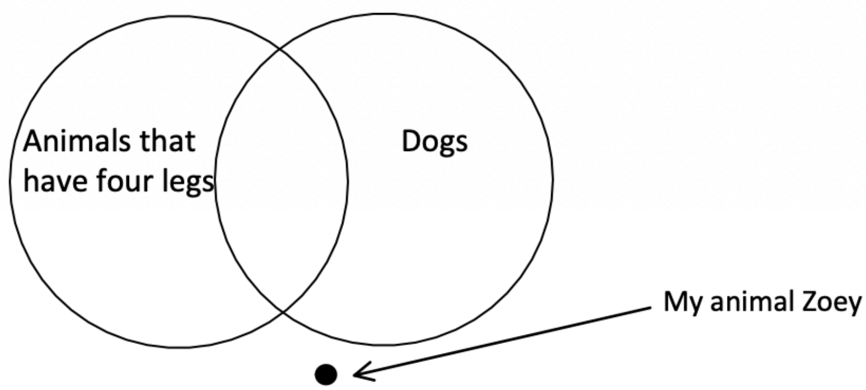
This syllogism is valid. No matter the location of Hillary (there is only one region in which to place Hillary), the conclusion is true whenever the premises are true.

c. At least one dog does not have four legs. My animal Zoey has three legs. Therefore, Zoey is a dog.

The statement "At least one A is not B" is the same as "Not all A are B" or "Some A are not B."



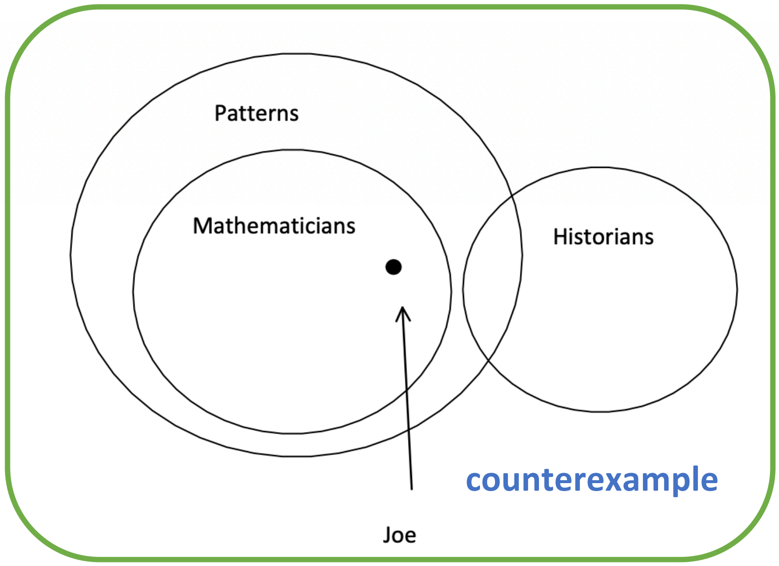
This Euler diagram suggests that this syllogism is true. However, all possible locations of Zoey must be considered before the syllogism is proven valid or invalid. Consider this Euler diagram:

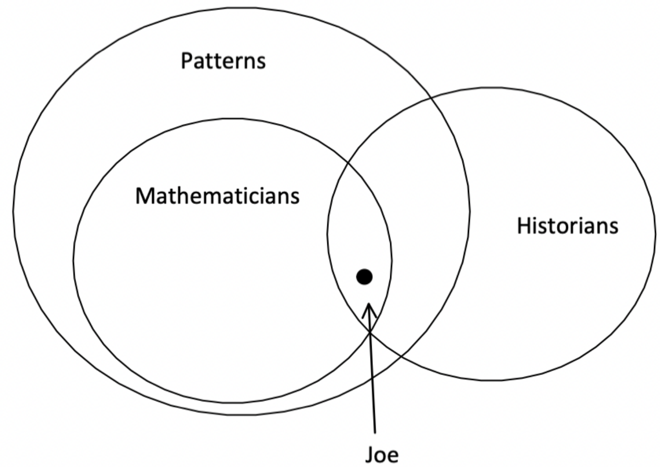


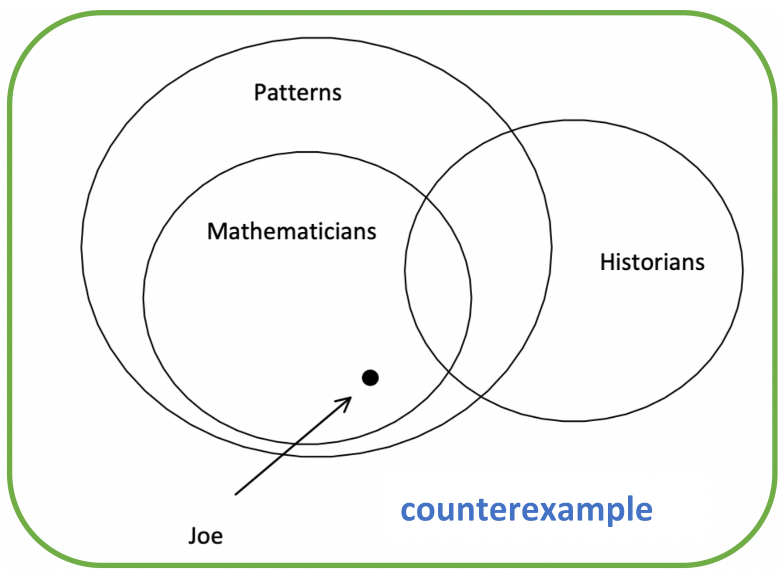
This Euler diagram shows that the syllogism is invalid. (For instance, what if Zoey is a cat with three legs?)

d. All mathematicians like patterns. Some historians like patterns. Joe is a mathematician. Therefore, Joe is a historian.

Consider a few possible Euler diagrams and locations of Joe:



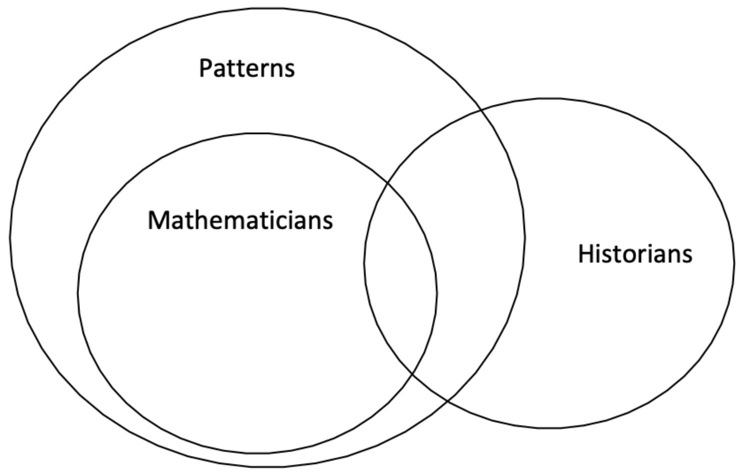




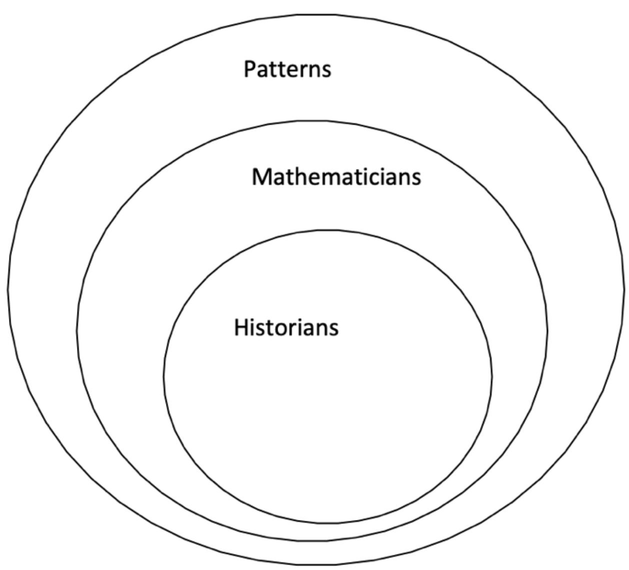
Either of the two Euler diagrams in a green box can prove this syllogism is invalid. In both diagrams, Joe is not a historian, which shows the conclusion can be false.

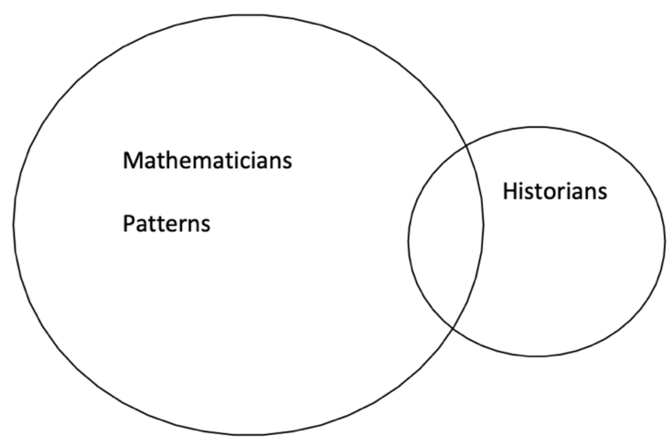
e. All mathematicians like patterns. Some mathematicians are historians. Therefore, some historians like patterns.

Consider the following Euler diagram:



In this diagram there is some overlap between historians and people who like patterns. This implies that some historians like patterns. To prove a syllogism valid, however, it is important to also consider some of the special cases to verify that the conclusion is always true. Consider the following two special cases:





While this is not an exhaustive list of all Euler diagrams, this is enough to provide evidence that the conclusion is always true (there is always overlap between historians and patterns) and therefore the syllogism is valid.

❖ **YOU TRY 2.8 A:** Prove the following syllogisms valid or invalid.

- **a.** All presidents of the U.S. have identified as men (thus far). Justin Timberlake identifies as male. Thus, Justin Timberlake was president of the United States.
- **b.** Some birds chirp when spring is here. There are birds chirping right now. Therefore, spring is here.
- **c.** All roses are red. I have a rose. Therefore, my rose is red.
- **d.** All newspaper editors have a college degree. Some newspaper editors work at the Chicago Tribune. Therefore, some people who work at the Chicago Tribune have a college degree.

Completing a Syllogism with a Conclusion

The question could be posed "What is a valid conclusion given this set of premises?"

To answer a question such as that, draw multiple Euler diagrams representing the cases where all premises are true. Examine all diagrams to see what is common among them. The common characteristic among them is the conclusion, which will, by design, be true in all cases.

➢ **EXAMPLE 2.8.4:** Given the following set of premises, identify the conclusion that creates a valid syllogism.

- **a.** All country music groups have a guitar player. Some country music groups have a harmonica player.

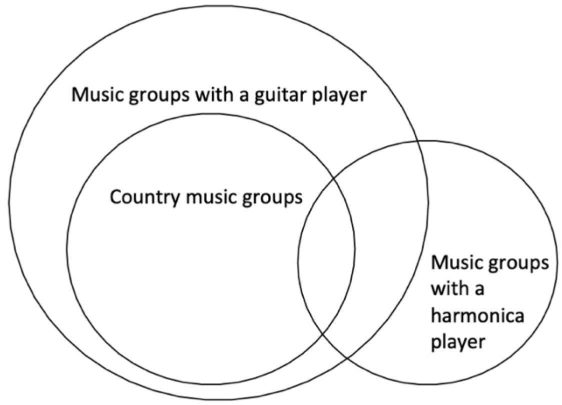
  What is the valid conclusion?
  1. All music groups with a guitar player have a harmonica player.
  2. Some music groups with a guitar player have a harmonica player.
  3. All country music groups have a harmonica player.
  4. Some country music groups do not have a harmonica player.

- **b.** All rainy days have clouds. On all days with clouds, I wear rain boots. It rains some days in spring.

  What is the valid conclusion?
  1. Some days in spring it does not rain.
  2. Some days in spring I do not wear my rain boots.
  3. Some spring days there are no clouds.
  4. Some days in spring I wear my rain boots.
  5. Some rainy days are not spring days (i.e. some rainy days happen in the summer or fall, for example).

SOLUTION:

a. Start by drawing an Euler diagram in which all premises are true.



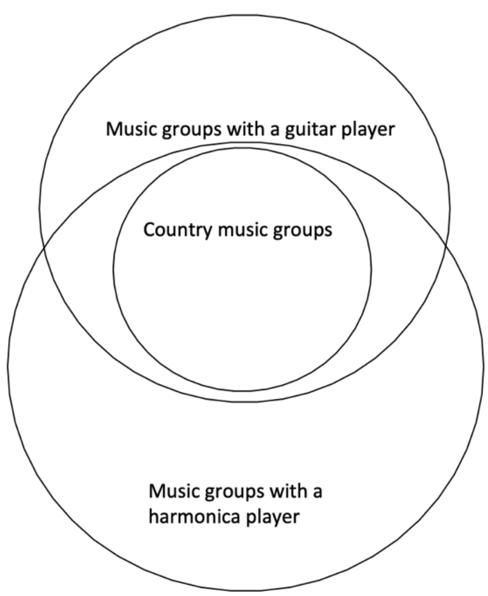
For a conclusion to create a valid syllogism, the conclusion must also be true in the Euler diagram drawn.

This Euler diagram proves conclusion 1 false, since there are some music groups with a guitar player without a harmonica player.

This Euler diagram also proves conclusion 3 false, since there are some county music groups without a harmonica player.

Conclusions 2 and 4 are true in the above Euler diagram. There are some country music groups with and some groups without a harmonica player. However, would conclusions 2 or 4 always be true?

Draw another Euler diagram in which all premises are true and see if it is possible to draw the diagram such that conclusion 2 or 4 is false.

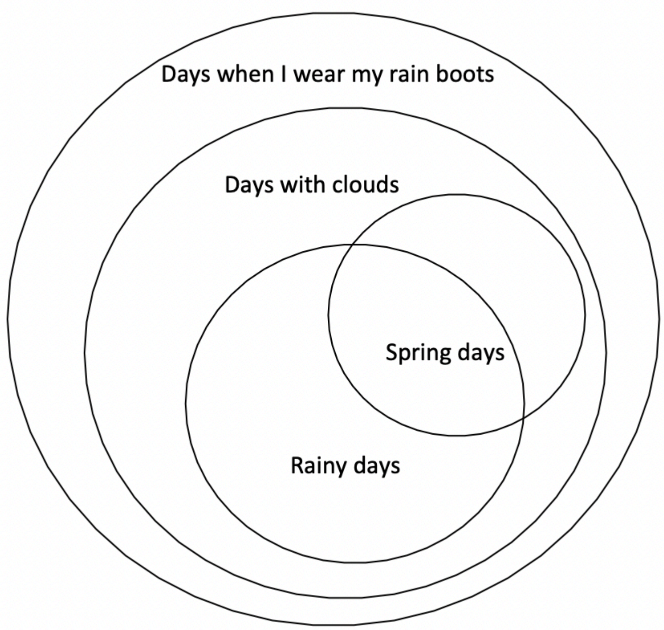


In this diagram, premise 1 and premise 2 are both true. However, all country music groups have a harmonica player in this diagram. This shows that conclusion 4 can be false. Hence conclusion 4 is not a valid conclusion.

Conclusion 2 is the conclusion that makes this syllogism valid. In both options shown, there are some music groups with a guitar player that have a harmonica player (namely, the country music groups).

NOTE: For a conclusion to make a syllogism valid, it would have to be true in every possible Euler diagram in which the premises are true. For conciseness, not every Euler diagram will be shown here.

b. Start by drawing an Euler diagram in which all premises are true.

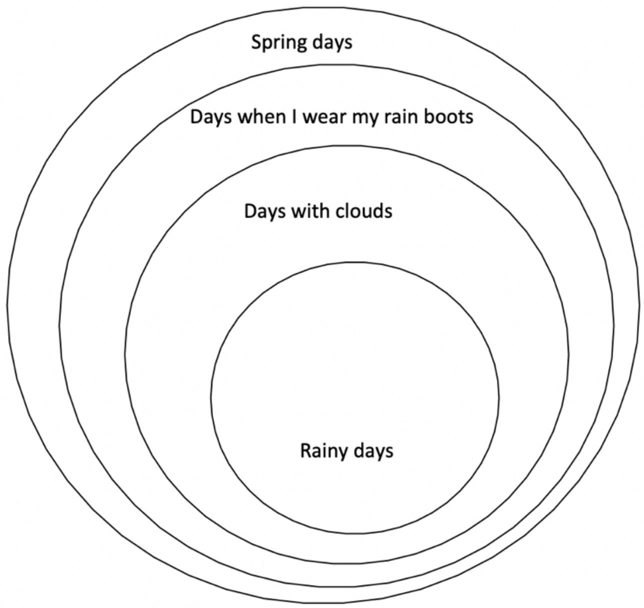


For a conclusion to create a valid syllogism, it must be true whenever the premises are true.

Conclusions 1, 4, and 5 are true in the above diagram. Some spring days it does not rain, some days in spring I wear my rain boots, and some rainy days are not spring days.

Conclusions 2 and 3 are false in the above diagram. In the above diagram, on all spring days I wear my rain boots and on all spring days there are clouds.

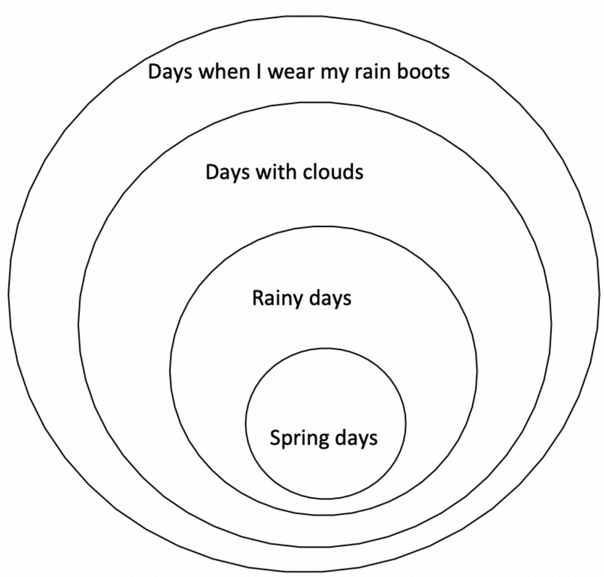
However, must conclusions 1, 4, and 5 always be true, no matter what the Euler diagram where the premises are true looks like? Draw another Euler diagram in which all premises are true. Consider this diagram:



In this diagram, all premises are true. Also, conclusions 1 and 4 are true. That is, some days in spring it does not rain and some days in spring I wear my rain boots.

Conclusion 5 is false because all rainy days are spring days.

Consider one additional diagram:



In this diagram, all premises are true. Conclusion 1 is false because on all spring days it rains.

Notice in all three diagrams shown conclusion 4 is true. Thus conclusion 4 is the conclusion that makes a valid syllogism.

- **YOU TRY 2.8B:** Given the following set of premises, identify the conclusion that creates a valid syllogism.

  a. All almonds have healthy fats. Some almonds are roasted.

  What is the valid conclusion?
  1. All roasted nuts have healthy fats.
  2. Some roasted nuts do not have healthy fats.
  3. Some roasted nuts have healthy fats.
  4. No roasted nuts have healthy fats.

  b. All cinnamon rolls are a form of enriched bread. All enriched breads are a form of bread. Some cinnamon rolls are baked as a tear-and-share. (This question is inspired by the Great British Baking Show.)

  What is the valid conclusion?
  1. Some tear-and-shares are not bread.
  2. Some tear-and-shares are not enriched bread.
  3. Some tear-and-shares are not cinnamon rolls.
  4. Some tear-and-shares are bread.

Quick Review

Syllogism – consists of two or more premises followed by a conclusion. Typically, a syllogism contains at least one quantifier.

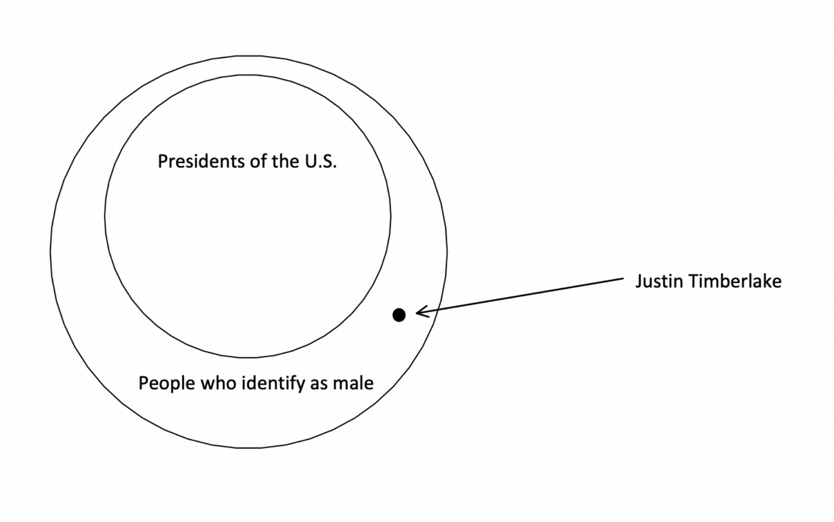
Counterexample – a case that shows the conclusion is not true for the given premises.

Assuming all premises are true, a syllogism:
- Is valid if the conclusion must be true.
- Is invalid if the conclusion can be false.

YOU TRY IT 2.8A SOLUTION:

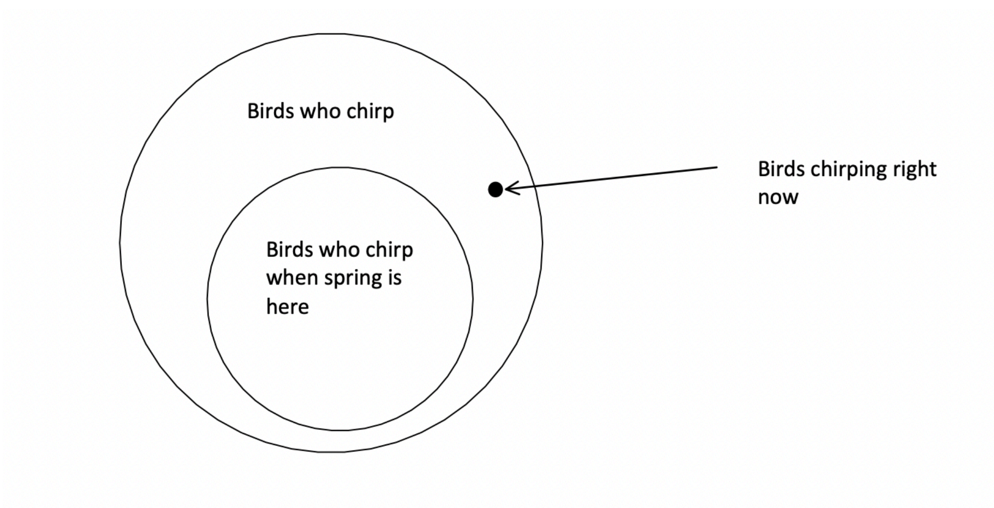
a. All presidents of the U.S. have publicly identified as male. Justin Timberlake identifies as male. Thus, Justin Timberlake was president of the United States.

This syllogism is false. Consider the following diagram that shows all presidents can publicly identify as male, Justin can identify as male, and Justin doesn't have to be a president.

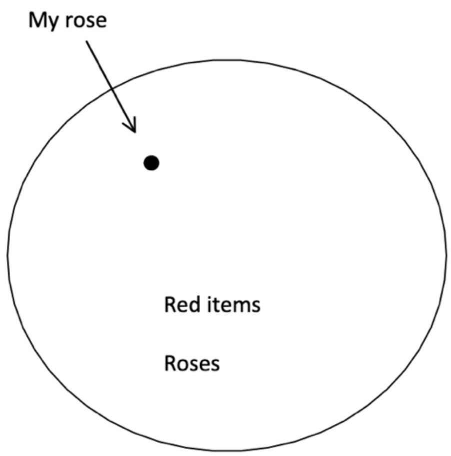


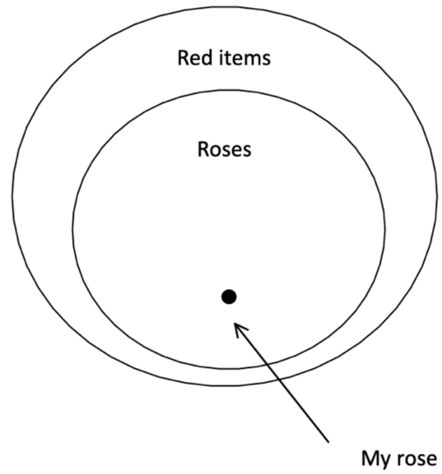
b. Some birds chirp when spring is here. A bird is chirping right now. Therefore, spring is here.

This syllogism is false. Consider the following diagram that shows the birds who chirp when spring is here. The chirping bird doesn't have to be that kind; therefore, spring doesn't have to be here.



c. All roses are red. I have a rose. Therefore, my rose is red.



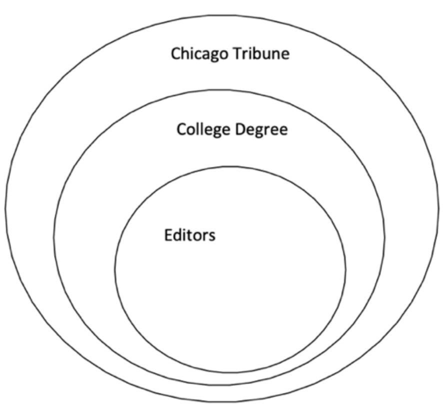


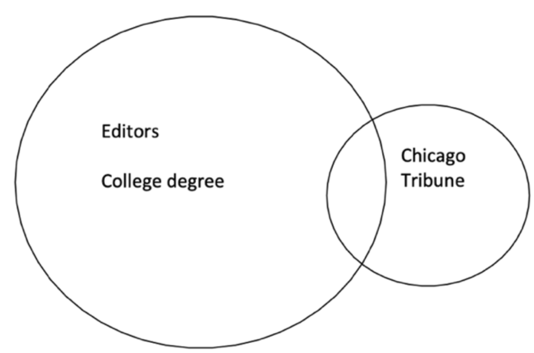
This is a valid syllogism. The conclusion is true whenever all premises are true.

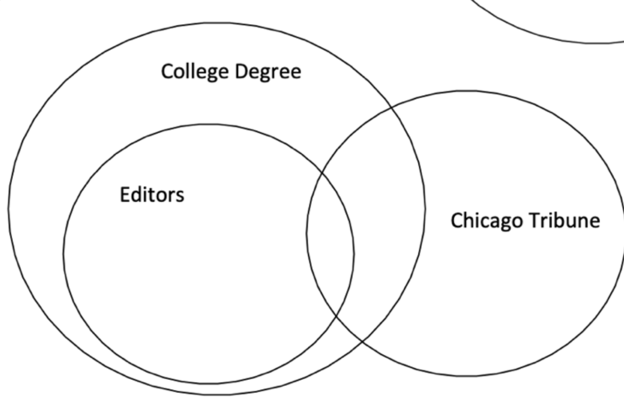
d. All newspaper editors have a college degree. Some newspaper editors work at the Chicago Tribune. Therefore, some people who work at the Chicago Tribune have a college degree.

Consider the following Euler diagrams. While this is not an exhaustive list of diagrams, it is enough to give evidence for the validity of this syllogism.

Further, however the circles are arranged, because all editors have a college degree and some editors work at the Chicago Tribune, it is true that some staff at the Chicago Tribune will have a college degree (namely, the editors).

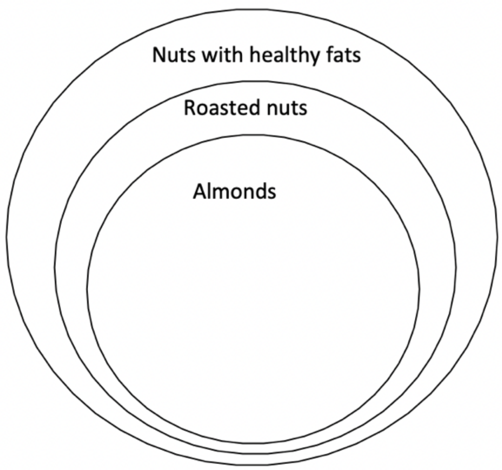


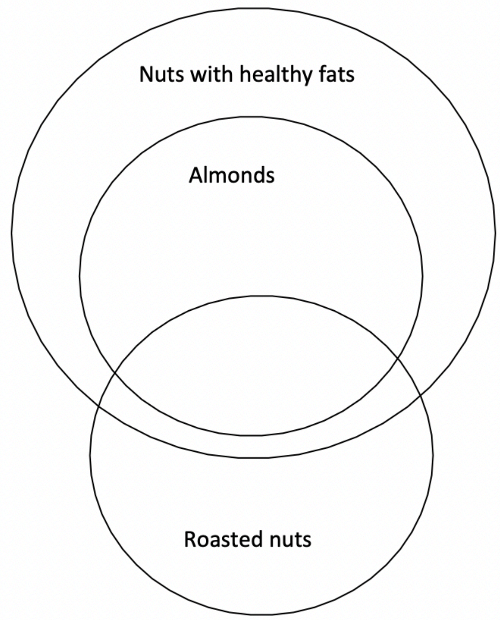




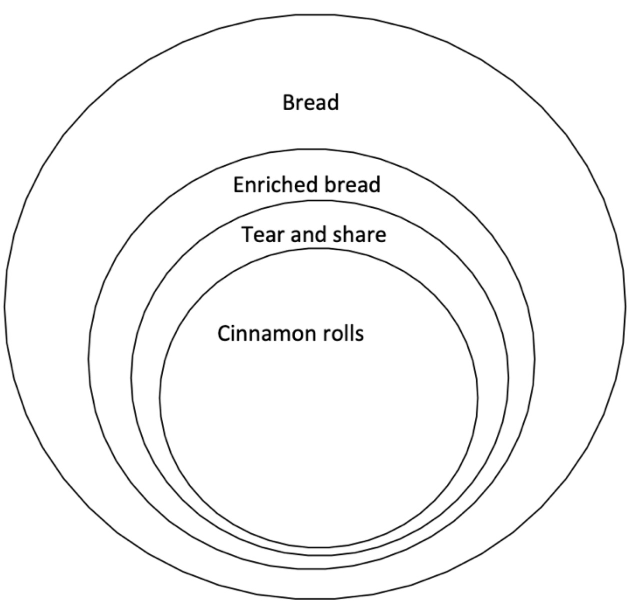
YOU TRY IT 2.8B SOLUTION:

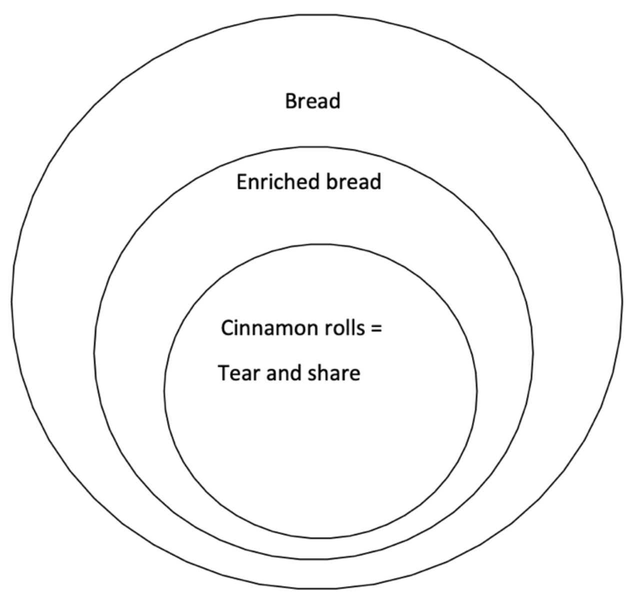
a. Conclusion 3 is the valid conclusion.





b. Conclusion 4 is the valid conclusion.





Section 2.8 Exercises

For exercises 1-2, state if the following strings of statements are syllogisms or arguments. Explain why.

1. My upstairs neighbors are loud or I don't sleep. My upstairs neighbors aren't loud. Therefore, I sleep.
2. All dogs go to heaven. Maggie is a dog. Therefore, Maggie goes to heaven.

For exercises 3-5, draw all possible Euler diagrams that represent the statement(s). (Here column 1 of Figure 2.8.1 will be used.)

3. Some gyms have a pool.
4. All gyms have treadmills. Some gyms have ellipticals.
5. No gyms serve donuts. All gyms have water fountains.

For exercises 6-20, prove the syllogism valid or invalid.

6. All gyms have treadmills. Some gyms have ellipticals. Therefore, my gym has an elliptical.
7. All gyms have treadmills. I work out in my basement (which isn't a gym). Therefore, my basement doesn't have a treadmill.
8. All gyms have treadmills. Where I work out doesn't have a treadmill. Therefore, I don't work out in a gym.
9. No person who eats chicken is a vegetarian. Brent eats chicken. Therefore, Brent is not a vegetarian.
10. No vegetarian eats chicken. Haley doesn't eat chicken. Therefore, Haley is a vegetarian.
11. No vegans consume animal products. Some vegans consume almond milk. Therefore, some people who consume animal products do not consume almond milk.
12. All soccer players wear shin guards. Ondro is a soccer player. Therefore, Ondro wears shin guards.
13. Some flowers are yellow. Daisies are flowers. Therefore, daisies are yellow.
14. No sharks have bones. The blacktip shark is a medium sized shark species. Therefore, blacktip sharks do not have bones.
15. All rectangles have four corners. All squares are rectangles. Therefore, all squares have four corners.

16. All birds have feathers. Some birds can fly. Penguins are birds. Therefore, penguins can fly.
17. All Elasmobranchs have ampullae of Lorenzini (special receptor organs for sensing electric fields). Sharks are Elasmobranchs. Stingrays are Elasmobranchs. Therefore, all stingrays are sharks.
18. All mathematicians are logical people. Some mathematicians are not social people. Therefore, some logical people are not social people.
19. All mathematicians are logical. Some athletes are mathematicians. John Urschel is both a mathematician and an athlete. Therefore, John Urschel is logical.
20. Some mathematicians have a Ph.D. No mathematician is illogical. Chris has a Ph.D. Therefore, Chris is not illogical.

For exercises 21-23, given the set of premises, identify the conclusion that creates a valid syllogism. Using an Euler diagram is recommended.

21. All postpositivists believe in one reality. Some postpositivists conduct mixed method research. What is the valid conclusion?
    a. All people who conduct mixed method research believe in one reality.
    b. All people who believe in one reality conduct mixed method research.
    c. Some people who believe in one reality conduct mixed method research.
    d. No one who believes in one reality conducts mixed method research.
22. Some stingrays can add and subtract 'one' in the number space from one to five. All stingrays are venomous. What is the valid conclusion?
    a. No venomous creatures can add and subtract 'one' in the number space from one to five.
    b. Some venomous creatures cannot add and subtract 'one' in the number space from one to five.
    c. Some creatures that can add and subtract 'one' in the number space from one to five are venomous.
    d. All creatures that can add and subtract 'one' in the number space from one to five are venomous.
23. All aquatic animals have some dependence on water. Some aquatic animals are mammals. What is the valid conclusion?
    a. No mammals have a dependence on water.
    b. All animals with a dependence on water are mammals.
    c. All animals with a dependence on water are aquatic.
    d. Some animals with a dependence on water are mammals.

Concept Extension

24. Syllogisms can sometimes be rewritten as arguments. Consider the syllogism “All sharks have teeth. Bugsby is a shark. Therefore, Bugsby has teeth.” The first statement can be rewritten as “If an animal is a shark, then an animal has teeth.” Rewriting the first statement that way, what form does this argument take? Is it a valid or invalid argument?

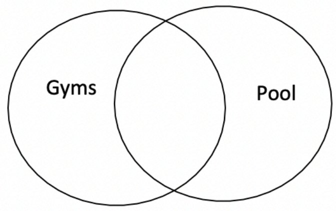
25. Syllogisms can sometimes be rewritten as arguments. Consider the syllogism “All pigs can fly. Charlotte is not a pig. Therefore, Charlotte cannot fly.” The first statement can be rewritten as “If an animal is a pig, then it can fly.” Rewriting the first statement that way, what form does this argument take? Is it a valid or invalid argument?

26. Syllogisms can sometimes be rewritten as arguments. Consider the statements “All squares are rectangles. All rectangles are polygons.” Provide a conclusion statement to make the syllogism valid.

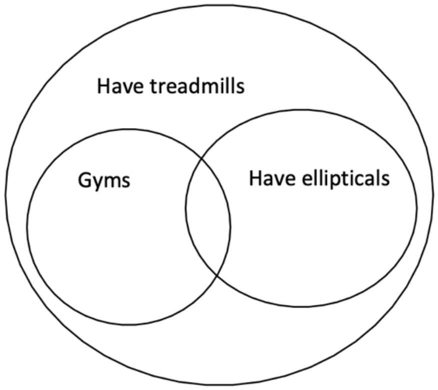
27. Syllogisms can sometimes be rewritten as arguments. Consider the statements “All gymnasts are strong. A gorilla is strong.” Provide a conclusion statement that makes the syllogism invalid.

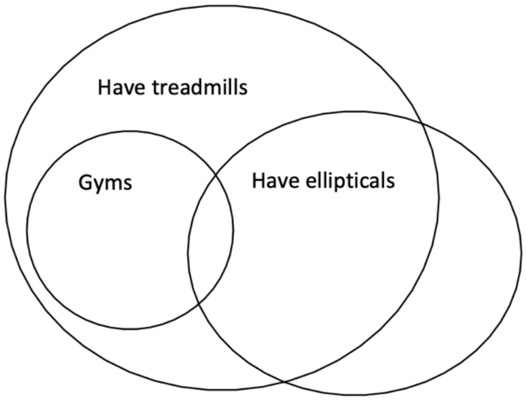
Section 2.8 | Exercise Solutions

1. This is an argument. It contains connectives and no quantifiers.
2. This is a syllogism. It contains two or more premises followed by a conclusion. It also contains a quantifier.
3. There is one possible Euler diagram shown below.

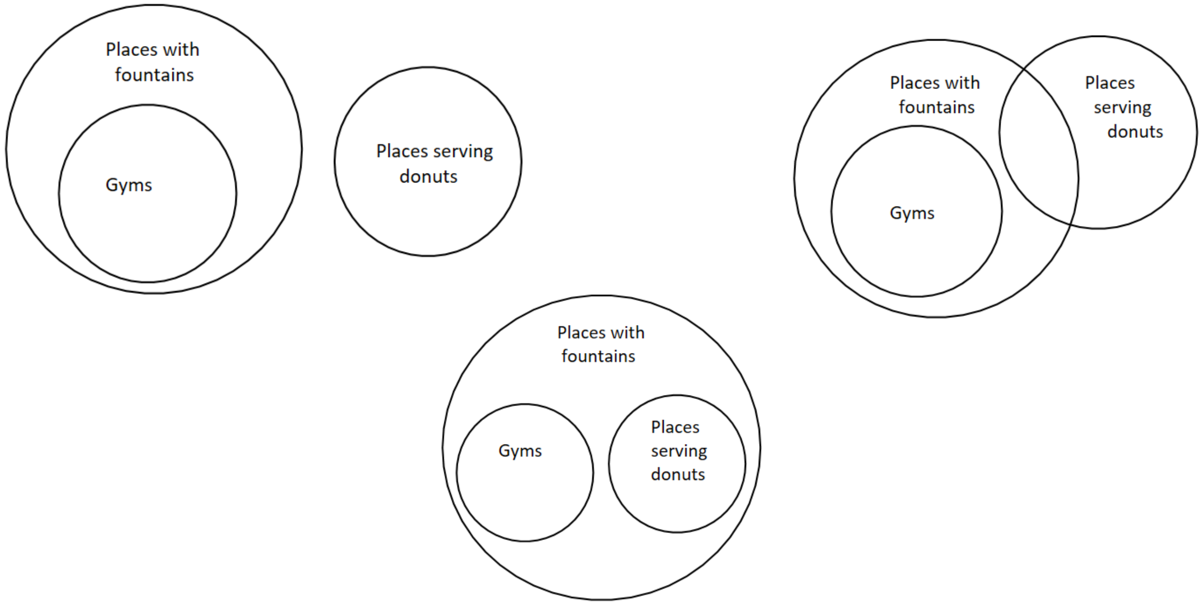


4. There are two possible Euler diagrams shown below.

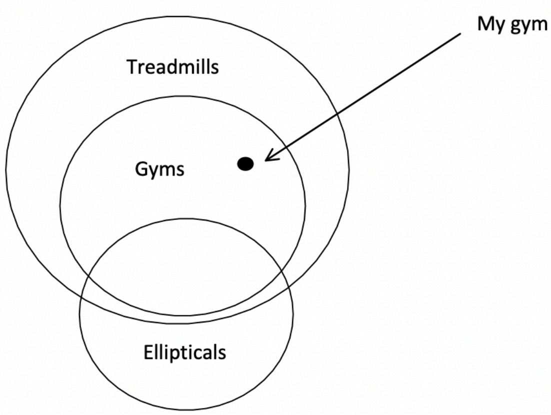




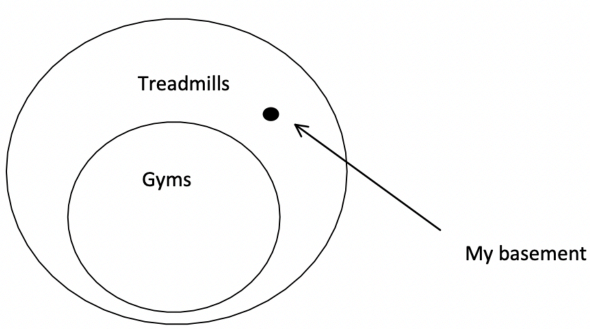
5. There are three possible Euler diagrams shown below.



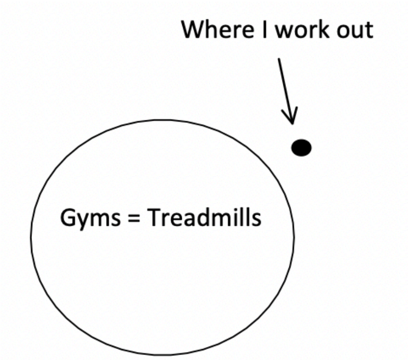
6. This syllogism is invalid. My gym doesn't have to have an elliptical. Consider the following Euler diagram.

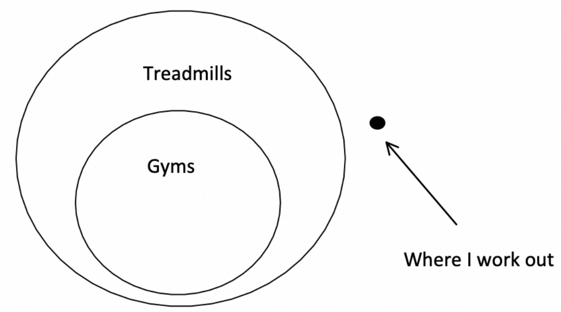


7. This syllogism is invalid. My basement can still have a treadmill and not be a gym. Consider the following Euler diagram as a counterexample.

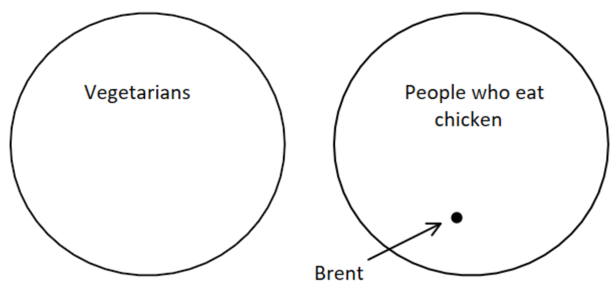


8. This syllogism is valid. Below are possible Euler diagrams. In each diagram, the conclusion is always true.

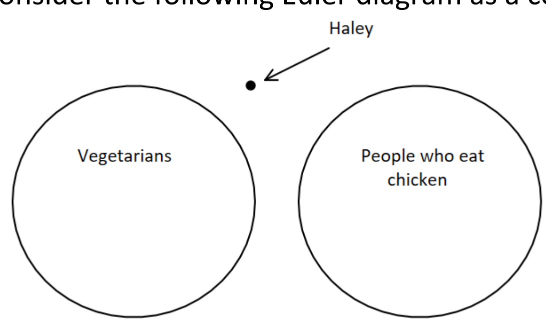




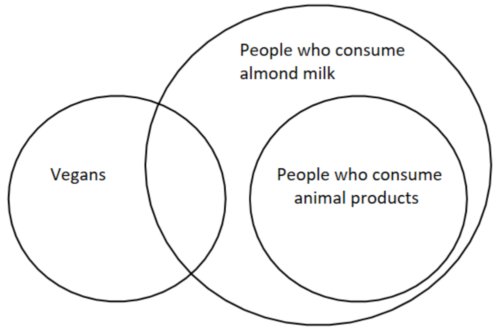
9. This syllogism is valid. Below is the only possible Euler diagram. In this diagram, the conclusion is true.



10. This syllogism is invalid. Just because Haley doesn't eat chicken doesn't mean that she is a vegetarian. Consider the following Euler diagram as a counterexample.

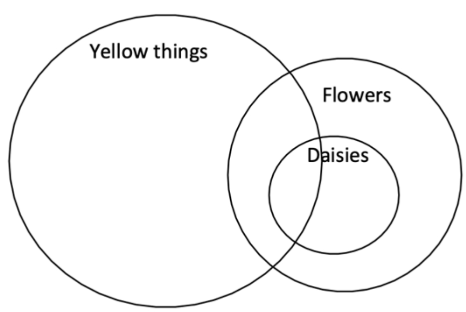


11. This syllogism is invalid. Based on the given information, a conclusion cannot be made that some people who consume animal products do not consume almond milk. Consider the following Euler diagram as a counterexample.

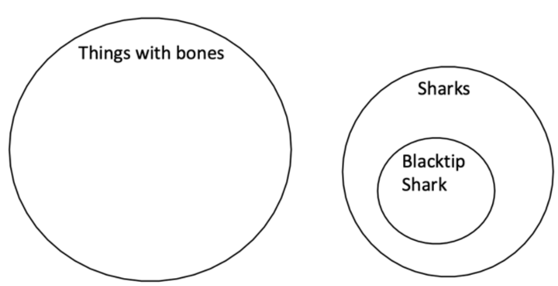


12. This syllogism is valid. Below is the only possible Euler diagram. In this diagram, the conclusion is true.

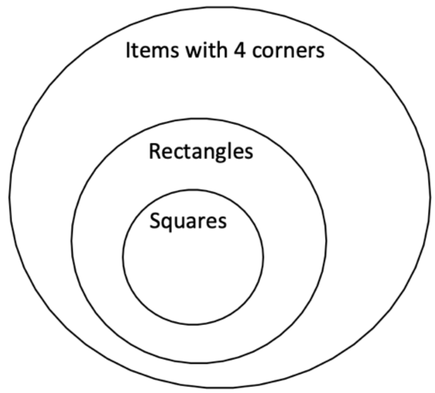
13. This syllogism is invalid. Based on the given information, a conclusion cannot be made that all daisies are yellow. Consider the following Euler diagram as a counterexample.



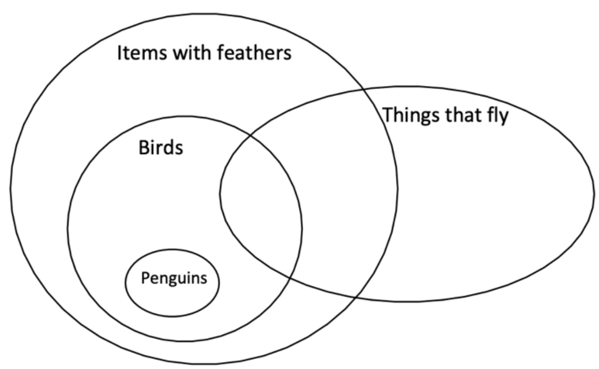
14. This syllogism is valid. Below is the only possible Euler diagram. In this diagram, the conclusion is true.



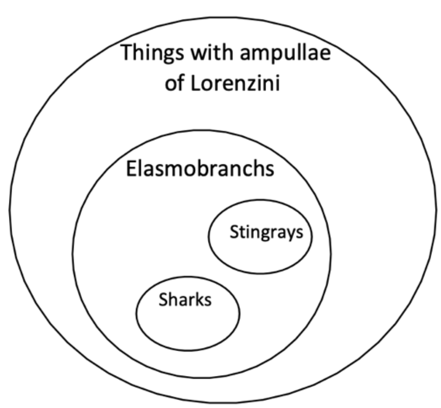
15. This syllogism is valid. Below is the only possible Euler diagram. In this diagram, the conclusion is true.



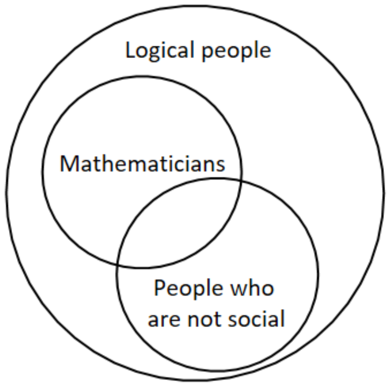
16. This syllogism is invalid. Based on the given information, a conclusion cannot be made that penguins can fly. Consider the following Euler diagram as a counterexample.

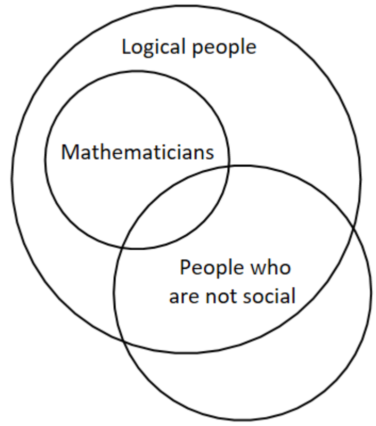


17. This syllogism is invalid. Based on the given information, a conclusion cannot be made that all daisies are yellow. Consider the following Euler diagram as a counterexample.

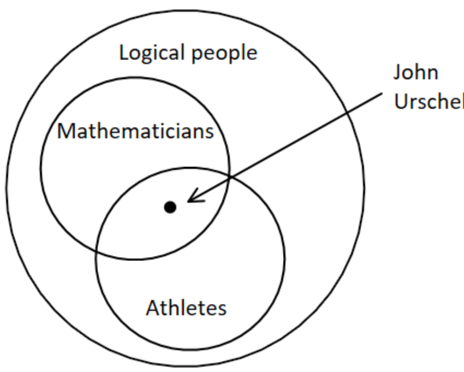


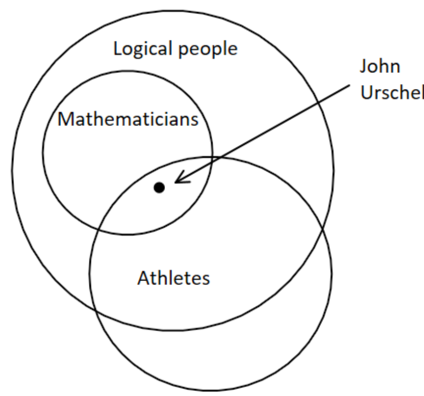
18. The syllogism is valid. There are two possible Euler diagrams, shown below. In each diagram, the conclusion is always true.



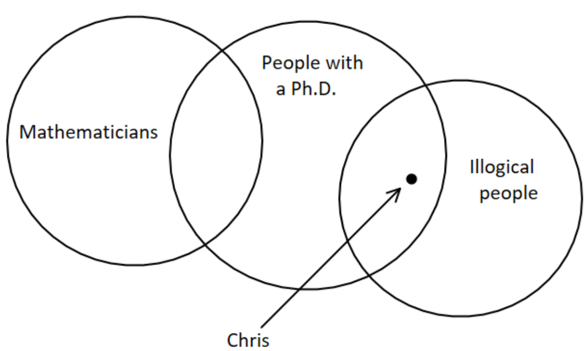


19. The syllogism is valid. There are two possible Euler diagrams, shown below. In each diagram, the conclusion is always true.

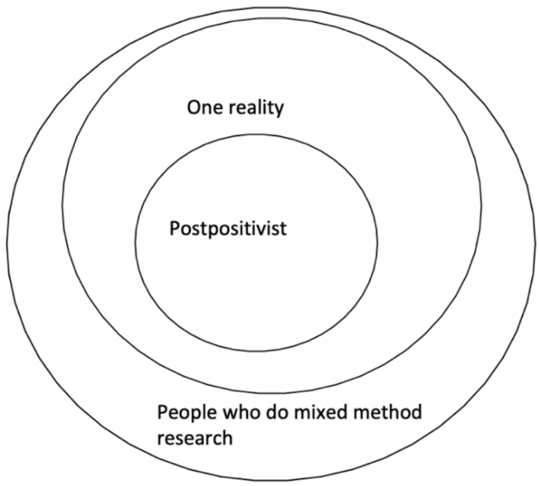




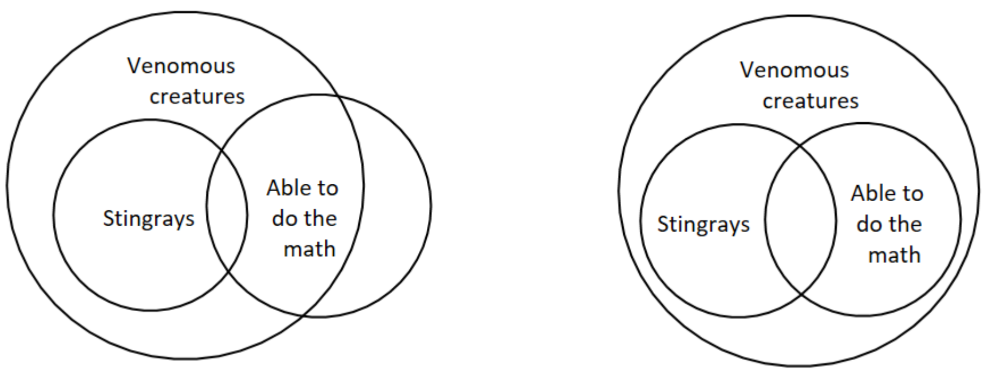
20. This syllogism is invalid. Based on the given information, a conclusion cannot be made that Chris is not illogical. Consider the following Euler diagram as a counterexample.



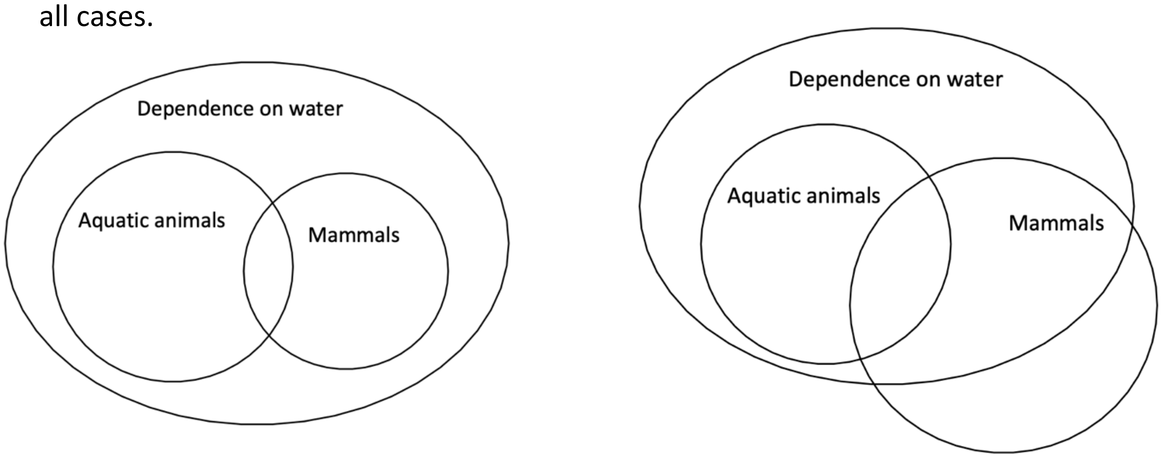
21. Only conclusion c, “Some people who believe in one reality conduct mixed method research,” is true in all cases.



22. There are two possible Euler diagrams that meet the given conditions shown below. Only conclusion c, “Some creatures that can add and subtract ‘one’ in the number space from one to five are venomous,” is true in all cases.



23. There are two possible Euler diagrams that meet the given conditions shown below. Only conclusion d, “Some animals with a dependence on water are mammals,” is true in all cases.



24. This is a valid argument. It is an example of direct reasoning.
25. This is a fallacy or an invalid argument. It is an example of fallacy of the inverse.
26. One possible conclusion statement to make the syllogism valid could be “All squares are polygons.”
27. One possible conclusion statement to make the syllogism invalid could be “All gymnasts are gorillas.”

Chapter 3

Financial Math

Section 3.A Fractions, Decimals, & Percent

Prerequisite Content

Percent

DEFINITION: A percent represents a part out of one hundred. The term **percent** means "per cent" which translate to "per hundred." A percent is written using the percent symbol %.

A percent represents a part out of one hundred. For example, 100% represents 100 out of 100 or the whole amount whereas 38% represents 38 out of 100. Listed in Figure 3.1.1 are some commonly used percentages along with their fraction and decimal equivalents.

FIGURE 3.1.1

Commonly Used Percent	Fraction Equivalent	Decimal Equivalent
100%	$\frac{100}{100}$	1
50%	$\frac{50}{100}$	0.5
25%	$\frac{25}{100}$	0.25
1%	$\frac{1}{100}$	0.01

Since 100% indicates $\frac{100}{100}$ or the whole amount, percent values greater than 100% would indicate more than a whole. For example, a person who earns a 5% raise will be making 100% plus 5%, or 105% of their previous salary which could be written as $\frac{105}{100}$. Percent values below 1% indicate a small fraction of the total amount. For example, $\frac{1}{2}$% represents $\frac{1}{2}$ out of 100 which could be written $\frac{\frac{1}{2}}{100}$ or $\frac{1}{200}$.

➢ **EXAMPLE 3.A.1:** Convert the percent to an equivalent, simplified fraction.

a. 35%

b. 72%

c. 210%

d. $\frac{3}{5}\%$

e. 0.32%

SOLUTION:

a. Given 35%, this means 35 per 100 or the fraction $\frac{35}{100} = \frac{7}{20}$.

b. Given 72%, this means 72 per 100 or the fraction $\frac{72}{100} = \frac{18}{25}$.

c. Given 210%, this means 210 per 100 or the fraction $\frac{210}{100} = \frac{21}{10}$ or $2\frac{1}{10}$ as a mixed number.

NOTE: A percent greater than 100% will be larger than 1 and can be written as an improper fraction or mixed number.

d. Given $\frac{3}{5}\%$, this means $\frac{3}{5}$ per 100 or the fraction $\frac{\frac{3}{5}}{100}$. Knowing that a fraction bar indicates divide, this can be written as

$$\frac{3}{5} \div 100 = \frac{3}{5} \div \frac{100}{1} = \frac{3}{5} \times \frac{1}{100} = \frac{3}{500}$$

Another possible method to write $\frac{3}{5}\%$ as a fraction would be to convert $\frac{3}{5}$ into its decimal equivalent first. It becomes,

$$\frac{3}{5}\% = (3 \div 5)\% = 0.6\% = \frac{0.6}{100} = \frac{0.6}{100}\left(\frac{10}{10}\right) = \frac{6}{1{,}000} = \frac{3}{500}$$

Multiply by 10 to eliminate the decimal within the fraction

e. Given 0.32%, this means 0.32 per 100 or the fraction $\frac{0.32}{100}$ and then simplify

$$0.32\% = \frac{0.32}{100} = \frac{0.32}{100}\left(\frac{100}{100}\right) = \frac{32}{10{,}000} = \frac{8}{2{,}500} = \frac{2}{625}$$

❖ **YOU TRY IT 3.A.A:** Convert the percent to an equivalent fraction.

a. 42%

b. $\frac{5}{8}\%$

c. 372%

There are two methods that could be used to reverse this process and convert a fraction into a percent. The first method utilizes equivalent fractions while the second method utilizes multiplication by 100%, which is equivalent to 1.

Method 1: Use equivalent fractions to convert $\frac{3}{20}$ into an equivalent percent.

Since the term percent means "per hundred," it is possible to consider the question, $\frac{3}{20}$ is equal to what percent? Written as an equation this becomes $\frac{3}{20}$ equals what number out of 100?

$$\frac{3}{20} = \frac{?}{100}$$

$$\frac{3}{20} = \frac{x}{100}$$

$$3\,(100) = x\,(20)$$

$$300 = 20x$$

$$15 = x$$

Therefore, $\frac{3}{20}$ is equivalent to $\frac{15}{100}$ or 15%.

Method 2: Use the fact that 100% = 1 to convert $\frac{3}{20}$ into an equivalent percent.

First begin by changing $\frac{3}{20}$ into its decimal equivalent, then multiply by 100%.

$$\frac{3}{20} = 3 \div 20 = 0.15 = 0.15 \times 1 = 0.15 \times 100\% = 15\%$$

Therefore, $\frac{3}{20}$ is equivalent to 15%.

➢ **EXAMPLE 3.A.2:** Use equivalent fractions (Method 1 above) to convert the fraction into an equivalent percent.

a. $\frac{6}{25}$

b. $\frac{3}{8}$

c. $\frac{17}{15}$

d. $\frac{1}{250}$

SOLUTION:

a. $\frac{6}{25} = \frac{x}{100}$ and then solve.

$$\frac{6}{25} = \frac{x}{100}$$
$$6(100) = 25x$$
$$600 = 25x$$
$$24 = x$$

Therefore, $\frac{6}{25}$ is equivalent to 24%.

b. $\frac{3}{8} = \frac{x}{100}$ and then solve.

$$\frac{3}{8} = \frac{x}{100}$$
$$3(100) = 8x$$
$$300 = 8x$$
$$37.5 = x$$

Therefore, $\frac{3}{8}$ is equivalent to 37.5% or $37\frac{1}{2}$%.

c. $\frac{17}{15} = \frac{x}{100}$ and then solve.

$$\frac{17}{15} = \frac{x}{100}$$
$$17(100) = 15x$$
$$1700 = 15x$$
$$113\frac{1}{3} = x$$

Therefore, $\frac{17}{15}$ is equivalent to $113\frac{1}{3}$%.

When dividing, this becomes $113.\overline{3}$ which is not a terminating decimal and does not have an exact decimal equivalent. The remainder should be written as a fraction to maintain equality.

d. $\frac{1}{250} = \frac{x}{100}$ and then solve.

$$\frac{1}{250} = \frac{x}{100}$$
$$1(100) = 250x$$
$$100 = 250x$$
$$0.4 = x$$

Therefore, $\frac{1}{250}$ is equivalent to 0.4% or $\frac{4}{10}$% which becomes $\frac{2}{5}$% when simplified.

❖ **YOU TRY IT 3.A.B:** Use equivalent fractions (Method 1 above) to convert the fraction into an equivalent percent.

a. $\frac{7}{10}$
b. $\frac{8}{5}$
c. $\frac{3}{200}$

➢ **EXAMPLE 3.A.3:** Use multiplication by 100% (Method 2 above) to convert the fraction into an equivalent percent.

a. $\frac{6}{25}$
b. $\frac{3}{8}$
c. $\frac{17}{15}$
d. $\frac{1}{250}$

SOLUTION:

a. First begin by changing $\frac{6}{25}$ into its decimal equivalent, then multiply by 100%.

$$\frac{6}{25} = 6 \div 25 = 0.24 = 0.24 \times 1 = 0.24 \times 100\% = 24\%$$

Therefore, $\frac{6}{25}$ is equivalent to 24%.

b. First begin by changing $\frac{3}{8}$ into its decimal equivalent, then multiply by 100%.

$$\frac{3}{8} = 3 \div 8 = 0.375 = 0.375 \times 1 = 0.375 \times 100\% = 37.5\%$$

Therefore, $\frac{3}{8}$ is equivalent to 37.5%.

c. First begin by changing $\frac{17}{15}$ into its decimal equivalent, then multiply by 100%.

$$\frac{17}{15} = 17 \div 15 = 1.1\overline{3}$$

The number $1.1333\overline{3}$ is not a terminating decimal and it does not have an exact decimal equivalent. Use an improper fraction and mixed number instead.

$$\frac{17}{15} = \frac{17}{15} \times 100\% = \frac{17}{15} \times \frac{100}{1}\% = \frac{1700}{15}\%$$
$$= 113\frac{5}{15}\%$$
$$= 113\frac{1}{3}\%$$

Therefore, $\frac{17}{15}$ is equivalent to $113\frac{1}{3}\%$.

d. First begin by changing $\frac{1}{250}$ into its decimal equivalent, then multiply by 100%.

$$\frac{1}{250} = 1 \div 250 = 0.004 = 0.004 \times 1 = 0.004 \times 100\% = 0.4\%$$

Therefore, $\frac{1}{250}$ is equivalent to 0.4%.

❖ **YOU TRY IT 3.A.C:** Use multiplication by 100% (Method 2 above) to convert the fraction into an equivalent percent.

a. $\frac{11}{20}$

b. $\frac{9}{4}$

c. $\frac{7}{500}$

It is possible to use the fact that 100% = 1 to convert a percent to a decimal or in return change a decimal to a percent. Multiplying a decimal by one does not change its value; however, it does change its appearance.

For example, to change the decimal 0.54 into an equivalent percent, multiply 0.54×1.

$$0.54 = 0.54 \times 1 = 0.54 \times 100\% = 54\%$$

Therefore, 0.54 is equivalent to 54%.

➢ **EXAMPLE 3.A.4:** Convert the decimal into an equivalent percent.

a. 0.385
b. 1.26
c. 0.048
d. 4.5

SOLUTION:

a. $0.385 = 0.385 \times 1 = 0.385 \times 100\% = 38.5\%$
b. $1.26 = 1.26 \times 1 = 1.26 \times 100\% = 126\%$
c. $0.048 = 0.048 \times 1 = 0.048 \times 100\% = 4.8\%$
d. $4.5 = 4.5 \times 1 = 4.5 \times 100\% = 450\%$

NOTE: Multiplying by 100% moves the decimal two places to the right and places the % symbol at the end of the expression.

❖ **YOU TRY IT 3.A.D:** Convert the decimal into an equivalent percent.

a. 0.735
b. 0.006
c. 2.81

It is also possible to use the fact, 100% = 1, to convert a percent into an equivalent decimal. For example, to change the percent 86.2% into an equivalent decimal, divide $86.2\% \div 1$.

$$86.2\% = \frac{86.2\%}{1} = \frac{86.2\%}{100\%} = \frac{86.2}{100} = 0.862$$

Therefore, 86.2% is equivalent to 0.862.

The percent symbol is eliminated leaving $\frac{86.2}{100}$

➢ **EXAMPLE 3.A.5:** Convert the percent into an equivalent decimal.

a. 54.9%
b. 217%
c. 0.35%

SOLUTION:

a. $54.9\% = \frac{54.9\%}{1} = \frac{54.9\%}{100\%} = \frac{54.9}{100} = 0.549$

b. $217\% = \frac{217\%}{1} = \frac{217\%}{100\%} = \frac{217}{100} = 2.17$

c. $0.35\% = \frac{0.35\%}{1} = \frac{0.35\%}{100\%} = \frac{0.35}{100} = 0.0035$

NOTE: Dividing by 100% moves the decimal two places to the left and eliminates the percent symbol.

❖ **YOU TRY IT 3.A.E:** Convert the percent into an equivalent decimal.

a. 249%
b. 73.5%
c. 0.36%

YOU TRY IT 3.A.A SOLUTION:

a. $\frac{21}{50}$
b. $\frac{1}{160}$
c. $\frac{93}{25}$ or $3\frac{18}{25}$

YOU TRY IT 3.A.B SOLUTION:

a. 70%
b. 160%
c. 1.5%

YOU TRY IT 3.A.C SOLUTION:

a. 55%
b. 225%
c. 1.4%

YOU TRY IT 3.A.D SOLUTION:

- **a.** 73.5%
- **b.** 0.6%
- **c.** 281%

YOU TRY IT 3.A.E SOLUTION:

- **a.** 2.49
- **b.** 0.735
- **c.** 0.0036

Section 3.B Using Logarithms to Solve for a Variable in the Exponent

Prerequisite Content

Logarithms

A logarithm is a way to answer the question, for example, "To what power must 4 be raised in order to get 16?" In this example, the answer is 2. But what if the question were "To what power must 4 be raised in order to get 12?" Only an educated guess could be made (somewhere between 1 and 2) without the use of logarithms. Logarithms are in essence a way to undo the operation of raising a known base to an unknown power.

DEFINITION: The quantity to which a base must be raised to produce a given value is called a **logarithm.**

Symbolically, $b^y = x \leftrightarrow \log_b x = y$, where y is a real number, x and b are positive real numbers, and $b \neq 1$.

Let's begin with the example of "4 to what power is equal to 16?" This can be rewritten as:

$$4^n = 16$$

Of course, $4^2 = 16$. With logarithms, one would rewrite that as:

$$\log_4 16 = 2$$

The example of "4 to what power is equal to 12?" becomes:

$$4^n = 12$$

$$\log_4 12 \approx 1.79$$

At certain points throughout this chapter, it will be necessary to solve for an exponent within an exponential equation. Logarithmic functions will be used in the solving process.

Using Logarithms to Solve for a Variable

Properties exist for rewriting logarithms in equivalent forms. One such property of interest is called the power property of logs.

> **Power Property of Logs**
>
> For any positive real numbers x and b, $b \neq 1$,
>
> $$\log_b(x^p) = p \cdot \log_b x.$$

Logarithms are the inverse operation of exponentiation. That is, logarithms can be used to bring a variable that is in the exponent out of the exponent position. This is essential because without logarithms, there isn't always a way to solve for a variable that is in the exponent position.

A log of any base may be used when solving an exponential equation. For the purposes of this text, the natural logarithm will be used.

DEFINITION: The quantity to which the number $e \approx 2.718282$ must be raised to produce a given value is called the **natural logarithm.**

Symbolically, the natural log of x is represented $\ln x$. Thus, $\log_e x = \ln x$.

- **EXAMPLE 3.B.1:** Use the power property of logs to solve the following.

  a. $5^x = 120$
  b. $10^{x+1} = 10{,}000$
  c. $2 \cdot 7^{3x} = 36$

  SOLUTION:

  a. Taking the natural log of both sides and using the power property of logs gives:

$$5^x = 120$$
$$\ln(5^x) = \ln(120)$$
$$x \cdot \ln(5) = \ln(120)$$
$$x = \frac{\ln(120)}{\ln(5)}$$
$$x \approx 2.974636$$

  Checking this result gives: $5^{2.974636} \approx 120$.

b. Taking the natural log of both sides and using the power property of logs gives:

$$10^{x+1} = 10{,}000$$
$$\ln(10^{x+1}) = \ln(10{,}000)$$
$$(x+1)\cdot\ln(10) = \ln(10{,}000)$$
$$x+1 = \frac{\ln(10{,}000)}{\ln(10)}$$
$$x = \frac{\ln(10{,}000)}{\ln(10)} - 1$$
$$x = 3$$

Checking this result gives: $10^{3+1} = 10^4 = 10{,}000$.

c. First, it is necessary to isolate the exponential expression by dividing both sides by 2. Then, taking the natural log of both sides and using the power property of logs gives:

$$2\cdot 7^{3x} = 36$$
$$7^{3x} = 18$$
$$\ln(7^{3x}) = \ln(18)$$
$$(3x)\cdot\ln(7) = \ln(18)$$
$$3x = \frac{\ln(18)}{\ln(7)}$$
$$x = \left(\frac{\ln(18)}{\ln(7)}\right)/3$$
$$x \approx 0.495119$$

Checking this result gives: $2\cdot 7^{3(0.495119)} \approx 36$.

❖ **YOU TRY IT 3.B.A:** Use the power property of logs to solve the following.

a. $4^x = 136$

b. $4\cdot 3^{x+1} = 108$

YOU TRY IT 3.B.A SOLUTION:

a. $\frac{\ln(136)}{\ln(4)} \approx 3.543731$

b. 2

Section 3.1 | Percent & Percent Change

Objectives

- Solve problems using percent increase or percent decrease
- Solve problems involving sales tax and income tax

The Mathematics of Finance chapter will begin with a discussion of percent and then continue with applications using percent calculations such as simple interest, sales tax, and income tax.

Applications Using Percent

Many daily activities use percent calculations. In many states, sales tax is added to the purchase price to determine the total amount paid. In Illinois, the sales tax was 6.25% in 2022. Banks advertise their services by mentioning the interest rates of their different types of bank accounts or loans. For example, a local bank offered car loan rates as low as 3.19%. Retail stores entice customers to purchase items by offering discounts on items. A store might advertise that the entire store is 25% off. When dining out, a service tip of 10% to 20% is common. In each of these scenarios, a percentage of the total is calculated by multiplying the decimal equivalent of the percent by the base or original amount.

NOTE: Throughout Chapter 3, intermediate steps will not be rounded. Intermediate steps will be shown rounded to six decimal places only so students can follow along. Rounding in intermediate steps will cause the final solution to be slightly different than if rounding is only done in the final answer.

➢ **EXAMPLE 3.1.1:** Sini found a new backpack with a purchase price of \$36.99. If the sales tax rate is 6.25%, find the amount of sales tax as well as the total amount needed to purchase the backpack.

SOLUTION:

Sometimes it is helpful to state the basic information in a concise sentence, which can then be converted into an equation.

The sales tax is 6.25% of the backpack price.

Convert 6.25% to its decimal equivalent 0.0625

The word "of" indicates multiply

$$\text{tax} = 0.0625 \times \text{backpack price}$$
$$\text{tax} = 0.0625 \times 36.99$$
$$\text{tax} = 2.311875$$

Use traditional rounding.

$$\text{tax} \approx \$2.31$$

The tax amount is \$2.31 and the backpack price is \$36.99 for a total amount of $\$2.31 + \$36.99 = \$39.30$ to purchase the backpack.

➢ **EXAMPLE 3.1.2:** Sports Outlet is having a sale on their camping equipment. A tent normally sells for \$149.95. However, during their summer camping sale, it is marked down 20% off.

a. Find the amount of discount.
b. Find the sale price for the camping tent.
c. If sales tax rate for the store is 7.5%, find the sales tax.
d. Find the total amount that should be paid for the camping tent including the discount and tax.

SOLUTION:

a. The discount is 20% of the tent price.
$$\text{discount} = 20\% \times \text{tent price} = 0.20 \times 149.95 = \$29.99$$
b. The sale price for the tent would be $149.95 - 29.99 = \$119.96$
c. The sales tax is 7.5% of the tent price.
$$\text{tax} = 7.5\% \times \text{tent sale price} = 0.075 \times 119.96 = 8.997 \approx 9.00$$
d. To find the total amount to pay for the camping tent, add the sale price and tax.
$$\text{total} = 119.96 + 9.00 = \$128.96$$

➢ **EXAMPLE 3.1.3:** Samuel and his younger sibling Micol went out to dinner. Together, they purchased food and drinks totaling \$27.82. The tax rate is 9.7% and they plan to leave a 15% tip.

a. Find the sales tax on the food and drink purchase.
b. If they plan to leave a tip worth 15% of their food and drink purchase (not including the tax), how much should they leave as a service tip?
c. What is the total price of their dinner, including tax and tip?

SOLUTION:

a. The sales tax is 9.7% of the food and drink purchase.

$$\text{tax} = 9.7\% \times \text{food and drink purchase} = 0.097 \times 27.82 \approx 2.69854 \approx \$2.70$$

b. The tip is 15% of the food and drink purchase.

$$\text{tip} = 15\% \times \text{food and drink purchase} = 0.15 \times 27.82 = 4.173 \approx \$4.17$$

c. Find the total price of their dinner.

$$\text{total} = 27.82 + 2.70 + 4.17 = \$34.69$$

❖ **YOU TRY IT 3.1.A:** Carlos wants to purchase a new pair of shoes for the school year. Originally the shoes sold for \$79.99; however, they are currently on sale for 30% off.

a. Find the amount of discount.
b. Find the sale price of the shoes.
c. If sales tax is 8.4%, find the sales tax amount.
d. How much will Carlos pay for the new shoes?

Percent Increase and Percent Decrease

❖ **EXAMPLE 3.1.4:** The local community college saw a 5.3% increase in student enrollment this year. Last year the enrollment 21,426 students. Find the number of students enrolled this year. Round your answer to the nearest whole number.

SOLUTION:

A 5.3% increase would mean that the new enrollment is 100% of the original enrollment plus an additional 5.3% increase of the original enrollment for a total of 105.3%.

New enrollment is 105.3% of the enrollment last year.

$$\text{new enrollment} = 105.3\% \times \text{last year} = 1.053 \times 21{,}426 = 22{,}561.578 \approx 22{,}562$$

The enrollment this year is 22,562 students.

➢ **EXAMPLE 3.1.5:** RJ just had a positive annual work review and received a 3% raise. After the 3% increase, RJ is now making \$47,132.80. Find the amount he was making before his raise.

SOLUTION:
A 3% increase would mean that the new salary is 100% of the original salary plus an additional 3% increase of the original salary for a total of 103%.

New salary is 103% of the salary last year.

$$\text{New salary} = 103\% \times \text{last year's salary}$$

Substitute known values

$$47{,}132.80 = 1.03 \times L$$

$$45{,}760 = L$$

Change 103% to its decimal equivalent 1.03 and use a variable for the unknown value of last year's salary

RJ's salary last year salary was \$45,760.

➢ **EXAMPLE 3.1.6:** According to a CBS News report[4] in October 2021, the North Atlantic right whale's population has seen an 8% decrease from 2019 to 2020. The North Atlantic area had 336 right whales in 2020. Find the number of right whales in 2019.

SOLUTION:
An 8% decrease would mean that the new population is 100% of the original population minus an 8% decrease of the original population for a total of $100\% - 8\% = 92\%$.

New population is 92% of last year's population.

$$\text{New population} = 92\% \times \text{last year's population}$$

$$336 = 0.92 \times L$$

$$365.2173913 = L$$

The North Atlantic right whale population was 365 whales in 2019.

[4] Cohen, L. (2021, October 26). *Population of critically endangered North Atlantic right whales hits near 20-year low - and humans are largely to blame*. CBS News. Retrieved October 4, 2022, from https://www.cbsnews.com/news/north-atlantic-right-whale-endangered-lowest-population-20-years/

➢ **EXAMPLE 3.1.7:** According to Statistica.com[5], the number of company-owned Starbucks in the United States increased from 7,559 in 2015 to 8,947 in 2021. Find the percent of increase in company-owned Starbucks stores. Round the answer to the nearest tenth of a percent.

SOLUTION:
First calculate the amount of increase in company-owned stores:
$8{,}947 - 7{,}559 = 1{,}388$ increase. To calculate the percent of increase, begin by making a fraction. Then convert the fraction into a percent.

$$\frac{\text{amount of increase}}{\text{starting amount}} = \frac{1{,}388}{7{,}559} \approx 0.18362217 \times 100\% \approx 18.4\% \text{ increase}$$

The number of company-owned Starbucks stores in the United States increased 18.4% from 2015 to 2021.

❖ **YOU TRY IT 3.1.B:**

a. There was a 6.9% increase in the price of a gallon of milk from 2020 to 2021. If the price of a gallon of milk in 2021 was $3.55, find the price of a gallon of milk in 2020.

b. The price of a gallon of milk was $1.12 in 1980. The price increased to $3.55 in 2021. Find the percent increase in the price of milk from 1980 to 2021. Round your answer to the nearest hundredth of a percent.

Federal Income Tax Applications Using Percent

Income tax paid to the IRS is calculated as a percentage of taxable income. Each year the government uses percentages to calculate income tax. Tax tables are used to display the tax percentage and calculation method. Below is an example of how the income tax tables are used to calculate the income tax due for the whole year.

As an example, Makayla is single and filing her income tax return. Her 2021 taxable income was $47,890. Using the selected federal income tax tables, find the amount of federal income tax she owes for the year. First, find the table for a person filing a single tax return (the top table).

Next, find the appropriate row that corresponds to Makayla's taxable income of $47,890.

For this example, the taxable income fits into the third row: over $40,525 but not over $86,375.

[5] Lock, S. (2022, July 27). *Number of Starbucks stores in the U.S. 2021*. Statista. Retrieved October 12, 2022, from https://www.statista.com/statistics/218360/number-of-starbucks-stores-in-the-us/

2021 Tax Rate Schedules [6]

Schedule X—If your filing status is Single

If your taxable income is: Over—	But not over—	The tax is:	of the amount over—
$0	$9,950	.... 10%	$0
9,950	40,525	$995.00 + 12%	9,950
40,525	86,375	4,664.00 + 22%	40,525
86,375	164,925	14,751.00 + 24%	86,375
164,925	209,425	33,603.00 + 32%	164,925
209,425	523,600	47,843.00 + 35%	209,425
523,600	....	157,804.25 + 37%	523,600

Schedule Y-1—If your filing status is Married filing jointly or Qualifying widow(er)

If your taxable income is: Over—	But not over—	The tax is:	of the amount over—
$0	$19,900	.... 10%	$0
19,900	81,050	$1,990.00 + 12%	19,900
81,050	172,750	9,328.00 + 22%	81,050
172,750	329,850	29,502.00 + 24%	172,750
329,850	418,850	67,206.00 + 32%	329,850
418,850	628,300	95,686.00 + 35%	418,850
628,300	....	168,993.50 + 37%	628,300

Read the tax expression and simplify:

The income tax is $4,664.00 plus 22% of the amount over $40,525

To find the amount over 40,525, take the entire taxable income and subtract 40,525 which is

47,890 – 40,525

The income tax is $4,664.00 plus 22% of the amount over $40,525

$$\text{Income tax} = 4{,}664.00 + 22\%\ (47{,}890 - 40{,}525)$$
$$\text{Income tax} = 4{,}664.00 + 0.22\ (7{,}365)$$
$$\text{Income tax} = 4{,}664.00 + 1{,}620.30$$
$$\text{Income tax} = \$6{,}284.30$$

Simplify inside parentheses and change 22% into its decimal equivalent 0.22

For 2021, Makayla must pay $6,284.30 in federal income tax.

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[6] (2022). *2021 Instructions 1040*. Prior Year Forms and Instructions. https://www.irs.gov/forms-pubs/prior-year

If Makayla was paying \$290 toward her 2021 income tax in each of her 24 paychecks for the year, does she still owe money or does she get a refund? Find the amount she still owes or the amount of refund.

For the entire year, Makayla's federal income tax bill is \$6,284.30. However, she has already paid \$290 in each of her 24 paychecks. Makayla has already paid $290 \times 24 = \$6{,}960$ through her paychecks during the year. Makayla has already paid more than her federal income tax bill. Since she has paid too much, she will receive a refund of $6{,}960 - 6{,}284.30 = 675.70$. Makayla's refund will be \$675.70

➢ **EXAMPLE 3.1.8:** Javi and Jess are married and filing a joint income tax return. They have a combined 2021 taxable income of \$72,500.

a. Find the amount of federal income tax they must pay.
b. If Javi and Jess were paying \$625 in each of their 12 monthly paychecks for the year, do they still owe money or do they receive a refund? Find the amount they still owe or the amount of refund.

SOLUTION:

a. Using the federal income tax tables, find the amount of federal income tax they owe for the year. First, find the table for married filing jointly. Next, find the appropriate row that corresponds to the taxable income of \$72,500. For this question, the taxable income fits into the second row: over \$19,900 but not over \$81,050. Read the tax expression and simplify:

2021
Tax Rate
Schedules

Schedule Y-1—If your filing status is **Married filing jointly** or **Qualifying widow(er)**

If your taxable income is: Over—	But not over—	The tax is:	of the amount over—
\$0	\$19,900	.... 10%	\$0
19,900	81,050	\$1,990.00 + 12%	19,900
81,050	172,750	9,328.00 + 22%	81,050
172,750	329,850	29,502.00 + 24%	172,750
329,850	418,850	67,206.00 + 32%	329,850
418,850	628,300	95,686.00 + 35%	418,850
628,300	....	168,993.50 + 37%	628,300

The income tax is $1,990.00 plus 12% of the amount over $19,900

To find the amount over 19,900, take the entire taxable income and subtract 19,900 which is 72,500 – 19,900 = 52,600

The income tax is $1,990.00 plus 12% of the amount over $19,900

$$\text{Income tax} = 1{,}990.00 + 12\%\ (72{,}500 - 19{,}900)$$

Simplify inside the parentheses and change 12% into its decimal equivalent 0.12

$$\text{Income tax} = 1{,}990.00 + 0.12\ (52{,}600)$$
$$\text{Income tax} = 1{,}990.00 + 6{,}312.00$$
$$\text{Income tax} = \$8{,}302.00$$

For the 2021 tax year, Javi and Jess must pay $8,302 in federal income tax.

b. For the entire year, Javi and Jess must pay a federal income tax bill of $8,302.00. However, they have already paid $625 in each of their 12 paychecks. They have already paid $625 \times 12 = \$7{,}500$ during the year through their paychecks. Since they have not paid enough to cover the entire tax bill, they still owe $\$8{,}302 - \$7{,}500 = \$802$.

❖ **YOU TRY IT 3.1.C:** Manny is single and filing his income tax return. His taxable income in 2021 was $108,250.

a. Find the amount of federal income tax Manny must pay.

b. If Manny paid $750 in each of his 26 paychecks for the year, does he still owe money or does he receive a refund? Find the amount he still owes or the amount of refund.

Interest rates are another application of percent. In the remaining sections of this chapter, percentage rates will be used to calculate interest for bank accounts and loans using simple interest, compound interest, as well as annuities and sinking funds. Interest rates and percentages will also be used to calculate loan payments and credit card fees.

YOU TRY IT 3.1.A SOLUTION:

- **a.** \$24.00
- **b.** \$55.99
- **c.** \$4.70
- **d.** \$60.69

YOU TRY IT 3.1.B SOLUTION:

- **a.** \$3.32 (Hint: 6.9% increase + 100% original price = 106.9%)
- **b.** 216.96%

YOU TRY IT 3.1.C SOLUTION:

- **a.** \$20,001 Hint: $14{,}751 + 0.24(108{,}250 - 86{,}375)$
- **b.** Manny still owes \$501.

Section 3.1 | Exercises

In Exercises 1 – 8, fill in the blank with >, <, or = to make the expression true.

1. 65% of 15 _____ 18% of 50
2. 25% of 40 _____ 40% of 25
3. 85% of 125 _____ 32% of 350
4. 40% of 150 _____ 10% of 600
5. 130% of 25 _____ 13% of 250
6. 0.26% of $15{,}500$ _____ 0.16% of $21{,}500$
7. 0.057% of $1{,}378$ ____ 0.059% of $1{,}350$
8. 213% of 53 ____ 220% of 47

In Exercises 9 – 16, use percentages to solve the application problems. Round answers to the nearest whole number.

9. A poll of 2,400 students enrolled in weightlifting at the local community college found that 62% had played sports during high school. Of the students surveyed, how many students played sports in high school?
10. When female sea turtles bury their eggs, the sex is determined by the temperature of sand. It is estimated that only 1,133,000 sea turtles remain. If 90% of all sea turtles are female, find the number of female sea turtles.
11. According to the Pew Research Center[7], in 2022 approximately 85% of Americans owned a smart phone. In 2022, the population of the United States was 332,403,650. Find the number of Americans who owned a smart phone in 2022.
12. According to the website Running With Grit[8], 0.05% of the US population has completed a marathon. In 2022, the population of the United States was 332,403,650. Find the number of Americans who completed a marathon based on the population in 2022.
13. A community college has a student population of 12,758 students. Out of those students, 37% participate in student clubs. How many students participate in clubs?

[7] Pew Research Center. (2022, October 7). *Mobile fact sheet*. Pew Research Center: Internet, Science & Tech. Retrieved October 12, 2022, from https://www.pewresearch.org/internet/fact-sheet/mobile/

[8] Runningwithgrit. (2022, January 7). *Statistics about running; facts about runners*. Running With Grit. Retrieved October 12, 2022, from https://runningwithgrit.com/statistics-about-running/

14. According to the U.S. Bureau of Labor Statistics[9], there were approximately 129,000 aerospace engineers in 2021, of which 11.09% were female. How many female aerospace engineers were there in in the U.S. in 2021?
15. Joseph found that 55.6% of people at his gym are on a diet. If there are 2,368 people registered at his gym, how many are on a diet at Joseph's gym?
16. Esperanza sold 0.357% of her stocks. If she had $1,453,974 worth of stocks, how much in stocks did she sell?

In Exercises 17 – 27, use percentages to solve sales tax application problems. Round answers to the nearest cent.

17. Jackson found a pair of waterproof winter snow boots with a purchase price of $89.99. If the sales tax rate is 3.45%, find the amount of sales tax as well as the total amount needed to purchase the snow boots.
18. Katie wants to purchase a new sketch pad. The purchase price is $15.10. If the sales tax rate is 5.9%, find the amount of sales tax as well as the amount needed to purchase the sketch pad.
19. Makena would like to upgrade her fish tank to a larger 55-gallon fish tank. The purchase price of the fish tank is $74.99. If the sales tax rate is 4.8%, find the amount of sales tax as well as the amount needed to purchase the fish tank.
20. Linus purchases a new Apple Watch for $329.99. If the sales tax rate is 6.25%, find the amount of sales tax and the amount Linus will need to purchase the watch.
21. Parker would like to get a new gaming chair that costs $82.99. The sales tax rate is 5.75%. Find the amount of sales tax and the amount Parker will need to purchase the chair.
22. Asia wants to purchase $1199.00 Louboutin shoes. The sales tax rate is 8.25%. Find the amount of the sales tax and the amount Asia will pay for the shoes.
23. Markita is considering buying a Burberry purse for $2150 with a 15% discount or a Louis Vuitton purse for $2300 with a 20% discount. Which bag would be cheaper and how much would she save?
24. Boden wants to purchase a new pair of tennis shoes. He found a pair that he liked at Sports Plus for a purchase price of $54.99 and the store is currently having a 20% off sale. He also found the same shoes at Shoe Mart for $81.99 with a current discount on shoes for 45% off. Which store should Boden choose to purchase the tennis shoes? How much will he save by using that store?

[9] U.S. Bureau of Labor Statistics. (2022, January 20). *Labor Force Statistics from the Current Population Survey, Table 11*. U.S. Bureau of Labor Statistics. Retrieved October 19, 2022, from https://www.bls.gov/cps/cpsaat11.htm

25. Lorelai plans to purchase a new sofa. Furniture Warehouse sells the sofa for $769.99 and is offering a 10% off sale. The same sofa can be found at Sam's Sofa World for $899.99 and is holding a 25% off sale. Which store should Lorelai choose to purchase her sofa? How much will she save by using that store?
26. Conrad will purchase either World Series tickets for $5,375 with a sale tax rate of 5.25% or World Cup tickets for $5,150 with a sales tax rate of 10.75%. If he will purchase the cheaper tickets, which purchase will he make and how much will he save?
27. Ellie needs a new computer for college. A tablet model costs $379.98 and is currently discounted at 15% off. A laptop model costs $412.75 and is currently discounted at 25% off. Which model is cheaper and by how much?

In Exercises 28 – 34, use percentages to solve the discount and sales tax application problems. Round all answers to the nearest cent.

28. At the local art supply store, a watercolor paint set usually sells for $97.99. However, there is weekend sale with 40% off all paints.
    a. Find the amount of discount.
    b. Find the sale price of the watercolor set.
    c. If sales tax rate is 3.75%, find the sales tax amount.
    d. Find the total amount that should be paid for the watercolor paint set including the discount and sales tax.

29. Remy wants to purchase a sweatshirt. The price tag shows the sweatshirt price is $35.99. There is a back-to-school sale, and all clothing is 20% off.
    a. Find the amount of discount.
    b. Find the sale price of the sweatshirt.
    c. If sales tax rate is 5.1%, find the sales tax amount.
    d. Find the total amount that should be paid for the sweatshirt including the discount and sales tax.

30. Denzel wants to purchase a longboard. He found one for $180.99. The website was having 15% off summer sale.
    a. Find the amount of discount.
    b. Find the sale price of the longboard.
    c. If sales tax rate is 6.8%, find the sales tax amount.
    d. Find the total amount that should be paid for the longboard including the discount and sales tax.

31. Nate wants to purchase a new guitar. The model he is interested in costs $209.99 but is on sale for 33% off.
    a. Find the amount of the discount.
    b. Find the sale price of the guitar.
    c. If sales tax rate is 5.75%, find the sales tax amount.
    d. Find the total amount that should be paid for the guitar including the discount and sales tax.

32. Adriana wants to purchase a hair dryer for her salon. The hair dryer costs $199.99 but is on sale for 10% off.
    a. Find the amount of the discount.
    b. Find the sale price of the hair dryer.
    c. If sales tax rate is 4.5%, find the sales tax amount.
    d. Find the total amount that should be paid for the hair dryer including the discount and sales tax.

33. Jose wants to purchase airline tickets. There is a 15% off discount on all airfare today. Round trip tickets to Paris cost $750 before the discount.
    a. Find the amount of discount.
    b. Find the sale price of the airline tickets.
    c. If sales tax rate is 6.25%, find the sales tax amount.
    d. Find the total amount that Jose will pay for the airline tickets including the discount and sales tax.

34. Xiaofei wants to purchase a television. There is a 57% off discount during Black Friday. The original price of the television is $2400.
    a. Find the amount of discount.
    b. Find the sale price of the television.
    c. If sales tax rate is 7.875%, find the sales tax amount.
    d. Find the total amount that Xiaofei will pay for the television including the discount and sales tax.

In Exercises 35 – 41, use percentages to find discounts and sales tax.

35. Christy and Beckham went to the local coffee shop for breakfast. Together they purchased food and drinks totaling $13.45.
    a. If the sales tax rate on food and drinks is 2.3%, find the sales tax amount on the food and drink purchase.
    b. If they plan to leave a tip worth 25% of their food and drink purchase (not including sales tax), how much should they leave as a tip?
    c. What is the total price of their breakfast, including tax and tip?

36. Marisol's family went to dinner. Together they purchased food and drinks totaling $85.36
    a. If the sales tax rate on food is 1.7%, find the sales tax amount on the food and drink purchase.
    b. If they plan to leave a tip worth 18% of their food and drink purchase (not including sales tax), how much should they leave as a tip?
    c. What is the total price of their dinner, including tax and tip?

37. Kameron plans to celebrate and decides to dine at a three-star Michelin restaurant. The tasting menu costs $357 per person.
    a. If the sales tax rate on food is 2.3%, find the sales tax amount on the food purchase.
    b. If they plan to leave a tip worth 20% of their food and drink purchase (not including sales tax), how much should they leave as a tip?
    c. What is the total price of their Michelin star dinner, including tax and tip?

38. A family went out for brunch to celebrate a holiday, and the bill for food and drinks was $275.36.
    a. If the sales tax rate on food is 3.5%, find the sales tax amount that they were charged.
    b. If gratuity of 18% of their food and drink purchase (not including sales tax) was automatically added to the bill, how much did they end up paying for gratuity?
    c. What was the total price of their brunch, including tax and gratuity?

39. On the way to class in the morning, a student spent $15.68 on fast food.
    a. If the sales tax rate on food was 1.75%, find the amount of sales tax.
    b. If the student tipped the workers 10% on their food purchase (not including sales tax), how much did they leave for a tip?
    c. What was the total price of their food, including tip and tax?

40. On date night, Manu and Kamla spent \$223.47 at an Italian restaurant.
    a. If the sales tax rate on their meals is 1.95%, find the sales tax amount on their meal.
    b. If they left a tip worth 28% of their meal (not including sales tax), how much did they leave as a tip?
    c. What is the total price of their date night meal, including tax and tip?

41. Fifteen coworkers went out to dinner. Their bill for food and drinks was \$525.05.
    a. If the sales tax rate on food and drinks was 2.3%, find the sales tax amount that they were charged.
    b. If there is an automatic gratuity of 12% on their food and drink purchase (not including sales tax), how much did they pay for gratuity?
    c. What was the total price of their dinner, including tax and gratuity?

In Exercises 42 – 49, use percentages to solve the percent increase or percent decrease application problems.

42. Kayla received a pay raise after working for her company for five years. After a 4.8% salary increase her new salary is \$86,564.80. Find her original salary before the raise.
43. The number of homes for sale increased 58% from winter to the spring. If there are 145 homes for sale in this spring, find the number of homes that were for sale in the winter before the increase. Round to the nearest whole number.
44. Jan has been practicing every day and has reduced her 100-meter dash time by 8%. If she had a personal best time today of 12.85 seconds, find her previous best time from last week.
45. Alan's Appliance store is having a 15% off sale. If the advertised sale price on a new television is \$368.99, find the original price of the television.
46. The student population of a college increased by 2.5% over the past year. If the college had 8,756 students at the end of the year, how many students did the college have at the beginning of the year? Round to the nearest whole number.
47. The average price of a gallon of gas in a city decreased by 11.25% over the past month. If the average price was \$4.26 at the end of the month, what was the average gas price at the beginning of the month?
48. After a diet, Vlad lost 13.7% of his weight. He now weighs 237 pounds. How much did he weigh before his diet (to the nearest pound)?
49. Tara has improved her bench press weight by 32%. If she can now bench press 63.5 pounds, what bench press weight did she have as a starting weight?

In Exercises 50 – 57, find the percent of increase or decrease.

50. The price of an art supply kit reduced from $97.99 to $29.99. Find the percent of decrease. Round your answer to the nearest tenth of a percent.
51. A new rectangular dining table set with six chairs is on sale for $549.99. The original price of the dining set was $612.00. Find the percent of decrease. Round your answer to the nearest tenth of a percent.
52. When writing her 25,000-word thesis, Jennifer found that using 11-point font takes approximately 95 pages. If the text size is increased to 12-point font, the number of pages increases to 120 pages. Find the percent of increase in total pages when the text size is increased.
53. Carmen's bowling score improved over the bowling season from 98 pins to 148 pins. What is the percent of increase in her score?
54. The number of users of a new social media platform increased from 1.25 million to 1.26 million. What is the percent of increase in the number of users of this platform?
55. After a city-wide pet adoption event, the number of cats in a shelter still needing to be adopted decreased from 57 to 27. What is the percent of decrease in the number of cats in this shelter waiting to be adopted?
56. Rita's cholesterol decreased from 207 mg/dl to 179 mg/dl. What is the percent decrease in her cholesterol?
57. Jalen's GPA increased from 3.17 to 3.63. What is the percent increase in Jalen's GPA?

In Exercises 58 – 65, use the tax table to find the federal income tax.

Schedule X—If your filing status is **Single**

If your taxable income is: Over—	But not over—	The tax is:	of the amount over—
$0	$9,950	.... **10%**	**$0**
9,950	40,525	**$995.00 + 12%**	**9,950**
40,525	86,375	**4,664.00 + 22%**	**40,525**
86,375	164,925	**14,751.00 + 24%**	**86,375**
164,925	209,425	**33,603.00 + 32%**	**164,925**
209,425	523,600	**47,843.00 + 35%**	**209,425**
523,600	....	**157,804.25 + 37%**	**523,600**

Schedule Y-1—If your filing status is **Married filing jointly** or **Qualifying widow(er)**

If your taxable income is: Over—	But not over—	The tax is:	of the amount over—
$0	$19,900	.... **10%**	**$0**
19,900	81,050	**$1,990.00 + 12%**	**19,900**
81,050	172,750	**9,328.00 + 22%**	**81,050**
172,750	329,850	**29,502.00 + 24%**	**172,750**
329,850	418,850	**67,206.00 + 32%**	**329,850**
418,850	628,300	**95,686.00 + 35%**	**418,850**
628,300	....	**168,993.50 + 37%**	**628,300**

58. Marcus is single and filing his income tax return. His taxable income in 2021 was $65,420.
    a. Find the amount of federal income tax he must pay.
    b. If Marcus paid $425 in each of his 24 semi-monthly paychecks for the year, does he still owe money or does he receive a refund? Find the amount he still owes or the amount of refund.
59. The Jeffersons are married and filing a joint income tax return. Together their taxable income in 2021 was $158,247.
    a. Find the amount of federal income tax they must pay.
    b. Together they paid $2,100 in each of their 12 monthly paychecks for the year. Do they still owe money or do they receive a refund? Find the amount they still owe or the amount of refund.
60. Khadija is single and filing her income tax return. Her taxable income in 2021 was $39,490.
    a. Find the amount of federal income tax she must pay.
    b. If Khadija paid $405 in each of her 12 monthly paychecks for the year, does she still owe money or does she receive a refund? Find the amount she still owes or the amount of refund.
61. Maci and Greg are married and filing a joint income tax return. Together their taxable income in 2021 was $178,630.
    a. Find the amount of federal income tax they must pay.
    b. Together they have paid $1,350 in each of their 26 biweekly paychecks for the year. Do they still owe money or do they receive a refund? Find the amount they still owe or the amount of refund.
62. In 2021, a couple filed a joint income tax return and had a combined taxable income of $74,537.
    a. Find the amount of federal income tax they paid.
    b. If together they paid $658 in each of their 12 monthly paychecks for the year, did they still owe money or did they receive a refund? Find the amount they still owed or the amount of refund.
63. In 2021, Jose worked a part-time job while he was a single college student. That year, his taxable income was $14,587.
    a. Find the amount of federal income tax he paid.
    b. If Jose paid $85 in each of his 26 biweekly paychecks for the year, did he still owe money or did he receive a refund? Find the amount he still owed or the amount of refund.

64. Mario and Luigi are married and file a joint income tax return. Their 2021 taxable income was $537,163.
    a. Find the amount of federal income tax they paid.
    b. If they paid $5,125 in each of their 26 biweekly paychecks for the year, did they still owe money or did they receive a refund? Find the amount they still owed or the amount of their refund.
65. Urszula worked two jobs in 2021. That year, her taxable income for both jobs was $157,923.
    a. Find the amount of federal income tax she paid.
    b. Urszula paid a total of $755 towards federal income tax from both of her jobs in each of her 12 monthly paychecks. Did she still owe money or did she receive a refund? Find the amount she still owed or the amount of refund.

Section 3.1 Exercise Solutions

1. >
2. =
3. <
4. =
5. =
6. >
7. <
8. <
9. 1,488
10. 1,019,700
11. 282,543,103
12. 166,202
13. 4,720
14. 14,306
15. 1317
16. $5,190.69
17. Tax: $3.10 Total: $93.09
18. Tax: $0.89 Total: $15.99
19. Tax: $3.60 Total: $78.59
20. Tax: $20.62 Total: $350.61
21. Tax: $4.77 Total: $87.76
22. Tax: $98.92 Total: $1,297.92
23. Burberry saves $12.50
24. Sports Plus saves $1.10
25. Sam's Sofa World saves $18.00
26. World Series saves $46.44
27. The laptop is $13.42 cheaper
28. a. $39.20 b. $58.79
    c. $2.20 d. $60.99
29. a. $7.20 b. $28.79
    c. $1.47 d. $30.26
30. a. $27.15 b. $153.84
    c. $10.46 d. $164.30
31. a. $69.30 b. $140.69
    c. $8.09 d. $148.78
32. a. $20.00 b. $179.99
    c. $8.10 d. $188.09
33. a. $112.50 b. $637.50
    c. $39.84 d. $677.34
34. a. $1,368 b. $1032
    c. $81.27 d. $1,113.27
35. Tax: $0.31 Tip: $3.36
    Total: $17.12
36. Tax: $1.45 Tip: $15.36
    Total: $102.17
37. Tax: $8.21 Tip: $71.40
    Total: $436.61
38. Tax: $9.64 Tip: $49.56
    Total: $334.56
39. Tax: $0.27 Tip: $1.57
    Total: $17.52
40. Tax: $4.36 Tip: $62.57
    Total: $290.40
41. Tax: $12.08 Tip: $63.01
    Total: $600.14
42. $82,600
43. 92 homes
44. 13.97 seconds
45. $434.11
46. 8,542 students
47. $4.80
48. 275 pounds
49. 48.11 pounds
50. 69.4%
51. 10.1%
52. 26.3%
53. 51.0%

54. 0.8%
55. 52.6%
56. 13.5%
57. 14.5%
58. a. $10,140.90
    b. $59.10 refund
59. a. $26,311.34
    b. $1,111.34 still owe
60. a. $4,539.80
    b. $320.20 refund
61. a. $30,913.20
    b. $4,186.80 refund
62. a. $8,546.44
    b. $650.44 still owe
63. a. $1,551.44
    b. $658.56 refund
64. a. $137,095.55
    b. $3,845.55 still owe
65. a. $31,922.52
    b. $22,862.52 still owe

Section 3.2 Simple Interest

Objectives

- Solve problems involving simple interest
- Solve simple interest formulas for present value, rate, and time

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NOTE: Throughout Chapter 3, intermediate steps will not be rounded. Intermediate steps will be shown rounded to six decimal places only so students can follow along. Rounding in intermediate steps will cause the final solution to be slightly different than if rounding is only done in the final answer.

Simple Interest

Simple interest is a common application of percents. Though compound interest is much more prevalent, mathematicians begin the study of interest with simple interest and then advance through compound interest because compound interest is based upon simple interest concepts.

Some definitions are required to start.

DEFINITION: **Principal,** also referred to as **present value**, is the amount of money that is borrowed for a loan, mortgage, or credit card, or the amount that is invested in a savings account.

DEFINITION: **Interest** is the amount of money paid to the investor if money is invested. **Interest** is the amount of money paid to the lender if money is borrowed.

Interest is a term heard often when discussing banking accounts, loans, credit cards, mortgages, etc. Interest is typically a percentage of the principal amount.

DEFINITION: **Interest rate** is the rate offered for a loan or a savings account. Interest rate is expressed as a percentage but is calculated using the decimal form. Typically interest rates are expressed as annual interest rates.

DEFINITION: **Time** is the length of the account. Time is typically expressed in years.

DEFINITION: **Amount** is the future value of the account, sometimes called future value or maturity value. The amount is principal plus interest.

DEFINITION: With **simple interest**, the principal amount is invested or borrowed once. The interest is calculated on the entire principal amount once for the entire account period. Simple interest is the result of a calculation in which the principal is multiplied by the given interest rate and time.

The Federal Deposit Insurance Corporation (FDIC) published in August 2022 that the national average annual interest rate on a savings account is just 0.08%.[10] Investors could get a much higher return on a more volatile investment – stocks, for example.

The average interest rate on a credit card is much more complicated to calculate given that banks and investors talk in terms of annual percentage rate (APR) or effective interest rate (EIR), both with different meanings and therefore different rates. However, both APR and EIR on credit cards are much higher than 0.08%. According to the 2021 Consumer Credit Card Market Report both nationwide average APR and EIR range from 15.6% to 25.7%.[11]

The authors mention this so that students are aware of current interest rates. However, for the purposes of many math textbooks, students will notice that interest rates do not necessarily line up with these stated current interest rates.

Now variables can be assigned. See Figure 3.2.1 for the variables that will be used in this text.

FIGURE 3.2.1

Quantity	Variable	Details
Interest	I	I is stated in dollars
Interest rate	r	r is in decimal form
Principal	P	P is stated in dollars
Time	t	t is usually considered in years
Amount	A	A is stated in dollars

Simple interest I is a rate or percent r of the principal P over a length of time t.

Simple Interest Formula

$$I = Prt$$

[10] From https://www.fdic.gov/resources/bankers/national-rates/index.html#footnote

[11] From https://files.consumerfinance.gov/f/documents/cfpb_consumer-credit-card-market-report_2021.pdf

- **EXAMPLE 3.2.1:** Given the following values, use the simple interest formula $I = Prt$ to find the remaining variable. For part b, round to the nearest thousandth of a percent.

  f. $P = \$200, r = 5.2\%, t = 4$ years
  g. $I = \$20, P = \$1{,}500, t = 5$ years

  SOLUTION:

  a. Use the formula $I = Prt$ and substitute $P = 200, r = 0.052, t = 4$.

$$\begin{aligned} I &= Prt && \text{Substitute values for } P, r, t \\ I &= 200 \cdot 0.052 \cdot 4 && \text{Multiply} \\ I &= 41.6 \end{aligned}$$

  The interest earned is \$41.60.

  b. Use the formula $I = Prt$, substitute $I = 20, P = 1{,}500, t = 5$, and solve for r.

$$\begin{aligned} I &= Prt && \text{Substitute values for } P, r, t \\ 20 &= 1{,}500 \cdot r \cdot 5 && \text{Multiply} \\ 20 &= 7{,}500 \cdot r && \text{Divide by } 7{,}500 \\ 0.002667 &\approx r \end{aligned}$$

  The interest rate is 0.267%.

The amount A can be found by adding the principal and interest.

Future Value for Simple Interest Formula

$$A = P + I$$

The future value formula for simple interest its often rewritten using variables P, r, and t for ease of calculation.

First substitute Prt for I, since $I = Prt$.

$$\begin{aligned} A &= P + I \\ A &= P + Prt \end{aligned}$$

Next, notice that both terms contain a common factor of P. This common factor can be factored out.

$$A = P + Prt$$
$$A = P(1 + rt)$$

This is how the future value formula for simple interest is usually displayed.

Future Value for Simple Interest Formulas

$$A = P + Prt$$
$$A = P(1 + rt)$$

These formulas are summarized in Figure 3.2.2.

FIGURE 3.2.2

Formula	Title
$I = Prt$	Simple Interest Formula
$A = P + I$ or $A = P(1 + rt)$	Future Value for Simple Interest Formula

- **EXAMPLE 3.2.2:** Given the following values, use the future value for simple interest formula $A = P(1 + rt)$ to find the remaining variable. For dollar amounts, round to the nearest cent. For all other amounts, round to the nearest hundredth.

  a. $P = \$251.37, r = 2.7\%, t = 4$ years
  b. $A = \$5{,}525, r = 0.5\%, t = 10$ years
  c. $A = \$115.32, P = \$100, t = 2$ years

  SOLUTION:

  a. Use the formula $A = P(1 + rt)$ and substitute $P = 251.37, r = 0.027, t = 4$.

  $$A = P(1 + rt)$$
  $$A = 251.37 \cdot (1 + 0.027 \cdot 4)$$
  $$A \approx 278.52$$

  The future value is \$278.52.

b. Use the formula $A = P(1 + rt)$, substitute $A = 5{,}525, r = 0.005, t = 10$, and solve for P.

$$\begin{aligned} A &= P(1 + rt) \\ 5{,}525 &= P(1 + 0.005 * 10) \\ 5{,}525 &= P \cdot 1.05 \\ 5{,}261.90 &\approx P \end{aligned}$$

The principal is \$5,261.90.

c. Use the formula $A = P(1 + rt)$, substitute $A = 115.32, P = 100, t = 2$, and solve for r.

$$\begin{aligned} A &= P(1 + rt) \\ 115.32 &= 100(1 + r \cdot 2) \quad \boxed{\text{Divide by 100}} \\ 1.1532 &= 1 + 2r \\ 0.1532 &= 2r \\ 0.0766 &= r \end{aligned}$$

The interest rate is 7.66%.

❖ **YOU TRY IT 3.2.A:** Given the following values and formula, find the remaining variable. For dollar amounts, round to the nearest cent. For all other amounts, round to the nearest hundredth.

a. Use the formula $I = Prt$ and the following values to find the remaining variable. $I = \$20, P = \$2{,}512, t = 3$

b. Use the formula $A = P(1 + rt)$ and the following values to find the remaining variable. $P = \$2{,}512, r = 4.7\%, t = 11$.

The following examples are more commonly seen in real world situations. In these situations, the formula and the values of the variables are not given; this is left to the problem-solver to determine.

➢ **EXAMPLE 3.2.3:** David is saving to buy a new car. David has \$4,000 today and he'd like to buy his new car is 2 years. He is going to invest his money in a simple interest certificate of deposit (CD). (A CD is like a savings account, only the investor is required to leave the money in the account for a predetermined length of time or else pay a penalty. CDs typically have higher interest rates than savings accounts.) The CD has an annual interest rate of 2.1%. How much money will David have at the end of the 2 years?

SOLUTION:

Begin by choosing an appropriate formula and stating the known variables. One formula that could be used is $A = P(1 + rt)$ since this is simple interest, the present value is known ($4,000), and the future value is unknown. The values of the known variables are $P = \$4{,}000, r = 0.021, t = 2$. The value of A is unknown.

$$A = P(1 + rt)$$
$$A = 4{,}000(1 + 0.021 \cdot 2)$$
$$A = 4{,}168$$

David will have $4,168 in two years.

- **EXAMPLE 3.2.4:**

  a. Ellen is borrowing money from her parents. Ellen's parents want to teach her about simple interest, so they are charging Ellen a nominal simple interest rate for borrowing money. Ellen is borrowing $250 for prom and her parents want her to pay them back $265 after 1 year. What annual simple interest rate are Ellen's parents charging Ellen?
  b. Suppose Ellen's parents want their money back after 6 months instead of 1 year and Ellen agrees. What annual simple interest rate are Ellen's parents charging Ellen in this case?

SOLUTION:

a. Begin by choosing an appropriate formula and stating the known variables. One formula that could be used is $A = P(1 + rt)$ since this is simple interest and we are told about the present value ($250) and the future value ($265). The variables are $A = \$265, P = \$250, t = 1$. The value of r is unknown.

$$A = P(1 + rt)$$
$$265 = 250(1 + r \cdot 1)$$
$$1.06 = 1 + r$$
$$0.06 = r$$

The interest rate that Ellen's parents are charging her is 6%.

NOTE: The formula $I = Prt$ could also be used if the interest is found first $I = 265 - 250 = 15$).

b. If Ellen's parents change the terms of the loan to 6 months instead of 1 year, the only variable that changes is $t = 0.5$ years. (When using these formulas, time is typically given in years.)

$$6\ months \cdot \frac{1\ year}{12\ months} = 0.5 \text{ year}$$

Continue with the formula $A = P(1 + rt)$.

$$\begin{gathered} A = P(1 + rt) \\ 265 = 250(1 + 0.5r) \\ 1.06 = 1 + 0.5r \\ 0.06 = 0.5r \\ 0.12 = r \end{gathered}$$

Ellen's parents are charging her a 12% interest rate.

- **EXAMPLE 3.2.5:** Prisha is borrowing money to finance her study-abroad semester. She finds a bank who will loan her $3,500 at an interest rate of 9.9% and charge her $4,8160.70 (principal plus interest) at the end of the loan period. What is the length of Prisha's loan period? In terms of years and months, what is the length of Prisha's loan period?

SOLUTION:

Begin by choosing an appropriate formula and stating the known variables. One formula that could be used is $A = P(1 + rt)$ since the present and future values are both known. (The formula $I = Prt$ could also be used if one first finds the interest $I = 3960 - 3500 = 460.$)

The known variables are $A = \$4{,}816.70, P = \$3{,}500, r = 0.099$ and the unknown variable is t.

$$\begin{gathered} A = P(1 + rt) \\ 4{,}816.70 = 3500(1 + 0.099t) \\ 1.3762 = 1 + 0.099t \\ 0.3762 = 0.099t \\ 3.8 = t \end{gathered}$$

The length of Prisha's loan period is 3.8 years. In terms of years and months, that is NOT equivalent to 3 years 8 months. Rather, the 0.8 of one year must be converted to months.

$$0.8\ years \cdot \frac{12\ months}{1\ year} = 9.6\ months$$

In other words, the length of Prisha's loan is approximately 3 years 9.6 months.

➢ **EXAMPLE 3.2.6:** Jia is at his bank and choosing between two accounts in which to invest his money. Jia has \$500 to invest today. The first account (account A) has a promotional \$15 bonus for opening an account with a modest 0.4% interest rate. The \$15 does not earn interest. The second account (account B) has a 4.1% interest rate with no promotional bonus.

a. If Jia invests his money for 1 year, which account is the better choice?
b. If Jia invests his money for 6 months, which account is the better choice?
c. For what length of time are the two accounts equal in future value?

SOLUTION:

a. Begin by choosing an appropriate formula and stating the known variables. One formula that could be used is $A = P(1 + rt)$. For both accounts $P = 500$, $t = 1$, and A is unknown. For account A, $r = 0.004$. For account B, $r = 0.041$.

For account A, start with the formula $A = P(1 + rt)$.

$$A = P(1 + rt)$$
$$A = 500(1 + 0.004 \cdot 1)$$
$$A = 502$$

Since account A comes with a \$15 bonus, Jia's future value is $502 + 15 = 517$.

With account A, Jia will have \$517 at the end of 1 year.

For account B, start with the formula $A = P(1 + rt)$.

$$A = P(1 + rt)$$
$$A = 500(1 + 0.041 \cdot 1)$$
$$A = 520.5$$

With account B Jia will have \$520.50 at the end of 1 year.

Hence account B is the better choice.

b. In part b, the only change is that $t = 0.5$ rather than $t = 1$.

For account A, start with the formula $A = P(1 + rt)$.

$$A = P(1 + rt)$$
$$A = 500(1 + 0.004 \cdot 0.5)$$
$$A = 501$$

Since account A comes with a \$15 bonus, Jia's future value is $501 + 15 = 516$.

With account A Jia will have \$516 at the end of 6 months.

For account B, start with the formula $A = P(1 + rt)$.

$$A = P(1 + rt)$$
$$A = 500(1 + 0.041 \cdot 0.5)$$
$$A = 510.25$$

With account B Jia will have \$510.25 at the end of 6 months.

Hence account A is the better choice.

c. First, it's important to realize that the two accounts being equal in value means that:

Future amount of account A = Future amount of account B.

Also remember the future value of account A is $A = P(1 + rt) + 15$ due to the \$15 promotional bonus.

The time t is unknown.

That is,

Due to the extra \$15, distribution is a more appropriate method instead of dividing by 500

$$A + 15 = A$$
$$P(1 + rt) + 15 = P(1 + rt)$$
$$500(1 + 0.004t) + 15 = 500(1 + 0.041t)$$
$$500 + 2t + 15 = 500 + 20.5t$$
$$515 + 2t = 500 + 20.5t$$
$$15 = 18.5t$$
$$0.81081 \approx t$$

So, in 0.81081 of 1 year, or 9.73 months, the future value of both accounts is equal. That is, if Jia invests for less than 9.73 months, account A is the better choice. If Jia invests for more than 9.73months, account B is the better choice. (Note this is with Jia's $500. If he has a different amount of money to invest, the time it takes for the future value of both accounts to be equal will differ.)

❖ **YOU TRY IT 3.2.B:** Jamal is saving up to buy his first car. Jamal found a simple interest savings account that has a 1.5% interest rate but costs $25 to open. Jamal has $2,000 to place into the savings account and plans to leave it there for 18 months before purchasing his car. Is it worth it for Jamal to open the account? How much money will Jamal make or lose by opening the account?

In the next section, the simple interest formulas and concepts will be used to develop compound interest formulas. Moving forward, the formula work becomes more complex. It is important to have a good grasp of the arithmetic and algebra in this section.

Quick Review

Quantity	Variable	Details
Interest	I	I is stated in dollars
Interest rate	r	r is in decimal form
Principal	P	P is stated in dollars
Time	t	t is usually considered in years
Amount	A	A is stated in dollars

Formula	Title
$I = Prt$	Simple Interest Formula
$A = P + I$ or $A = P(1 + rt)$	Future Value for Simple Interest Formula

YOU TRY IT 3.2.A SOLUTION:

a. The interest rate on the account is 0.265%.

b. The future value on the account is $3,810.70.

YOU TRY IT 3.2.B SOLUTION:

The future value on Jamal's account is $2,045. Since Jamal spent $25 to open the account, he would only earn $20 interest over 18 months. ($2045 - 2000 - 25 = 20$).

Jamal would have to decide if it is worth it to him. If he's not going to touch the money for 18 months, then putting it in this savings account is better than holding onto it at home where it does not earn any interest.

Section 3.2 | Exercises

In exercises 1 – 8, given the following values, use the simple interest formula $I = Prt$ to find the remaining variable. When solving for t, state the answer in years and months (months rounded to the nearest tenth). When solving for r, round to the nearest hundredth.

1. $P = 515, r = 2.3\%, t = 4$ years
2. $P = 1425, r = 3.75\%, t = 7$ years
3. $I = 12.75, r = 3.3\%, t = 10$ years
4. $I = 100{,}000, r = 0.75\%, t = 6$ months
5. $P = 1{,}200, I = 15.25, t = 3$ months
6. $P = 630, I = 21.84, t = 8$ months
7. $P = 1{,}500, I = 23.45, r = 2.5\%$
8. $P = 10{,}575, I = 2{,}350, r = 8.25\%$

In exercises 9 – 16, given the following values, use the future value for simple interest formula $A = P(1 + rt)$ to find the remaining variable. When solving for t, state the answer in years and months (months rounded to the nearest tenth). When solving for r, round to the nearest hundredth.

9. $P = 1{,}534, r = 0.25\%, t = 13.51$ years
10. $P = 285, r = 1.65\%, t = 4.5$ years
11. $A = 2{,}312, r = 1.3\%, t =$ 3 years 4 months
12. $A = 947.34, r = 3.8\%, t =$ 5 years 7 months
13. $A = 1{,}500, P = 1{,}200, t = 3.7$ years
14. $A = 500{,}000, P = 200{,}000, t = 10$ years
15. $A = 1{,}075.2, P = 875, r = 1.3\%$, find t in months and years
16. $A = 5{,}723.39, P = 5{,}240, r = 2.7\%$, find t in months and years

In exercises 17 – 26, solve the application problems.

17. Lucas is investing his tip earnings from the last 6 months of his waiter job in a savings account. Lucas has $7,200 in tip earnings saved up. The savings account earns 2.5% simple interest. How much interest will Lucas have at the end of 6 months? What is the future value on Lucas's account?

18. Tom received a total of $2,658 for his graduation. He would like to invest this in an online savings account that offers 1.75% simple interest. How much will Tom have in the account after 5 years? How much interest will he earn during this time?
19. Rebs has a savings account earning 3.7% simple interest. She has $825.32 in the account now. How much did she have in the account when she opened it 18 months ago?
20. Monica wants to have $5,000 in her savings account. The account earns 4.5% simple interest. If she leaves the money in her account for 2 years and 9 months, find the amount she initially invested in the account.
21. David and Esteban are saving for when they take their band on tour. David puts $800 in a savings account that earns 4.2% simple interest. Six months later, Esteban puts $925 in the account. How much is in their account another 6 months later (so at the end of 1 year total)?
22. Tabitha wants to invest $460 in a savings account earning 3.25% simple interest for 2 years. She then invests $250 more into the account for an additional 3 years. Find the amount in the account at the end of the five years.
23. Syed is comparing savings accounts earning simple interest. One savings account (account A) advertises “Invest with us and your $100 could become $110 at the end of 1 year!” The other savings account (account B) advertises “Deposit your money here and we guarantee to double your money at the end of 10 years!” Compare the interest rates of the two accounts.
24. Samantha invests her money in an account earning simple interest. She invests $275 and has $302.74 at the end of 2 years. What simple interest rate does her account earn?
25. Mateo would like to invest the money he made over summer break in a savings account. He has $2,400 to invest and found an account offering 1.6% simple interest. How long would it take for his money to double?
26. Dharati would like to save $3,000 to take a trip. She invests $2,500 now in an account that earns 2.96% simple interest. How long will it take for her to achieve her goal? Give the answer in years and months.

Before calculators and technology were ubiquitous, it was simpler for banks to calculate interest out of 360 days per year rather than 365 (it simplified calculations). Using 360 days in a year is commonly known as banker’s rule or ordinary interest. Any part of a year was treated as that number of days in question divided by 360. In exercises 27 and 28, use the banker’s rule to calculate simple interest.

27. Jessica invests $1,250 for 2 years and 2 months (those 2 months being January and February – January with 31 days and February with 28 days). The account earns 0.87% simple interest. How much is in the account at the end of the 2 years and 2 months?
28. Simon invested $5,420 into a savings account that earns 1.3% simple interest. Using banker’s rule, how much is in the account at the end of 1 year and 147 days?

In exercises 29 – 32, solve the following application problems.

29. \$100 is invested in an account earning 3% simple interest for 2 years. Find the future value of the account in two ways.
    a. First use $I = Prt$ to find the interest earned. Then use $A = P + I$ to find the future value.
    b. Use $A = P(1 + rt)$ to find the future value.
    c. Do the methods give the same solution? Which method do you prefer and why?
30. \$100 is invested in an account earning 3% annual simple interest for 6 months. Find the future value of the account in two ways.
    a. First, leave the interest rate in terms of an annual interest rate and convert the time to years. Then find the future value.
    b. Second, convert the annual interest rate into a monthly interest rate. Then find the future value using both an interest rate and time in terms of months.
    c. Do the methods give the same solution? Why do you think mathematicians generally advise using an annual interest rate and time in years?
31. Anuli is saving her money to purchase a car and is comparing two different savings accounts. The first account (account A) has a promotional bonus of \$50 for opening an account with an interest rate of 0.75%. The promotional bonus does not earn interest. The second account (account B) does not offer a promotional bonus; however, it does offer a higher interest rate of 2.25% simple interest. If Anuli plans to invest \$3,500 for three years, which account is the better choice?
32. Ozzie would like to save money for a new laptop and is deciding between two online savings accounts. Account A offers a simple interest rate of 2.25% and has a fee of \$25 to open an account. Account B is free to open and offers a simple interest rate of 0.75%.
    a. If Ozzie invests \$500 for 2 years and 6 months, which account is better?
    b. If Ozzie invests \$500 for 4 years and 3 months, which account is better?
    c. If Ozzie invests \$500, for what length of time are the two accounts equal in future value? Give the solution in years and months.

Section 3.2 | Exercise Solutions

1. $47.38
2. $374.06
3. $38.64
4. $26,666,666.67
5. 5.08%
6. 5.2%
7. 7.5 months
8. 2 years and 8.3 months
9. $1,585.81
10. $306.16
11. $2,215.97
12. $781.53
13. 6.76%
14. 15%
15. 17 years and 7.2 months
16. 3 years and 5 months
17. Interest: $90. Future value: $7,290.
18. Amount in account: $2,890.58.
    Interest earned: $232.58.
19. $781.92
20. $4,449.39
21. $1,778.03
22. $809.13
23. 10% for either account
24. 5.04%
25. 62 years and 6 months
26. 6 years and 9.081 months
27. $1,273.53
28. $5,519.23
29. a. $106
    b. $106
    c. Same answer. The method choice is a personal preference.
30. a. $101.5
    b. $101.5
    c. Same answer. Answers will vary, but one possible reason is convention; calculations are simpler when institutions follow the same convention.
31. Account A: $3,628.75 (3,578.75 + 50 bonus)
    Account B: $3,736.25
    Account B is the better choice.
32. a. Account B ($509.38) is better than account A ($503.13).
    b. Account A ($522.81) is better than account B ($515.94).
    c. It will take 3 years and 4 months.

Section 3.3 | Compound Interest

Objectives

- Solve problems involving compound interest
- Solve compound interest formulas for principal, rate, and time
- Solve problems using Annual Percentage Yield (APY)

In the previous section, simple interest accounts were discussed in which a one-time lump sum (principal) is deposited and interest for the account is calculated once. The discussion will progress to interest accounts in which the principal is deposited while interest for the account is calculated at regular intervals throughout the length of the account.

Compound Interest

DEFINITION: **Compound Interest** is when the principal amount is invested once and interest is calculated on the entire principal at regular intervals throughout the length of the account.

Before presenting the formula for compound interest, it is important to view an example of a compound interest account. Suppose Max invested \$100 into a compound interest account and left the money in the account for three months. Interest is calculated monthly with an annual interest rate of 12%. How much money will Max have at the end of the three months?

	Principal in the account	Value of the account at the end of the month
Month 1	\$100	$A = P(1+rt) = 100\left(1+0.12*\frac{1}{12}\right) = 101$
Month 2	\$101	$A = P(1+rt) = 101\left(1+0.12*\frac{1}{12}\right) = 102.01$
Month 3	\$102.01	$A = P(1+rt) = 102.01\left(1+0.12*\frac{1}{12}\right) \approx 103.03$

At the end of three months, Max will have \$103.03.

The following table shows the calculations in terms of variables rather than with specific values substituted.

	Principal in the account	Value of the account at the end of the month
Month 1	P	$A = P(1+rt)$
Month 2	$P(1+rt)$	$A = [P(1+rt)](1+rt) = P(1+rt)^2$
Month 3	$P(1+rt)^2$	$A = [P(1+rt)^2](1+rt) = P(1+rt)^3$

Two new variables are necessary for the compound interest formula. The variable $\boldsymbol{m}$ is used to represent the number of times interest is compounded per year. For example, interest could be compounded annually ($m = 1$), semiannually ($m = 4$), quarterly ($m = 4$), monthly ($m = 12$), bimonthly (twice a month, $m = 24$), biweekly (every other week, $m = 26$), or daily ($m = 365$). The variable $\boldsymbol{n}$ represents the total number of compounding periods over the life of the account. If an account has interest compounded quarterly for five years, then interest would be compounded $4 \times 5 = 20$ times over the five years of the account. A summary of the variables used in the compound interest formula can be found in Figure 3.3.1.

FIGURE 3.3.1

Quantity	Variable	Details
Interest rate	r	r is in decimal form
Principal	P	P is stated in dollars
Time	t	t is usually considered in years
Amount	A	A is stated in dollars
Number of compounds per year	m	Common values of m are annually (1), quarterly (4), monthly (12), and daily (365).
Total number of compounding periods	n	n is found by taking $m \cdot t$

Compound Interest Formula

$$A = P\left(1 + \frac{r}{m}\right)^n$$

Note: $n = mt$

NOTE: Throughout Chapter 3, intermediate steps will not be rounded. Intermediate steps will be shown rounded to six decimal places only so students can follow along. Rounding in intermediate steps will cause the final solution to be slightly different than if rounding is only done in the final answer.

Now it is possible to compute Max's account balance using the compound interest formula from the previous page. As a reminder, Max invested \$100 into a compound interest account and kept the money in the account for three months. Interest is calculated monthly with an annual interest rate of 12%. How much money will Max have at the end of the three months?

Start by listing the variables and amounts provided. In this case, $P = 100, r = 12\%$, $m = 12$, and $n = 3$ to represent the three times interest was calculated during the account.

$$A = P\left(1 + \frac{r}{m}\right)^n$$

$$A = 100\left(1 + \frac{0.12}{12}\right)^3$$

Substitute values

$$A = 100(1 + 0.01)^3$$

Simplify in the parentheses

$$A = 100(1.01)^3$$

$$A = 100(1.030301)$$

Compute the exponent

$$A = 103.0301$$

Multiply

The amount in Max's account after three months would be \$103.03 which is the same answer obtained using repeated monthly interest calculations at the beginning of this section. Using the compound interest formula will be more efficient when an account compounds interest multiple times.

- **EXAMPLE 3.3.1:** Harry deposits \$1,200 into a savings account that earns 3% annual interest compounded quarterly.
  a. Find the amount in the account at the end of five years.
  b. Find the amount of interest earned.

  SOLUTION:

  a. Start by listing the known values: $P = 1{,}200, r = 3\%, m = 4$ (quarterly), $n = 4 \times 5 = 20$. Then solve for the remaining variable.

$$A = P\left(1 + \frac{r}{m}\right)^n$$

$$A = 1{,}200\left(1 + \frac{0.03}{4}\right)^{20}$$

$$A = 1{,}200(1 + 0.0075)^{20}$$
$$A = 1{,}200(1.0075)^{20}$$

$$A \approx 1{,}200(1.161184)$$

$$A \approx 1{,}393.420971$$

The account will have \$1,393.42 at the end of five years.

b. To find the interest earned, subtract the amount deposited from the amount in the account: $1{,}393.42 - 1{,}200 = 193.42$. Over five years, the account will earn \$193.42 in interest.

➢ **EXAMPLE 3.3.2:** Mila deposits \$4,600 into a savings account that earns 1.8% annual interest compounded semiannually.

a. Find the amount in the account at the end of 20 years.

b. Find the amount of interest earned.

SOLUTION:

a. Start by listing the known values: $P = 4{,}600, r = 1.8\%, m = 2$ (semiannually), $n = 2 \times 20 = 40$. Then solve for the remaining variable.

$$A = P\left(1 + \frac{r}{m}\right)^n$$

$$A = 4{,}600\left(1 + \frac{0.018}{2}\right)^{40}$$

$$A = 4{,}600(1 + 0.009)^{40}$$
$$A = 4{,}600(1.009)^{40}$$

$$A \approx 4{,}600(1.431023)$$

$$A \approx 6{,}582.706338$$

The account will have \$6,582.71 at the end of 20 years.

b. To find the interest earned, subtract the amount deposited from the amount in the account: $6{,}582.71 - 4{,}600 = 1{,}982.71$. Over twenty years, the account will earn \$1,982.71 in interest.

❖ **YOU TRY IT 3.3.A:** Ellie deposits \$2,740 into a savings account that earns 4.2% annual interest compounded monthly.

d. Find the amount in the account at the end of six years.

e. Find the amount of interest earned.

➢ **EXAMPLE 3.3.3:** Sam is saving money for a big vacation in 3 years after graduation. He plans to set aside money into an account earning 5.62% interest compounded monthly. He predicts that he will need $1,700 for his vacation.

a. Find the amount Sam should deposit in the savings account today so that it will grow to have $1,700 in 3 years.

b. Find the amount of interest earned.

SOLUTION:

a. In this case, the amount of the account is known to be $1,700; however, the principal is unknown. Start by listing the known values: $A = 1{,}700, r = 5.62\%$, $m = 12\ (\text{monthly}), n = 12 \times 3 = 36$. Then solve for the remaining variable.

$$A = P\left(1+\frac{r}{m}\right)^n$$

$$1{,}700 = P\left(1+\frac{0.0562}{12}\right)^{36}$$

Substitute values

$$1{,}700 \approx P(1+0.004683)^{36}$$

Simplify in the parentheses

$$1{,}700 \approx P(1.004683)^{36}$$

$$1{,}700 \approx P(1.183181)$$

Compute the exponent

$$\frac{1{,}700}{1.183180822} \approx \frac{P(1.183181)}{1.183181}$$

Divide to isolate the P variable

$$P \approx 1{,}436.804898$$

Sam will need to deposit $1,436.80 now to have a total of $1,700 in the bank account in three years.

b. To find the interest earned, subtract the amount deposited from the amount in the account: $1{,}700 - 1{,}436.80 = 263.20$. Over three years, the account will earn $263.20 in interest.

❖ **YOU TRY IT 3.3.B:** Julia needs $2,400 to purchase a new sofa in 5 years when she purchases her new home. She plans place money into a savings account that earns 2.75% annual interest compounded quarterly.

a. Find the amount Julia should deposit in the savings account today so that it will grow to have $2,400 in 5 years.

b. Find the amount of interest earned.

➢ **EXAMPLE 3.3.4:** Marius has \$1,400 to deposit into an account. The bank employee tells him it will take 7 years for his money to grow to \$2,184. If interest is compounded monthly, what is the annual interest rate the bank is offering for this savings account? Round the answer to the nearest hundredth of a percent.

SOLUTION:

In this case, the principal and amount of the account are known. However, the rate is unknown. Start by listing the known values: $A = 2{,}184, P = 1{,}400, m = 12,$ $n = 12 \times 7 = 84$. Then solve for the remaining variable.

$$A = P\left(1 + \frac{r}{m}\right)^n$$

$$2{,}184 = 1{,}400\left(1 + \frac{r}{12}\right)^{84}$$ Substitute values

$$\frac{2{,}184}{1{,}400} = \frac{1{,}400\left(1 + \frac{r}{12}\right)^{84}}{1{,}400}$$ Divide

$$1.56 = \left(1 + \frac{r}{12}\right)^{84}$$

$$(1.56)^{\frac{1}{84}} = \left[\left(1 + \frac{r}{12}\right)^{84}\right]^{\frac{1}{84}}$$ Take the 84th root of each side. This is the same as raising each side to the $\frac{1}{84}$th power.

$$1.005308 \approx 1 + \frac{r}{12}$$

$$0.005308 \approx \frac{r}{12}$$ Subtract 1

$$0.063695 \approx r$$ Multiply by 12

$$6.369499\% \approx r$$ Convert to a percent

The annual interest rate of the bank account is 6.37%.

❖ **YOU TRY IT 3.3.C:** Layla made \$8,500 while working in the summer. She would like to have \$10,000 in her bank account. The bank employee tells her it will take 5 years. If interest is compounded quarterly, what is the annual interest rate the bank is offering for this savings account? Round the answer to the nearest hundredth of a percent.

➢ **EXAMPLE 3.3.5:** Chantal wants to know how long it takes to make her \$500 deposit increase to \$600 using an account with an annual interest rate of 3% compounded annually. How long would it take for her deposit to increase to \$600? (Hint: Review solving with logarithms in the prerequisite Section 3.B before Chapter 3). Round the answer to the nearest hundredth of a percent.

SOLUTION:

In this case, the principal and amount of the account are known. However, the time is unknown. Start by listing the known values: $A = 600, P = 500, r = 3\%, m = 1$, $n = 1 \times t$. Then solve for the remaining variable.

$$A = P\left(1 + \frac{r}{m}\right)^n$$

$$600 = 500\left(1 + \frac{0.03}{1}\right)^{1t}$$

Substitute values

$$600 = 500(1 + 0.03)^t$$
$$600 = 500(1.03)^t$$

Simplify in the parentheses

$$\frac{600}{500} = \frac{500(1.03)^t}{500}$$

Divide

$$1.2 = (1.03)^t$$

$$\ln(1.2) = \ln(1.03)^t$$

Take the natural log of each side

$$\ln(1.2) = t\ln(1.03)$$

Use power rule of logarithms

$$\frac{\ln(1.2)}{\ln(1.03)} = \frac{t\ln(1.03)}{\ln(1.03)}$$

Divide

$$6.168097 \approx t$$

Without any additional payments, Chantal will need 6.17 years for her money to increase from \$500 to \$600.

❖ **YOU TRY IT 3.3.D:** Simon deposited his graduation gifts of \$350 into an account earning an annual interest rate of 4.2% compounded quarterly. He plans to purchase a new Apple Watch for \$399. Without any additional deposits, how long would it take for his deposit to increase to \$399? Round your answer to the nearest hundredth.

In the next example, interest that is compounded daily will be considered. For the purposes of this text, daily compounding will assume 365 days in a year. However, some banks utilize ordinary interest (also called banker's interest) where interest is compounded 360 times instead of 365 times for ease of calculation.

- **EXAMPLE 3.3.6:** Eiley deposits $1,050 into a savings account that earns 2.3% annual interest compounded daily.
  a. Find the amount in the account at the end of 7 years.
  b. Find the amount of interest earned.

SOLUTION:

a. Start by listing the known values: $P = 1{,}050, r = 2.3\%, m = 365\ \text{(daily)}$, $n = 365 \times 7 = 2555$. Then solve for the remaining variable.

$$A = P\left(1 + \frac{r}{m}\right)^n$$

$$A = 1{,}050\left(1 + \frac{0.023}{365}\right)^{2555}$$

$$A \approx 1{,}050(1 + 0.000063014)^{2555}$$

$$A \approx 1{,}050(1.174679)$$

$$A \approx 1233.412961$$

When simplifying $\frac{0.023}{365}$ a calculator will probably give the answer 6.301370 E -5. This is written in scientific notation and is equivalent to 0.000063014.

The account will have $1,233.41 at the end of 7 years.

b. To find the interest earned, subtract the amount deposited from the amount in the account: $1{,}233.41 - 1{,}050 = 183.41$. Over seven years, the account will earn $183.41 in interest.

- **YOU TRY IT 3.3.E:** Bryson deposits $1,800 into a savings account that earns 3.8% annual interest rate compounded daily.
  a. Find the amount in the account at the end of 10 years.
  b. Find the amount of interest earned.

Compound Continuously

So far in this text examples have included accounts where interest was compounded annually, semiannually, quarterly, monthly, and daily. Is it possible to compound interest more often? By the hour or by the minute? Although compounding by the hour or minute are not typical examples, there is a specific function for **compounding continuously**. When interest is compounded continuously, interest is calculated every fraction of a second possible. In continuous compounding, it would be impossible to count the number of times interest is calculated. Continuous compounding is the upper bound or maximum amount of interest that can be calculated for an account. The formula for continuous compounding uses an irrational number $\boldsymbol{e}$ which is approximated as $e \approx 2.71828$.

Compound Continuously Formula

$$A = Pe^{rt}$$

A = total amount of the account
P = principal
r = annual interest rate (as a decimal)
t = time (in years)

- **EXAMPLE 3.3.7:** Sean deposits $10,300 into a savings account that earns 5.2% annual interest compounded continuously.
  a. Find the amount in the account at the end of 5 years.
  b. Find the amount of interest earned.

SOLUTION:

a. Start by listing the known values: $P = 10{,}300, r = 5.2\%, t = 5$. Then solve for the remaining variable.

$$A = Pe^{r\,t}$$
$$A = 10{,}300e^{0.052\times 5}$$
$$A = 10{,}300e^{0.26}$$
$$A \approx 10{,}300(1.296930087)$$
$$A \approx 13{,}358.37989$$

Simplify the exponent

Calculate the exponent expression

Multiply

The account will have $13,358.38 at the end of five years.

b. To find the interest earned, subtract the amount deposited from the amount in the account: $13{,}358.38 - 10{,}300 = 3{,}058.38$. Over five years, the account will earn \$3,058.38 in interest.

❖ **YOU TRY IT 3.3.F:** Aaliyah deposits \$5,600 into a savings account that earns 2.1% annual interest compounded continuously.

a. Find the amount in the account at the end of 8 years.

b. Find the amount of interest earned.

Like Example 3.3.5, it is possible to use logarithms to solve for either rate or time when compounding continuously.

➢ **EXAMPLE 3.3.8:** Gia deposits \$3,120 into a savings account and would like for her deposit to accumulate to \$4,000 in 5 years. If interest is compounded continuously, find the annual interest rate of the bank account. Round the answer to the nearest hundredth of a percent.

SOLUTION:

Start by listing the known values: $A = 4{,}000, P = 3{,}120,\ t = 5$. Then solve for the remaining variable.

$$
\begin{aligned}
A &= Pe^{r\,t} \\
4{,}000 &= 3{,}120e^{r\times 5} \\
\frac{4{,}000}{3{,}120} &= \frac{3{,}120e^{r\times 5}}{3{,}120} && \text{Divide} \\
1.282051282 &= e^{5r} \\
\ln(1.282051282) &\approx \ln(e^{5r}) && \text{Take the natural logarithm of both sides} \\
\ln(1.282051282) &\approx 5r\ln(e) && \text{Use power rule of logarithms} \\
\ln(1.282051282) &\approx 5r && \text{Simplify} \\
\frac{\ln(1.282051282)}{5} &\approx \frac{5r}{5} && \text{Divide} \\
0.0496922719 &\approx r \\
4.96922719\% &\approx r && \text{Convert to a percent}
\end{aligned}
$$

The account has an annual interest rate of 4.97%.

❖ **YOU TRY IT 3.3.G:** Kate deposits \$540 into a savings account earning 3.5% interest compounded continuously. Find the amount of time needed for the account to accumulate to \$700. Round the answer to the nearest hundredth.

Comparing Accounts

➢ **EXAMPLE 3.3.9:** Arun worked all summer and plans to deposit his \$5,200 earnings into an account. The bank offers two different savings accounts. Account A earns 3.4% annual interest compounded semiannually (two times a year) while account B earns 3.38% annual interest compounded monthly. If Arun plans to leave his savings in the account for 2 years, which account should he choose?

SOLUTION:

Find the amount in each account.
For account A, list the known values: $P = 5{,}200, r = 3.4\%, m = 2, n = 2 \times 2 = 4$.
For account B, list the known values: $P = 5{,}200, r = 3.38\%, m = 12, n = 12 \times 2 = 24$.

Account A	Account B
$A = P\left(1 + \frac{r}{m}\right)^n$	$A = P\left(1 + \frac{r}{m}\right)^n$
$A = 5{,}200\left(1 + \frac{0.034}{2}\right)^4$	$A = 5{,}200\left(1 + \frac{0.0338}{12}\right)^{24}$
$A = 5{,}200(1.017)^4$	$A \approx 5{,}200(1.0028166667)^{24}$
$A \approx 5{,}200(1.069753736)$	$A \approx 5{,}200(1.069835582)$
$A \approx \$5{,}562.719425$	$A \approx \$5{,}563.145027$

Arun should choose account B to have the higher amount in his bank account at the end of two years.

Comparing the final amount in the account is not the only way to compare different interest rates and number of compound periods. It is enough to compare the **effective interest rate** or **annual percentage yield** for one year. To develop the formula for annual percentage yield, begin with the compound interest formula and let $P = 1$ and $t = 1$ to represent what happens to \$1 in one year. Because time is set as one year, the values of m and n are equal: $n = m \times t = m \times 1 = m$. At the end of the formula, the one that is subtracted represents the initial 100% of principal that was deposited, leaving only the percent of increase during the year.

DEFINITION: **Annual percentage yield (APY)** is the one-year simple interest rate equivalent for an account that utilizes compounding interest.

Annual Percentage Yield (APY) Formula

$$APY = \left(1 + \frac{r}{m}\right)^m - 1$$

- **EXAMPLE 3.3.10:** Kadin is opening a bank account and he is offered two different interest rates. Account A offers an annual interest rate of 6.15% compounded semiannually while account B offers an annual interest rate of 6.1% compounded monthly. Which account should Kadin choose?

SOLUTION:

Find the annual percentage yield for each account.
For account A, list the known values: $r = 6.15\%, m = 2$.
For account B, list the known values: $r = 6.1\%, m = 12$.

APY for Account A	APY for Account B
$APY = \left(1 + \frac{r}{m}\right)^m - 1$	$APY = \left(1 + \frac{r}{m}\right)^m - 1$
$APY = \left(1 + \frac{0.0615}{2}\right)^2 - 1$	$APY = \left(1 + \frac{0.061}{12}\right)^{12} - 1$
$APY = (1 + 0.03075)^2 - 1$	$APY \approx (1 + 0.005083333)^{12} - 1$
$APY \approx 1.062445563 - 1$	$APY \approx 1.06273469 - 1$
$APY \approx 0.062445563$	$APY \approx 0.06273469$
$APY \approx 6.24\%$	$APY \approx 6.27\%$

Account A with interest at 6.15% compounded semiannually would be equivalent to a single simple interest calculation of 6.24%. Account B with interest at 6.1% compounded monthly would be equivalent to a single simple interest calculation of 6.27%. Kadin should choose account B which has the higher simple interest rate equivalent.

NOTE: The APY calculation does not require the principal amount or the length of time for the account to determine the equivalent simple interest rate.

❖ **YOU TRY IT 3.3.H:** Clark was given two savings account options from Bank of Metropolis. Account A earns 4.72% annual interest compounded semiannually while account B earns 4.7% annual interest compounded monthly. Which account should Clark choose? Round the interest rates to the nearest hundredth of a percent.

Quick Review

Compound Interest Formula

$$A = P\left(1 + \frac{r}{m}\right)^{n}$$

Note: $n = mt$

Compound Continuously Formula

$$A = Pe^{rt}$$

Annual Percentage Yield (APY) Formula

$$APY = \left(1 + \frac{r}{m}\right)^{m} - 1$$

YOU TRY IT 3.3.A SOLUTION:

a. $\$3{,}523.72$
b. $\$783.72$

YOU TRY IT 3.3.B SOLUTION:

a. $\$2092.67$
b. $\$307.33$

YOU TRY IT 3.3.C SOLUTION:

3.26%

YOU TRY IT 3.3.D SOLUTION:

3.14 years

YOU TRY IT 3.3.E SOLUTION:

a. $\$2{,}632.06$
b. $\$832.06$

YOU TRY IT 3.3.F SOLUTION:

a. $\$6{,}624.45$

b. $\$1{,}024.45$

YOU TRY IT 3.3.G SOLUTION:

7.41 years

YOU TRY IT 3.3.H SOLUTION:

Account A has $APY \approx 4.78\%$ while account B has $APY \approx 4.80\%$. Clark should choose account B to have the most money in his account.

Section 3.3 Exercises

In Exercises 1 – 8, use the compound interest formula to solve for the missing variable. Round answers to the nearest hundredth.

1. $P = 3{,}100, r = 2.5\%, m = 12, t = 7$
2. $P = 17{,}500, r = 4.1\%, m = 4, t = 10$
3. $A = 5{,}900, r = 3.7\%, m = 2, t = 5$
4. $A = 11{,}250, r = 2.8\%, m = 4, t = 8$
5. $A = 3{,}800, P = 2{,}000, r = 2.6\%, m = 12$
6. $A = 125{,}000, P = 100{,}000, r = 3.25\%, m = 1$
7. $A = 10{,}000, P = 9{,}200, m = 2, t = 10$
8. $A = 57{,}805, P = 55{,}100, m = 12, t = 3$

In Exercises 9 – 24, use the compound interest formula to solve for the missing variable. Round answers to the nearest hundredth.

9. Nola deposited \$3,240 into a savings account that earns 3.3% annual interest rate compounded monthly.
   a. Find the amount in the account after 15 years.
   b. Find the amount of interest earned.

10. William received a holiday bonus from work and decides to deposit the whole \$8,500 check into a savings account that earns 2.9% annual interest rate compounded quarterly.
    a. Find the amount in the account after 9 years.
    b. Find the amount of interest earned.

11. Erik received a \$500 bonus from work. He deposited the money into a savings account that earns 4.9% annual interest compounded daily.
    a. Find the amount in the account after 6 years.
    b. Find the amount of interest earned.

12. Manuel deposited \$1,875 into a savings account that earns 6.42% annual interest rate compounded semiannually.
    a. Find the amount in the account after 10 years.
    b. Find the amount of interest earned.

13. Martina deposited \$2,875 into a savings account that earns 5.1% annual interest compounded daily. Find the amount in the account at the end of 3 years.
    a. Find the amount in the account after 3 years.
    b. Find the amount of interest earned.

14. Haley invests \$500 when she is 20 years old in an account with a 3.75% annual interest rate compounded annually.
    a. Find the amount in the account when she is 50 years old.
    b. Find the amount of interest earned.

15. LeBron invests \$1 million in an account paying 5.5% annual interest compounded semiannually.
    a. Find the amount in the account after 10 years.
    b. Find the amount of interest earned.

16. Oksana invests \$125,650 in an account paying 4.75% annual interest compounded quarterly.
    a. Find the amount in the account after 17 years.
    b. Find the amount of interest earned.

17. Selma needs \$4,500 in 2 years to pay for college tuition. How much should she invest today into a savings account that earns 4.25% annual interest rate compounded semiannually?
    a. Find the principal amount.
    b. Find the amount of interest earned.

18. Noah plans to purchase a used car for about \$12,000 in five years. How much should he invest today into a savings account that earns 6.7% annual interest rate compounded quarterly?
    a. Find the principal amount.
    b. Find the amount of interest earned.

19. Eva needs \$8,500 in 8 years to put a down payment on a new condo. How much should she invest today into a savings account that earns 5.85% annual interest rate compounded daily?
    a. Find the principal amount.
    b. Find the amount of interest earned.

20. Lucia plans to purchase a TV when she moves into a new apartment after college in 3 years. She expects that she will need \$475. How much should she invest today into a savings account that earns 2.95% annual interest rate compounded semiannually?
    a. Find the principal amount.
    b. Find the amount of interest earned.

21. Maeve dreams of buying a new boat. She is planning ahead so that she can purchase a boat costing \$23,500 in 10 years. How much should she invest today into a savings account that earns 3.85% annual interest rate compounded monthly?
    a. Find the principal amount.
    b. Find the amount of interest earned.

22. Arjun would like to take a trip to Europe in three years that will cost \$3,000. How much should Arjun invest now at 4.5% annual interest compounded annually to accomplish this?
    a. Find the principal amount.
    b. Find the amount of interest earned.

23. Kono will be purchasing a condominium in Honolulu in eight years. She needs a down payment of \$70,000. How much must she invest now into an account offering an annual interest rate of 5.25% compounded monthly?
    a. Find the principal amount.
    b. Find the amount of interest earned.

24. Sean needs \$100,000 for college in ten years. How much must he invest now into an account offering an annual interest rate of 4.75% compounded quarterly?
    a. Find the principal amount.
    b. Find the amount of interest earned.

In Exercises 25 – 31, use the compound interest formula and information given to solve for the interest rate. Round answers to the nearest hundredth of a percent.

25. Asher received \$1,460 for his Bar mitzvah. He needs \$1,700 to enroll in a summer outdoor adventure camp in two years. If interest is compounded monthly, what is the annual interest rate that he needs to have enough money for the summer camp?

26. Kai received $465 for his graduation celebration. He needs $500 to buy tickets to Lollapalooza next year (1 year). If interest is compounded daily, what is the annual interest rate that he needs to have enough money to purchase tickets?

27. Maria wants to pay $250 for an extreme driving experience in four years when she turns 16. She currently has $197 in her bank account. If interest is compounded semiannually, what is the annual interest rate that she needs to have enough money in four years?

28. Adam made $3,250 during the summer. He needs $5,000 when he graduates in three years to be able to take an international trip. If interest is compounded quarterly, what is the annual interest rate the bank is offering for this savings account?

29. Sofia wants to purchase a record player for her mom's 60^{th} birthday celebration in two years. She has $278 saved but needs $310. If interest is compounded daily, what is the annual interest rate that she needs in order to have enough money to purchase the record player?

30. Miguel made an investment of $25,000 that grew to $50,000 in 25 years. What was the interest rate of this investment if interest was compounded annually?

31. Chase wants $500,000 when he retires in 10 years. He has $150,000 to invest in a savings account. What annual interest rate must he get if he wants interest compounded daily?

In Exercises 32 – 37, use the compound interest formula and information given, to solve for time. Round answers to the nearest hundredth.

32. Jana has $825 saved. If she uses an account that earns 7.46% interest compounded monthly, how long will it take for her money to grow to $1,000?

33. Meredith won the lottery and received a check for $687,390. She would like to say she has a million dollars in her account. If she uses an account that earns 4.85% interest compounded quarterly, how long will it take for her money to grow to $1,000,000?

34. Bill saved $17.25 from his lemonade stand. He wonders how long it will take for the money to grow to $20. If he uses an account that earns 5.42% interest compounded monthly, how long will it take for his account to grow to $20?

35. Tobias is saving money to take a helicopter ride to view the Grand Canyon. He knows that he needs \$350 for the 45-minute tour. He currently has \$217 saved. If his bank account offers 3.65% annual interest compounded daily, how much time will he have to wait to purchase the helicopter tour ticket?

36. Li invests \$10,000 in an account with a rate of 2.75% compounded semiannually. How many years will it take for the investment to double?

37. Lorenzo invested \$37,500 in an account with an annual rate of 4.35% compounded daily. How many years will it take for the investment to be \$200,000?

In Exercises 38 – 47, use the compound continuously formula to solve for the missing variable. Round answers to the nearest hundredth.

38. Jazz deposited \$2,850 into a savings account earning 3.75% interest compounded continuously. Find the amount in the account at the end of 7 years.
    a. Find the amount at the end of the account.
    b. Find the amount of interest earned.

39. Sandro needs \$5,000 in 8 years to purchase a used car. He plans to put his savings into an account earning 4.8% interest compounded continuously. Find the amount he needs to invest so that his money grows to \$5,000.
    a. Find the amount invested.
    b. Find the amount of interest earned.

40. Rohan has saved \$18,450 to purchase a new motorcycle. He needs \$19,700. If he plans to put his savings into an account earning 2.95% interest compounded continuously, how much time will it take for his account to have \$19,700?

41. Brandy would like to have \$100,000 in her savings account by the time she is 50. By age 18, she has currently saved \$48,500. She plans to put her savings into an account that compounds interest continuously. What interest rate is needed for her to meet her goal?

42. Micah deposited \$12,500 into a savings account earning 5.14% interest compounded continuously. Find the amount in the account at the end of 20 years.
    a. Find the amount at the end of the account.
    b. Find the amount of interest earned.

43. Oliver would like to have $1,500 to purchase a new computer when he starts college in three years. He currently has $1,300 in savings. If he puts his money in a savings account compounding continuously, find the interest rate needed for him to meet his goal.

44. Laura has saved $3,270. She plans to purchase a new Vespa scooter once she has $3,900 in her bank account. If she plans to put her savings into an account earning 4.75% interest compounded continuously, how much time will it take for her to meet her goal?

45. Jordan would like to have $2,500 in his bank account at the end of college in four years. He plans to put his savings into an account earning 3.5% interest compounded continuously. Find the amount he needs to invest so that his money grows to $2,500.
    a. Find the amount invested.
    b. Find the amount of interest earned.

46. Abishola wants $60,000 to purchase a new car in three years. She will be depositing $47,000 in an account with interest compounded continuously. What interest rate must she receive to purchase the car?

47. Mohammed invested $50,000 in an account at 4.15% interest compounded continuously. Find the amount in the account after 5 years.
    a. Find the amount at the end of the account.
    b. Find the amount of interest earned.

In Exercises 48 – 54, use the APY formula to find the equivalent simple interest rate. Round answers to the nearest hundredth of a percent.

48. Find the equivalent simple interest rate for investing money into an account earning 3.75% interest compounded quarterly.

49. Find the equivalent simple interest rate for investing money into an account earning 5.25% compounded semiannually.

50. Find the equivalent simple interest rate for investing money into an account earning 6.45% compounding monthly.

51. Find the equivalent simple interest rate for investing money into an account earning 3% compounded daily.

52. Mariana is comparing savings accounts. Bank A is offering a savings account earning 3.94% compounding annual. Bank B is offering a savings account earning 3.93% compounding daily. Which savings account offers a higher rate?

53. Puk is saving money toward purchasing her first home. Bank Forever is offering a savings account earning 2.18% compounding monthly. Bank of Savings is offering a savings account earning 2.22% compounding semiannually. Which savings account should Puk use?

54. Brijesh is starting his first savings account. First Bank of Math is offering a savings account earning 6.25% compounding quarterly. International Savings Bank is offering a savings account earning 6.2% compounded monthly. Which savings account should Brijesh use?

Section 3.3 | Exercise Solutions

1. Total: $3,692.09 Interest: $592.09
2. Total: $26,314.34 Interest: $8,814.35
3. Principal: $4,911.81 Interest: $988.19
4. Principal: $8,999.32 Interest: $2,250.68
5. 24.71 years
6. 6.98 years
7. 0.84%
8. 1.60%
9. a. $5,311.60 b. $2,071.60
10. a. $11,024.55 b. $2,524.55
11. a. $670.88 b. $170.88
12. a. $3,527.25 b. $1,652.25
13. a. $3,350.27 b. $475.27
14. a. $1,508.74 b. $1,008.74
15. a. $1,720,428.43 b. $720,428.43
16. a. $280,407.35 b. $154,757.35
17. a. $4,136.99 b. $363.01
18. a. $8,607.91 b. $3,392.09
19. a. $5,323.35 b. $3,176.65
20. a. $435.05 b. $39.95
21. a. $16,000.45 b. $7,499.55
22. a. $2,628.89 b. $371.11
23. a. $46,035.43 b. $23,964.57
24. a. $62,362.76 b. $37,637.24
25. 7.63%
26. 7.26%
27. 6.05%
28. 14.62%
29. 5.45%
30. 2.81%
31. 12.05%
32. 2.59 years
33. 7.78 years
34. 2.74 years
35. 13.10 years
36. 25.38 years
37. 38.48 years
38. a. $3,705.50 b. $855.50
39. a. $3,405.66 b. $1594.34
40. 2.22 years
41. 2.26%
42. a. $34,943.37 b. $22,443.37
43. 4.77%
44. 3.71 years
45. a. $2,173.40 b. $326.60
46. 8.14%
47. a. $61,529.89 b. $11,529.89
48. 3.80%
49. 5.32%
50. 6.64%
51. 3.05%
52. Bank B 4.01%
53. Bank of Savings 2.21%
54. First Bank of Math 6.40%

Section 3.4 | Annuities & Sinking Funds

Objectives

- Calculate the future value of an annuity
- Calculate the present value of an annuity
- Calculate the future value of an annuity due

NOTE: Throughout Chapter 3, intermediate steps will not be rounded. Intermediate steps will be shown rounded to six decimal places only so students can follow along. Rounding in intermediate steps will cause the final solution to be slightly different than if rounding is only done in the final answer.

Annuities

In the discussion about simple interest and compound interest thus far, individuals have invested a lump sum once and let the money sit in the account for a certain number of years earning interest. However, very few people have a large sum of money to invest in a savings account at any given point in time.

Banks and investment firms recognize this problem; hence annuities are used to counteract this dilemma. With an annuity, an individual makes a regular payment (typically monthly, though not necessarily) over a course of time. With each additional payment, the principal in the account increases. The account is earning compound interest as well. In this way, individuals can reap the benefits of compound interest without needing a large initial lump sum of money.

A few definitions are required before the discussion continues further.

DEFINITION: An **annuity** is an account for investing money at regular intervals throughout the length of the account. Interest is also compounded at regular intervals throughout the length of the account.

For the purposes of this text, money is invested and interest is compounded at the same interval.

DEFINITION: An annuity in which the repeated investment is made at the end of the compounding period is called an **ordinary annuity**.

DEFINITION: An annuity in which the repeated investment is made at the beginning of the compounding period is called an **annuity due**.

DEFINITION: The sum of all payments and interest that accumulated over the length of the investment is called the **future value** of an annuity.

Suppose Gail has \$100 to invest in an annuity every month for three months. Her annual interest rate is 12%. Gail is investing in an annuity due, where her payment is made at the beginning of the compounding period. How much money does Gail have in her annuity at the end of the three months?

We will procced using the compound interest formula.

	Principal in the account	Value of the account at the end of the month
Month 1	100	$A = P(1+rt) = 100\left(1+0.12*\frac{1}{12}\right) = 101$
Month 2	$101 + 100 = 201$	$A = P(1+rt) = 201\left(1+0.12*\frac{1}{12}\right) = 203.01$
Month 3	$203.01 + 100 = 303.01$	$A = P(1+rt) = 303.01\left(1+0.12*\frac{1}{12}\right) \approx 306.04$

Thus, the amount in Gail's account at the end of the 3 months is \$306.04. This process would become tedious if the annuity lasted years instead of months.

Based on the compound interest formula from Section 3.3, mathematicians developed the future value of an annuity formula. Figure 3.4.1, variables are presented. The first five variables are repeated from a previous section; the remaining variables are new.

FIGURE 3.4.1

Quantity	Variable	Details
Interest	I	I is stated in dollars
Interest rate	r	r is in decimal form
Principal	P	P is stated in dollars
Time	t	t is usually considered in years
Amount	A	A is stated in dollars
Regular payment	R	R is stated in dollars
Number of compounds per year	m	Common values of m are annually (1), quarterly (4), monthly (12), and daily (365).
Total number of compounding periods	n	n is found by taking $m \cdot t$

Future Value of an Ordinary Annuity Formula

$$A = R\left(\frac{\left(1+\frac{r}{m}\right)^{n}-1}{\frac{r}{m}}\right)$$

Note: $n = mt$

Future Value of an Annuity Due Formula

$$A = R\left(\frac{\left(1+\frac{r}{m}\right)^{n+1}-1}{\frac{r}{m}}\right) - R$$

Note: $n = mt$

- **EXAMPLE 3.4.1:** Use the appropriate future value of an annuity formula and the given variables to solve for the missing value. Round to the nearest cent.

  $R = 250, r = 2.7\%, m = 12, n = 60$. Payments are made at the beginning of the compounding period.

SOLUTION:

Since payments are made at the beginning of the compounding period, this is annuity due. Substitute the appropriate variables.

Substitute values

$$A = 250\left(\frac{\left(1+\frac{0.027}{12}\right)^{61}-1}{\frac{0.027}{12}}\right) - 250$$

Simplify in the parentheses and denominator

$$A = 250\left(\frac{1.00225^{61}-1}{0.00225}\right) - 250$$

Simplify the exponent and subtract 1

$$A \approx 250\left(\frac{0.146938}{0.00225}\right) - 250$$

Multiply and divide

$$A \approx 16{,}326.44984 - 250$$

Subtract 250

$$A \approx 16{,}076.449840$$

That is, the future value of this annuity is \$16,076.45.

A quick way to see if your solution is reasonable would be to find the total amount of money invested and compare that to the future value of the account. In this example, $250 * 60 = \$15{,}000$ was invested. So, with the addition of the compound interest the account will earn, a future value of \$16,076.45 makes sense.

❖ **YOU TRY IT 3.4.A:** Use the appropriate future value of an annuity formula and the given variables to solve for the missing variable. Round to the nearest cent.

$R = 342.12, r = 1.7\%, m = 4, n = 40$. Payments are made at the beginning of the compounding period.

An account similar to an annuity, called a sinking fund, is used when the regular payment is unknown. For instance, suppose Sue knows in 5 years they'll want to buy a car. They know they'll have to purchase a $20,000 car; what regular payment must Sue make into the account for the 5 years to have $20,000 in the end? This type of fund is called a sinking fund.

Sinking Funds

DEFINITION: A **sinking fund** is an account in which money is invested and interest is compounded at multiple regular intervals to fund a future purchase of a known amount. With a sinking fund, the regular investment amount is unknown.

NOTE: In this text, all sinking funds will be solved with the ordinary annuity formula.

A sinking fund is a way of letting interest add to principal instead of a loan where interest is additional money paid.

➢ **EXAMPLE 3.4.2:** Use the appropriate future value of an annuity formula and the given variables to solve for the missing value. Round to the nearest cent.

$$r = 3\%, t = 4, m = 12, A = 5{,}000$$

SOLUTION:

First find the total number of compounding periods, $n = m \cdot t = 12 \cdot 4 = 48$. Since R is unknown, this is a sinking fund and the ordinary annuity formula is appropriate. Substitute the appropriate variables.

$$5{,}000 = R\left(\frac{\left(1 + \frac{0.03}{12}\right)^{48} - 1}{\frac{0.03}{12}}\right)$$
$$5{,}000 = R\left(\frac{1.0025^{48} - 1}{0.0025}\right)$$
$$5{,}000 \approx R\left(\frac{0.127328}{0.0025}\right)$$
$$5{,}000 \approx R \cdot 50.931208$$
$$R \approx 98.171635$$

The monthly payment on this annuity is \$98.17.

To check if the solution is reasonable, multiply the monthly payment times the number of payments ($98.17 * 48 = \$4{,}712.16$). The future value of the account is \$5,000 so, with the addition of compound interest, this number makes sense.

The careful reader will notice that variables n, r, m haven't been solved for yet. This is on purpose; solving for a variable in the exponent (namely, n) requires the use of logarithms. Solving for r or m by formula is beyond the scope of this text.

In the following example, it will be necessary to solve for n.

➢ **EXAMPLE 3.4.3:** Use the appropriate future value of an annuity formula and the given variables to solve for the missing value. Round to the nearest payment.

a. $A = 23{,}432.12, R = 150, m = 12, r = 3.2\%$, payments are made at the beginning of the compounding period.
b. $A = 23{,}432.12, R = 150, m = 12, r = 3.2\%$, payments are made at the end of the compounding period.

SOLUTION:

a. Since payments are made at the beginning of the compounding period, this is an annuity due. Substitute the given values and solve for n.

$$23{,}432.12 = 150\left(\frac{\left(1+\frac{0.032}{12}\right)^{n+1}-1}{\frac{0.032}{12}}\right) - 150$$

Add 150 then divide by 150

$$157.214133 \approx \frac{1.002667^{n+1}-1}{0.002667}$$

Take the natural log of each side

This brings the $n+1$ out of the exponent

Compute natural logs

Divide by 0.002663

$$\begin{aligned}
0.419238 &\approx 1.002667^{n+1}-1\\
1.419238 &\approx 1.002667^{n+1}\\
\ln 1.419238 &\approx \ln 1.002667^{n+1}\\
\ln 1.419238 &\approx (n+1)\cdot \ln 1.002667\\
0.350120 &\approx (n+1)\cdot 0.002663\\
131.469940 &\approx n+1\\
130.469940 &\approx n
\end{aligned}$$

The total number of payments for this annuity is approximately 131 payments. Here the number is rounded up to ensure that the investor has the required amount of money or more in their account. Recall that this is total number of payments. To convert to the number of years, take $\frac{130.469940}{12} = 10.872495$ years.

Thus, it will take 131 payments of $150 to reach the future value of $23,432.12.

b. Since payments are made at the end of the compounding period, this is an ordinary annuity. Substitute the given values and solve for n.

$$23{,}432.12 = 150\left(\frac{\left(1+\frac{0.032}{12}\right)^n - 1}{\frac{0.032}{12}}\right)$$

$$156.214133 \approx \frac{1.002667^n - 1}{0.002667}$$

$$0.416571 \approx 1.002667^n - 1$$
$$1.416571 \approx 1.002667^n$$
$$\ln 1.416571 \approx n \cdot \ln 1.002667$$
$$0.348239 \approx n \cdot 0.002663$$
$$130.763734 \approx n$$

The total number of payments for this annuity is approximately 131 payments or 10.9 years or 10 years 8 months' worth of payments.

Notice that, with all other variables held constant, an ordinary annuity and an annuity due provide slightly different values for number of payments needed. This difference is due to the regular investment being made at the beginning or the end of the compounding period.

❖ **YOU TRY IT 3.4.B:** Use the appropriate future value of an annuity formula and the given variables to solve for the missing variable. Round to the nearest payment.

$A = 17{,}000, R = 125, m = 4, r = 1.75\%$, payments are made at the end of the compounding period.

In real world applications, the values of the variables aren't listed out; it is left to the problem-solver to determine the value of each variable and which formula to use. The following examples illustrate that.

➢ **EXAMPLE 3.4.4:** Haris is opening an annuity to begin saving for retirement. Haris doesn't have much to save every month, but he knows that it will add up. Haris has $75 he can save every month. He puts this money into an ordinary annuity that has an interest rate of 2.5%. Haris plans on contributing to this annuity every month for 20 years. Round to the nearest cent.

a. How much money will Haris have at the end of 20 years?
b. How much of this money is principal?
c. How much of this money is interest?

SOLUTION:

a. First, decide if ordinary annuity or annuity due is appropriate. The problem states this is an ordinary annuity. Calculate $n = mt = 20 * 12 = 240$. Second, identify all the given variable values: $R = 75, r = 2.5\%, m = 12, n = 240$. The value of A is unknown. Using the appropriate formula, substitute the given values.

$$A = 75\left(\frac{\left(1+\frac{0.025}{12}\right)^{240}-1}{\frac{0.025}{12}}\right)$$

$$A \approx 75\,(310.974708)$$
$$A \approx 23{,}323.103116$$

The future value on Haris' account is $23,323.10. To determine if the solution is reasonable, multiply the monthly payment by the number of payments ($75 \cdot\ 240 = 18{,}000$). So, with the addition of compound interest, this answer is reasonable.

b. Haris invests $75 for 240 payments: $75 \cdot 240 = 18{,}000$. Thus, $18,000 is principal.

c. To find the total interest, take the future value minus the total principal amount invested: $23{,}323.10 - 18{,}000 = 5{,}323.10$. Haris will earn $5,323.10 in interest.

The authors hope this demonstrates the power of saving for retirement when a person is young – even a little bit every month makes a large difference.

➢ **EXAMPLE 3.4.5:** Ruth would like to buy a car in 5 years. She anticipates she will need $18,000 to buy her car. She would like to invest in a sinking fund that earns 4.2% interest compounded monthly to reach her goal. Round to the nearest cent.

a. How much must Ruth invest every month so that in 5 years she will have $18,000 to buy her car? Assume this is an ordinary annuity.
b. How much of the future value of $18,000 is principal?
c. How much is interest?

SOLUTION:

a. Since this is a sinking fund, use the ordinary annuity formula. Calculate $n = mt = 5 * 12 = 60$. Second, identify all the given variable values: $A = 18{,}000, r = 4.2\%, m = 12, n = 60$. The value of R is unknown. Using the appropriate formula, substitute the given values.

$$18{,}000 = R \cdot \frac{\left(\left(1 + \frac{0.042}{12}\right)^{60} - 1\right)}{\frac{0.042}{12}}$$

$$18{,}000 \approx R \cdot 66.635949$$
$$270.124464 \approx R$$

The monthly payment Ruth needs to make is $270.13. (The answer is rounded up to ensure that Ruth reaches or exceeds her needed $18,000.) To determine if the solution is reasonable, multiply the monthly payment times the number of payments $(270.13 \cdot 60 = 16{,}207.80)$. So, with the addition of compound interest, a future value of $18,000 is reasonable.

b. Ruth invests $270.13 for 60 payments: $60 \cdot 270.13 = \$16{,}207.80$. Thus, $16,207.80 of the $18,000 is principal.

c. To find total interest earned, take the future value minus the total principal invested: $18{,}000 - 16{,}207.80 = \$1{,}792.20$. Thus $1,792.20 of the future value of $18,000 is interest.

❖ **YOU TRY IT 3.4.C:** Hector would like to buy a motorcycle in 3 years. He anticipates he will need $10,000 to buy his motorcycle. He would like to invest in a sinking fund that earns 4.2% interest compounded monthly to reach his goal. Round to the nearest cent.

a. How much must Hector invest every month so that in 3 years he will have $10,000 to buy his motorcycle?
b. How much of the future value of $10,000 is principal?
c. How much is interest?

➢ **EXAMPLE 3.4.6:** Dottie's mom starts an ordinary annuity with an interest rate of 2.5% for them. Dottie's mom invests \$100 every month into this annuity for 5 years. When Dottie turns 21, Dottie's mom signs the annuity over to them. Dottie, being 21, doesn't make payments into the annuity but lets the money sit and earn compound interest for 5 years. How much money does Dottie have in the account at the end of those 10 years? Round to the nearest cent.

SOLUTION:

The problem states this is an ordinary annuity. This problem will be completed in two parts: (1) the first 5 years during which Dottie's mom is making regular payments and (2) the second 5 years during which Dottie is not making regular payments.

Part 1: Calculate $n = mt = 5 * 12 = 60$. Second, identify all the given variable values: $R = 100, r = 2.5\%, m = 12, n = 60$. The value of A is unknown. Using the appropriate formula, substitute the given values.

$$A = 100\left(\frac{\left(1 + \frac{0.025}{12}\right)^{60} - 1)}{\frac{0.025}{12}}\right)$$

$$A = 6{,}384.05$$

At the end of the first 5 years Dottie has \$6,384.05 in the account.

Part 2: Calculate the amount of money Dottie has after letting this \$6,384.05 sit in the account for 5 years without making additional payments. This is compound interest where $P = 6{,}384.05, r = 2.5\%, m = 12, t = 5$.

$$A = P\left(1 + \frac{r}{m}\right)^{n}$$

$$A = 6{,}384.05\left(1 + \frac{0.025}{12}\right)^{60}$$

$$A = 7{,}233.135812$$

That is, Dottie will have \$7,233.14 at the end of the second 5 years.

Quick Review

Future Value of an Ordinary Annuity Formula

$$A = R\left(\frac{\left(1+\frac{r}{m}\right)^{n} - 1}{\frac{r}{m}}\right)$$

Note: $n = mt$

Future Value of an Annuity Due Formula

$$A = R\left(\frac{\left(1+\frac{r}{m}\right)^{n+1} - 1}{\frac{r}{m}}\right) - R$$

Note: $n = mt$

YOU TRY IT 3.4.A SOLUTION:

$$A = 342.12\left(\frac{\left(1+\frac{0.017}{4}\right)^{40+1} - 1)}{\frac{0.017}{4}}\right) - 342.12$$

$$A = 14{,}945.71$$

The future value of the annuity is $14,945.71.

YOU TRY IT 3.4.B SOLUTION:

$n = 106.95$ quarters or 26.74 years

YOU TRY IT 3.4.C SOLUTION:

a. Hector needs to invest $261.14 every month.
b. Hector invests $261.14 every month for 3 years, so he contributes $9,401.04 of principal.
c. He will earn $598.96 in interest.

Section 3.4 | Exercises

In exercises 1 – 10, use the appropriate formula for an annuity and the given variables to solve for the missing variable. Round to the nearest hundredth or cent.

1. Payments are made at the end of the compounding period.
   $R = 150, r = 2.7\%, m = 12, t = 5$ years
2. Payments are made at the beginning of the compounding period.
   $R = 65, r = 1.8\%, m = 4, t = 20$ years
3. Payments are made at the beginning of the compounding period.
   $R = 75, r = 0.9\%, m = 6, n = 120$
4. Payments are made at the beginning of the compounding period.
   $R = 1{,}000, r = 3.75\%, m = 1, n = 7$
5. Payments are made at the end of the compounding period.
   $A = 12{,}000, r = 5.4\%, m = 12, n = 120$
6. Payments are made at the end of the compounding period.
   $A = 8{,}500, r = 3.1\%, m = 4, t = 10$
7. Payments are made at the end of the compounding period.
   $A = 5{,}234.73, R = 200, r = 2.12\%, m = 4$
8. Payments are made at the beginning of the compounding period.
   $A = 9{,}475, R = 450, r = 1.85\%, m = 2$
9. Payments are made at the beginning of the compounding period.
   $A = 850, R = 125, r = 1.99\%, m = 12$
10. Payments are made at the end of the compounding period.
    $A = 100{,}000, R = 1{,}500, r = 4.15\%, m = 1$

In exercises 11 – 25, solve the application problems using the appropriate annuity formula.

11. William's parents started a college savings fund for him. William is young; he will start college in 15 years. Suppose his parents invest $100 every month into an annuity due earning 1.75% interest compounded monthly.
    a. How much money will William have in his college account in 15 years?
    b. How much of this future value is principal?
    c. How much of this future value is interest?

12. Maya wants to save or retirement. She plans to invest $175 each month into an ordinary annuity for 35 years. The annuity earns 2.4% compounded monthly.
    a. How much money will Maya have in her retirement account in 35 years?
    b. How much of this future value is principal?
    c. How much of this future value is interest?
13. At age 20, Caedon decided his goal is to purchase a home worth $400,000 when he is 35. If he puts money into an annuity earning 3.6% interest compounded monthly, find the monthly payment that he needs to have enough money in his account.
14. Daniel has $250 to invest every month into an ordinary annuity earning 2.88% interest compounded monthly. How long will it be until they have $12,000 to buy a new car with?
15. Kaia decided to invest $250 at the beginning of each month into an annuity that earns 2.8% compounded monthly. After 5 years of making monthly payments, Kaia is unable to continue making payments. However, she leaves the money in the account to earn 2.8% interest compounded monthly for another 20 years. Find the amount in the account at the end of 25 years. Also find the amount of interest earned.
16. Matthew would like to invest the tips he makes from his job. For the next 3 years, he deposits $1,000 at the end of each quarter into an account earning 1.99% interest compounded quarterly. At the end of the 3 years, he withdraws his money and deposits the lump-sum into an account earning 3.75% compounded monthly for 5 more years (he makes no additional deposits at that time). What is the future value of Matthew's investment at the end of the 8 years? How much interest did he earn?
17. Which investment is better?
    A: Invest $200 each month into an ordinary annuity at 3% interest compounded monthly for 15 years.
    B: Invest $600 each quarter into an ordinary annuity at 1% interest compounded quarterly for 15 years.
18. Which investment is better?
    A: Invest $100 each month into an ordinary annuity at 6% interest compounded monthly for 20 years.
    B: Invest $100 each month into an annuity due at 6% interest compounded monthly for 20 years.
19. Kiara decides to stop having her nails done for a while. She wants to invest her monthly manicure money into an annuity instead. She has $180 to invest every month. She finds two annuities she could invest in – annuity A has a promotional $25 bonus for opening the annuity a 2.5% interest rate (the $25 bonus will not earn interest). Annuity B has an interest rate of 3.9%. Both are ordinary annuities and both are compounded monthly. If Kiara wants to invest for 1 year, which annuity is a better choice for her?
20. Jadyn would like to invest $50 at the beginning of each month into an account and has two options. Account A involves an initial fee of $75 to open the account and offers 5.25% interest compounded monthly. (Meaning the first month Jadyn would pay the $50 deposit as well as the $75 fee.) Account B is free to open but offers an interest rate of 5% compounded monthly. If Jadyn plans on investing for 4 years, which account would give a larger future value?

21. Jose would like to save \$5,000 to take a cruise with his family. How many years would it take Jose to achieve this goal if he deposits \$500 at the end of each quarter in an account earning 1.99% interest compounded quarterly?
22. How many years would it take to retire as a millionaire if a person invests \$250 at the end of each month into an account earning 4.15% compounded monthly?
23. The bank would prefer to pay the customer the least amount of interest possible; this makes good business sense. With all other variables held constant, which type of annuity do lending institutions prefer – an ordinary annuity or an annuity due? (Hint – this could be answered in two ways: (1) intuitively or (2) choosing easy values (e.g. $R = 100, r = 1\%$, etc.) and calculating and comparing the future value with both types of annuities.)
24. Which earns more interest – investing a \$6,000 lump sum into an account earning monthly compound interest for 5 years or investing \$100 every month into an ordinary annuity earning the same compound interest rate? (Hint – pick a value for r and perform both calculations.)
25. To retire as a millionaire, how much do you have to invest monthly for 40 years into an ordinary annuity that earns 4.5% interest compounded monthly?

Section 3.4 | Exercise Solutions

1. $9,624.22
2. $6,270.58
3. $9,867.57
4. $8,132.55
5. $75.64
6. $182.08
7. 24.58 compounding periods, or 6.14 years
8. 19.16 compounding periods, or 9.58 years
9. 6.76 compounding periods, or 6.76 months, or 0.56 years
10. 32.61 compounding periods, or 32.61 years
11. a. $20,596.48 b. $18,000 c. $2,596.48
12. a. $115,012.16 b. $73,500 c. $41,512.16
13. $1,679.22
14. 45.485229 compounding periods, or 3.790436 years or 3 years 9.485229 months
15. Amount: $22,199.27; Interest: $13,199.27
16. Future Value: $14,873.13; Interest: $2,873.13
17. Account A: $45,394.54; Account B: $38,788.03; Investment A is better.
18. Account A: $46,204.09; Account B: $46,435.11; Investment B is better.
19. Annuity A ($2,209.92) and annuity B ($2,199.03). Therefore, annuity A is the smarter choice.
20. Account A: $2,600.81; Account B: $2,661.79; Account B is better.
21. 9.78 quarterly payments, or 2.45 years, or 2 years and 5.4 months
22. 65.1 years
23. Lending institutions prefer an ordinary annuity.
24. The $6,000 lump sum will earn more interest.
25. $745.63

Brief Guide to Excel and TI-83/TI-84 Calculator

There are many technologies available to solve these problems. In general, compound interest problems, annuity problems, sinking fund problems, and mortgage payment problems are called time-value-of-money problems (TVM). Two common technologies used to solve these problems are Excel and a TI-83/TI-84 calculator. These will both be discussed here.

Excel

Excel is a powerful spreadsheet program that is part of the Microsoft Office suite. Excel has built-in TVM functions.

Variables, this text	Variables, Excel	Details
I	Not defined in Excel	-
r	Not defined in Excel	To find r using Excel, one must calculate $\frac{r}{m} \cdot m$
$\frac{r}{m}$	Rate	Displayed as the rate per compounding period rather than the annual interest rate. Displayed as a decimal.
P	pv	Principal or present value
t	Not defined in Excel	To find t using Excel, one must calculate $\frac{n}{m}$
A	fv	Amount or future value
R	pmt	Regular payment
m	Not defined in Excel	-
n	nper	Number of compounding periods

To distinguish between annuity and annuity due, Excel uses a “type” variable. Type is entered as 0 if the payment is made at the end of the compounding period and 1 if the payment is made at the beginning of the compounding period.

Excel has a few built-in functions to assist with TVM problems:

Function	Solves for
Pv(rate, nper, pmt, [fv], [type])	Principal or present value
Fv(rate, nper, pmt, [fv], [type])	Amount or future value
Pmt(rate, nper, pv, [fv], [type])	Regular payment
Nper(rate, pmt, pn, [fv], [type])	Total number of payments
Rate(nper, pmt, pv, [fv], [type], [guess])	Interest rate per period

Lastly, Excel recognizes cash flow. For example, if cash is flowing towards an individual that is a positive value in Excel. If cash is flowing away from an individual that is a negative value in Excel.

- **EXAMPLE 1:** What is the future value of an account with an interest rate of 4.2%, compounded quarterly, if $1,000 is invested for 7 years?

  SOLUTION:

  Identify all appropriate variable: $pv = 1{,}000, \frac{r}{m} = \frac{0.042}{4} = 0.0105$, $nper = 28, pmt = 0$ (since there is no regular payment, only a lump sum investment when the account is opened). For compound interest with no regular payment, the type variable is left blank.

  Then substitute all variables into the appropriate formula.

  ```
  =FV(0.0105, 28, 0, 1000)
  ```

  The future value of the account is $1,339.73.

- **EXAMPLE 2:** (This is Example 3.4.1a.) Use Excel to solve for the missing variable.

  $R = 250, r = 2.7\%, m = 12, n = 60$. Payments are made at the beginning of the compounding period.

  SOLUTION: Identify all appropriate variables: $Pv = 0, \frac{r}{m} = \frac{0.027}{12} = 0.00225$, $nper = 60, pmt = 250, type = 1$.

  Then substitute all variables into the appropriate formula.

  ```
  =FV(0.00225, 60, 250, 0, 1)
  ```

  The future value of the annuity is $16,076.45.

- **EXAMPLE 3:** (This is similar to Example 3.4.5.) Ruth would like to buy a car in a few years. She anticipates she will need $18,000 to buy her car. She would like to invest in a sinking fund that earns 4.2% interest compounded monthly to reach her goal. If Ruth has $200 to invest every month, how long will it take her to reach $18,000?

SOLUTION: Identify all appropriate variables: $Pv = 0, fv = 18{,}000$, $\frac{r}{m} = \frac{0.042}{12} = 0.0035, pmt = 200, type = 0$.

Then substitute all variables into the appropriate formula. Pmt will be negative since cash is flowing away from Ruth.

```
=NPER(0.035, -200, 0, 18000, 0)
```

The total number of payments is equal to 78.38. Remember, payments are made monthly so that is $\frac{78.38}{12} = 6.53$ years.

To use Excel to solve a loan problem, such as a mortgage problem, let pv = the value of the loan and fv = 0 since the loan borrower wishes to pay off the loan, or bring its balance down to 0.

TI-83 or -84 calculator

A TI-83 or -84 calculator comes with a built-in TVM solver. To access the TVM solver, proceed to Apps→ Finance → TVM solver.

Variables, this text	Variables, TI-83 or -84	Details
I	Not defined on TVM solver	-
r	$I\%$	This is entered as a percent, not as a decimal. For example, 4.2% is entered as 4.2.
P	PV	Principal or present value
t	Not defined on TVM solver	To find t using TVM solver, one must calculate $\frac{n}{m}$
A	FV	Amount or future value
R	PMT	Regular payment
m	$P/Y \text{ and } C/Y$	Excel recognizes both payments per year (P/Y) and compounds per year (C/Y). For the purposes of this text, these are always the same number.
n	N	Number of payments

All known variables should be entered. Then, with the cursor on the unknown variable, type in "alpha-enter" which has the calculator solve for the unknown variable.

- **EXAMPLE 4:** What is the future value of an account with an interest rate of 4.2%, compounded quarterly, if $1,000 is invested for 7 years?

  SOLUTION:

  Identify all appropriate variables.

N	28
$I\%$	4.2
PV	1,000
PMT	0
FV	
$\frac{P}{Y}$	4
$\frac{C}{Y}$	4
PMT	END

  (Here PMT is set to END by default.)

  The future value of the account is $1,339.73.

- **EXAMPLE 5:** (This is Example 3.4.1.) Use TVM solver to solve for the missing variable.

  $R = 250, r = 2.7\%, m = 12, n = 60$. Payments are made at the beginning of the compounding period.

  SOLUTION: Identify all appropriate variables.

N	60
$I\%$	2.7
PV	0
PMT	250
FV	
$\frac{P}{Y}$	12
$\frac{C}{Y}$	12
PMT	BEGIN

  The future value of the annuity is $16,076.45.

- **EXAMPLE 6:** (This is similar to Example 3.4.5.) Ruth would like to buy a car in a few years. She anticipates she will need $18,000 to buy her car. She would like to invest in a sinking fund that earns 4.2% interest compounded monthly to reach her goal. If Ruth has $200 to invest every month, how long will it take her to reach $18,000?

N	
$I\%$	4.2
PV	0
PMT	-200
FV	18,000
$\frac{P}{Y}$	12
$\frac{C}{Y}$	12
PMT	END

N is 78.38. That is, it will take Ruth 78.38 compounding periods or 6.53 years to save for her car.

Section 3.5 | Loans and Mortgages

Objectives

- Calculate installment and add-on loan payments
- Determine mortgage payment amounts

Over one's lifetime a person might get loans to purchase high value items such as automobiles or homes. In this section, loans, loan payment amounts, and applying down payments will be discussed.

Loans

DEFINITION: An **installment loan** is money borrowed that is paid back with interest over a set time period at regularly scheduled intervals.

Often loans are paid back monthly but can have shorter pay-back periods such as weekly or longer pay-back periods such as quarterly or yearly. Typically, loans are issued by financial institutions called lenders. Interest is applied to the loan so that the borrower pays back the amount they borrow plus the interest charged by the lender. Loans are beneficial for those people who would like to purchase items when they do not have all of the money necessary to purchase said item.

The following formula can be used to calculate the payment on an installment loan.

Installment Loan Payment Formula

Given the principal (borrowed) amount of a loan P, the interest rate r (in decimal form), the number of payments within one year m, and the time t (in years), the installment payment can be calculated by:

$$\text{Installment Payment Amount} = \frac{P\left(\frac{r}{m}\right)}{1-\left(1+\frac{r}{m}\right)^{-mt}} = \frac{P\left(\frac{r}{m}\right)}{1-\left(1+\frac{r}{m}\right)^{-n}}$$

Note: $n = mt$

NOTE: The installment payment amount above has the same formula as the R in the repeated payment for annuities in Section 3.4.

NOTE: Payments are always rounded up. This is to ensure the entire amount of the loan is paid back.

NOTE: Throughout Chapter 3, intermediate steps will not be rounded. Intermediate steps will be shown rounded to six decimal places only so students can follow along. Rounding in intermediate steps will cause the final solution to be slightly different than if rounding is only done in the final answer.

- **EXAMPLE 3.5.1:** Find the payment on an installment car loan if the purchase price of the car is $25,000 over 5 years at an interest rate of 7.3%. Parts a and b below provide a different scheduled interval for each payment.
  a. Monthly
  b. Bimonthly
  c. Which payment, monthly or bimonthly, is better for the borrower? Why?

  SOLUTION:

  a. Monthly Payment:

$$\frac{P\left(\frac{r}{m}\right)}{1-\left(1+\frac{r}{m}\right)^{-mt}} = \frac{25{,}000\left(\frac{0.073}{12}\right)}{1-\left(1+\frac{0.073}{12}\right)^{-12*5}} \approx$$

$$\frac{152.083333}{1-(1.006083)^{-60}} \approx \frac{152.083333}{0.305035} \approx \$498.58$$

  b. Bimonthly Payment:

$$\frac{P\left(\frac{r}{m}\right)}{1-\left(1+\frac{r}{m}\right)^{-mt}} = \frac{25{,}000\left(\frac{0.073}{24}\right)}{1-\left(1+\frac{0.073}{24}\right)^{-24*5}} \approx$$

$$\frac{76.041667}{1-(1.003042)^{-120}} \approx \frac{76.041667}{0.305419} \approx \$248.98$$

  c. For a borrower who pays bimonthly payments, in one month the borrower will make two payments of $248.98 which means the borrower will pay $497.96 per month. The difference between the two payments is $0.62. The bimonthly payment will save borrowers $0.62 per month. That is a savings of $37.20 (60 months times $0.62) over the life of the installment loan.

- **EXAMPLE 3.5.2:** Using the loan terms from Example 3.5.1, how much interest did the borrower pay on the monthly and the bimonthly car loan?

SOLUTION: To find the interest paid on the loan, first calculate the total amount paid to the lender, then subtract the amount of money borrowed. The total amount paid on the monthly loan is ($498.58)(60) = $29,914.80. The borrower paid 60 payments of $498.58 to the lender because the loan was for five years with 12 monthly payment per year. The total interest paid is $29,914.80 − $25,000 = $4,914.80.

The total amount paid on the bimonthly loan is $248.98(120) = $29,877.60. The borrower paid 120 payments of $248.98 to the lender because the loan was for five years with 24 payments per year. The total interest paid is $29,877.60 – $25,000 = $4,877.60.

As noted in Example 3.5.1, the monthly loan costs the borrower $4,914.80 − $4,877.60 = $37.20 more in interest than the monthly loan.

❖ **YOU TRY IT 3.5.A:** Find the payment on an installment loan issued for $40,000 over four years at an interest rate of 5.7%. Parts a, b, and c below provide a different scheduled interval for each payment.

a. Monthly
b. Quarterly
c. Yearly

❖ **YOU TRY IT 3.5.B:** Using the loan terms from You Try it 3.5.A, how much interest is paid on each loan?

a. Monthly
b. Quarterly
c. Yearly

DEFINITION: An **add-on interest loan** is a loan where the interest and principal amount of the loan are combined and then the total is divided by the number of payments.

Add-On Interest Loan Payment Formula

Given the principal (borrowed) amount of a loan P, the total simple interest amount I, and the number of total payments for the entire loan n, the add-on interest payment can be calculated by:

$$\text{Add-On Interest Payment Amount} = \frac{P+I}{n}$$

Note: The Simple Interest Formula is $I = Prt$ (Section 3.2).

Add-on interest loans can be used for short-term loans, purchasing appliances, auto loans, or cash checking businesses. Typically, add-on interest loans have a much higher cost for a borrower than installment loans.

➢ **EXAMPLE 3.5.3:** Find the monthly payment for a $1,000 add-on interest loan issued over 4 years with 6% interest.

SOLUTION:

$$I = Prt = 1000(0.06)(4) = \$240$$

$$\frac{P+I}{n} = \frac{1000+240}{48} \approx \$25.84$$

The principal (borrowed) amount is $1,000, so the simple interest on the loan is $\$1{,}000(0.06)(4) = \$240$. To find the total amount to repay add the principal amount to the interest, which is $\$1{,}000 + \$240 = \$1{,}240$. To determine the monthly installment of this add-on interest loan, divide the principal and interest by 48 (12 months times 4 years). The monthly payment is $\$1{,}240/48$ which is approximately $25.84 per month. The answer should be rounded up to the nearest penny to ensure that the total loan amount is paid.

➢ **EXAMPLE 3.5.4:** Find the payment on an add-on car loan if the purchase price of the car is $25,000 over 5 years at an interest rate of 7.3%. Parts a and b below provide a different scheduled interval for each payment.

a. Monthly
b. Bimonthly
c. Which payment, monthly or bimonthly, is better for the borrower? Explain.

SOLUTION:

a. Interest = $\$25{,}000(5)(0.073) = \$9{,}125$
Principal + Interest = $\$25{,}000 + \$9{,}125 = \$34{,}125$
Total Payments = $12(5) = 60$
Monthly Payment = $\frac{\$34{,}125}{60} = \$568.75$

b. Interest = $\$25{,}000(5)(0.073) = \$9{,}125$
Principal + Interest = $\$25{,}000 + \$9{,}125 = \$34{,}125$
Total Payments = $24(5) = 120$
Bimonthly Payment = $\frac{\$34{,}125}{120} = \$284.38$

c. For the bimonthly payment, the borrower pays two payments of \$284.38 which is \$568.76 per month. Note that the initial bimonthly payment was rounded up, which accounts for the penny difference from the monthly payment of \$568.75 in part a. So, the borrower will be paying the same amount per month for either loan. This is because the interest for an add-on interest loan is added at the beginning of the loan.

❖ **YOU TRY IT 3.5.C:** Find the monthly payment on an add-on interest loan issued for \$14,500 over 5 years at an interest rate of 4.25%.

➢ **EXAMPLE 3.5.5:** Using the payment amounts and interest paid from Examples 3.5.2 and 3.5.4, which monthly payment is better for the borrower, the monthly add-on car loan or the monthly installment car loan? Explain why. How much interest was paid by the borrower for each loan? Which loan is better for the borrower? Compare the differences.

SOLUTION: The borrower would pay the lender \$4,914.80 in interest for the installment loan and \$9,125 in interest for the add-on interest loan. The monthly installment loan is better for the borrower because the borrower pays \$4,914.80 in interest. The monthly add-on interest loan would be better for the lender because the borrower pays \$9,125 in interest. The borrower would pay \$9,125 − \$4,914.80 = \$4,210.20 more interest if they chose the add-on interest loan. The payment amount on the installment loan is \$498.58 whereas the payment amount on the add-on loan is \$568.75. The borrower's monthly loan payment would be lower if they chose the installment loan. The difference in the loan payments would be \$568.75 − \$498.58 = \$70.17 per month. Since the installment loan payment is lower than the add-on interest loan payment, the installment loan again would be better for the borrower.

Mortgages

DEFINITION: **Mortgages** are a type of installment loan used to purchase a home.

Typically, mortgages are repaid over 15, 20, or 30 years; however, terms can be shorter or longer. When purchasing a home, the borrower's credit history is checked. Based on the credit history, a financial institution (lender) will offer an interest rate for the mortgage. When purchasing a home, the buyer usually needs a down payment.

DEFINITION: A **down payment** is an amount of money that a buyer pays upfront to the financial institution (lender) to go towards the purchase of the home and loan.

If a person would like to purchase a $400,000 home and the lender requires a 20% down payment, the buyer would need to pay the financial institution ($400,000)(0.20) = $80,000 in order to get the mortgage. Then, the principal amount of the loan would be $400,000 - $80,000 = $320,000. The bank would give the home purchaser (the borrower), a mortgage loan for $320,000.

Borrowers have options when choosing a mortgage. There are different types of mortgages such as variable rate and fixed rate mortgages.

DEFINITION: **Variable rate mortgages** or adjustable rate mortgages (ARMs) have interest rates that vary over set time periods. Rates can vary (increase or decrease) weekly, monthly, or yearly.

DEFINITION: **Fixed rate mortgages** have the same interest rate for the entire term of the mortgage.

In this textbook, only fixed rate mortgages will be discussed.

The formula for a fixed rate mortgage is the same as the formula for an installment loan.

- **EXAMPLE 3.5.6:** Suppose George would like to buy a $350,000 condominium. His bank will offer him a mortgage for 15 years at an interest rate of 4% with a down payment of 25%. George will be making monthly payments.
  a. What is the amount of the down payment?
  b. What is the amount of the mortgage?
  c. What is George's monthly mortgage payment?
  d. How much interest will George pay over the life of the loan?

  SOLUTION:

  a. The down payment amount will be $350,000(0.25) = $87,500.

  b. The amount borrowed will be $350,000 - $87,500 = $262,500.

  c. The monthly mortgage payment will be

$$\frac{P\left(\frac{r}{m}\right)}{1-\left(1+\frac{r}{m}\right)^{-nt}} = \frac{262{,}500\left(\frac{0.04}{12}\right)}{1-\left(1+\frac{0.04}{12}\right)^{-12*15}} \approx$$

$$\frac{875}{1-(1.003333)^{-180}} \approx \frac{875}{1-0.5493595} \approx \$1{,}941.69$$

d. To find the amount of interest paid, first find the total amount paid over the life of the loan, then subtract that amount from the principal amount of the loan. The borrower paid 180 payments of $1,941.69, so 180(1,941.69) = $349,504.20. The principal amount of the loan was $262,500, so the interest paid over 15 years is $349,504.20 - $262,500 = $87,004.20.

❖ **YOU TRY IT 3.5.D:** Suppose Jane would like to buy a $550,000 house. Her bank will offer her a mortgage for 30 years at an interest rate of 3.75% with a down payment of 20%. Jane will be making monthly payments. Find the following:

a. What will be Jane's down payment amount?
b. What amount will Jane borrow from the bank?
c. What will be Jane's monthly mortgage payment?
d. How much interest will Jane pay over the life of the loan?

FIGURE 3.5.1

Advantages and Disadvantages of Loans	
Advantages: • Can make large purchases without having the money • Ability to pay back loans over a period of time • Helps build credit history if payments are on time	**Disadvantages:** • Paying interest on money borrowed • Missed payments negatively impact credit history • Easy to get into debt • Having too many loans at the same time can be problematic

Quick Review

- Installment Payment Amount $= \dfrac{P\left(\frac{r}{m}\right)}{1-\left(1+\frac{r}{m}\right)^{-mt}} = \dfrac{P\left(\frac{r}{m}\right)}{1-\left(1+\frac{r}{m}\right)^{-n}}$
- Add-On Interest Payment Amount $= \dfrac{P+I}{n}$

YOU TRY IT 3.5.A SOLUTION:

d. Monthly: $933.91
e. Quarterly: $2,813.52
f. Yearly: $11,464.47

YOU TRY IT 3.5.B SOLUTION:

- **a.** Monthly: $4,827.68
- **b.** Quarterly: $5,016.32
- **c.** Yearly: $5,857.88

YOU TRY IT 3.5.C SOLUTION: $293.03

YOU TRY IT 3.5.D SOLUTION:

- **a.** Down Payment: $110,000
- **b.** Amount Borrowed: $440,000
- **c.** Monthly Payment: $2,037.71
- **d.** Interest Paid: $293,575.60

Section 3.5 | Exercises

In Exercises 1 – 8, find the payment and total amount of interest paid on each installment loan given the following conditions.

1. A $35,000 four-year car loan at an interest rate of 4.3% with quarterly payments.
2. A $70,000 seven-year boat loan at an interest rate of 10.1% with bimonthly payments.
3. A $15,000 student loan for 15 years at an interest rate of 2.7% with yearly payments.
4. A $5,000 personal loan for 3 years at an interest rate of 9.9% with monthly payments.
5. A $12,000 motorcycle loan for 5 years at an interest rate of 7.2% with quarterly payments
6. A $9,250 car loan for 5 years at an interest rate of 6.2% with bimonthly payments.
7. An $18,000 personal loan for 2 years at an interest rate of 11.2% with daily payments (strange, but it could happen)
8. A $3,500 personal loan for 2 years at an interest rate of 5.4% with quarterly payments.

In Exercises 9 – 16, find the payment and total amount of interest paid on each add-on interest loan given the following conditions.

9. A $35,000 four-year car loan at an interest rate of 4.3% with quarterly payments.
10. A $70,000 seven-year boat loan at an interest rate of 10.1% with bimonthly payments.
11. A $15,000 student loan for 15 years at an interest rate of 2.7% with yearly payments.
12. A $5,000 personal loan for 3 years at an interest rate of 9.9% with monthly payments.
13. A $1,100 iPhone loan for 2 years at an interest rate of 3.5% with monthly payments
14. A $2,700 personal loan for 5 years at an interest rate of 5.8% with bimonthly payments.
15. A $2,500 loan for a Onewheel for 3 years at an interest rate of 4.8% with quarterly payments
16. A $10,400 car loan for 4 years at an interest rate of 6.5% with monthly payments.

In Exercises 17 – 24, find each of the following given the conditions of each situation:

   a. What is the amount of the down payment?
   b. What is the amount of the mortgage?
   c. What is the monthly mortgage payment?
   d. How much interest will be paid over the life of the loan?

17. Dylan would like to buy a condominium downtown. Their lender offered them $325,000 at 6.3% for 20 years with a 20% down payment.
18. Mindy bought a summer home on a lake for $800,000. She has a 10-year mortgage at an interest rate of 7.5%. She paid a 25% down payment.

19. A recent college graduate would like to purchase a $430,000 townhome. Their lender offered a 30-year mortgage at 4.75% with a 15% down payment.
20. Angela purchased a $1,000,000 house. She paid a 30% down payment. She has a 15-year mortgage with an interest rate of 2.25%.
21. Barack is ready to buy a summer home in Hawaii. He will purchase a $500,000 house and make a 10% down payment. He gets a 30-year mortgage with an interest rate of 4.0%.
22. Jackson purchased his first home for $225,000. He made a 15% down payment. He decided on a 20-year mortgage with a 3.52% interest rate.
23. Cindy and Dan are buying a new house for $430,000. They are ready to make a 20% down payment. Their lender offered them a 30-year mortgage with a 4.1% interest rate.
24. Cassandra and Maria are buying a new townhome for $307,000. Their lender is requiring a 10% down payment with the rest financed at 2.79% interest over 15 years.

Concept Review:

25. If all loan terms are equal, which is better for the consumer: installment loans or add-on loans?
26. Students are often confused with add-on interest loans about the apparent conflicting values of t and n. For instance, suppose Mary took out an add-on interest loan for $1,000 with an interest rate of 5%. She makes monthly payments for 2 years. In the calculations $I = 1{,}000(0.05)(2) = 100$ and the monthly payment is $\frac{1{,}000+100}{24}$. Why is $t = 2$ yet $n = 24$?
27. Jazmine is buying a house for $400,000 using a fixed 5.2% interest rate 20 years. The lender is offering either a monthly mortgage payment or a bimonthly payment. Which payment interval pays less interest over the life of the loan? How much money is saved over the life of the loan?

Section 3.5 Exercise Solutions

1. Quarterly Payment: $2,392.73; Total Interest: $3,283.68
2. Bimonthly Payment: $582.02; Total Interest: $27,779.36
3. Yearly Payment: $1,229.40; Total Interest: $3,441
4. Monthly Payment: $161.11; Total Interest: $799.96
5. Quarterly Payment: $719.80; Total Interest: $2,396
6. Bimonthly Payment: $89.75; Total Interest: $1520
7. Daily Payment: $27.53; Total Interest: $2,096.90
8. Quarterly Payment: $464.50; Total Interest: $216
9. Quarterly Payment: $2,563.75; Total Interest: $6,020
10. Bimonthly Payment: $711.25; Total Interest: $49,490
11. Yearly Payment: $1,405; Total Interest: $6,075
12. Monthly Payment: $180.14; Total Interest: $1,485
13. Monthly Payment: $49.05; Total Interest: $77
14. Bimonthly Payment: $29.03; Total Interest: $783
15. Quarterly Payment: $238.34; Total Interest: $360
16. Monthly Payment: $273; Total Interest: $2,704
17. (a) $65,000 (b) $260,000 (c) $1,908 (d) $197,920
18. (a) $200,000 (b) $600,000 (c) $7,122.11 (d) $254,653.20
19. (a) $64,500 (b) $365,500 (c) $1,906.63 (d) $320,886.80
20. (a) $300,000 (b) $700,000 (c) $4,585.60 (d) $125,408
21. (a) $50,000 (b) $450,000 (c) $2,148.37 (d) $323,413.20
22. (a) $33,750 (b) $191,250 (c) $1,111.14 (d) $75,423.60
23. (a) $86,000 (b) $344,000 (c) $1,662.21 (d) $254,395.60
24. (a) $30,700 (b) $276,300 (c) $1,880.30 (d) $62,154
25. If all loan terms are equal, installment loans are better for the consumer because the consumer will pay less interest on an installment loan than on an add-on interest loan.
26. The variable t refers to the total length of the loan. In this case, that is 2 years. The variable n refers to the total number of payments. In this case, that is 24 months (2 years times 12 monthly payments).
27. Monthly payment: $2,684.22; Total Interest: $244,212.80
    Bimonthly payment: $1,341.29; Total Interest: $243,819.20
    Bimonthly payment saves $393.60

Section 3.6 Amortization

Objectives

- Calculate loan payments
- Create amortization schedules

For installment loans such as mortgages or automobile loans, amortization schedules are issued. An amortization schedule is a detailed table reflecting how much of each payment is applied to interest and how much is applied principal along with the new loan balance. This is done for every payment period. For each amortization schedule, the following information is necessary for each payment made: payment number, payment amount, interest charged, amount applied towards the principal, and new principal balance.

To find the interest amount allocated from each loan payment, use the simple interest formula $I = Prt$ (Section 3.2). For the principal P, use the new principal balance from the line before. For the first entry of the amortization schedule, only list the original mortgage or loan amount under the new principal balance. For the rate r, use the rate given divided by the number of payment periods each year. For example, given a 4% rate on a monthly mortgage, the rate used to find the monthly interest is (new principal amount)$(0.04)(\frac{1}{12}$ year). The rate should be expressed in decimal form and time is considered a fraction of the year. For the beginning row of the amortization table only the initial principal amount is listed.

➢ **EXAMPLE 3.6.1:** Zara just got a $250,000 mortgage on her townhouse. Find the first three lines of her amortization schedule given that her monthly payment is $1,200 on a 4% loan.

Original Mortgage Loan Amount

SOLUTION:

Payment Number	Payment Amount	Interest Charged	Amount Towards Principal	New Principal Balance
				$250,000
1	$1,200	$833.34	$366.66	$249,633.34
2	$1,200	$832.12	$367.88	$249,265.46
3	$1,200	$830.89	$369.11	$248,896.35

$\$250{,}000(0.04)\left(\frac{1}{12}\right) = \$833.34$

$\$1{,}200 - \$833.34 = \$366.66$

$\$250{,}000 - \$366.66 = \$249{,}633.34$

NOTE: For loans, interest charged is rounded up because the lender receives any additional fractional amount of money paid.

- **EXAMPLE 3.6.2:** Bhakti has a \$50,000 car loan. Create an amortization schedule for the first three payments on her loan issued at 5.5% given that she makes \$500 monthly payments on the loan.

  SOLUTION:

Payment Number	Payment Amount	Interest Charged	Amount Towards Principal	New Principal
				\$50,000
1	\$500	\$229.17	\$270.83	\$49,729.17
2	\$500	\$227.93	\$272.07	\$49,457.10
3	\$500	\$226.68	\$273.32	\$49,183.78

- **YOU TRY IT 3.6.A:** Create an amortization schedule for the first four payments on a \$400,000 Ferrari car loan where the monthly payment is \$2,700 on a 6% loan.

NOTE: Throughout Chapter 3, intermediate steps will not be rounded. Intermediate steps will be shown rounded to six decimal places only so students can follow along. Rounding in intermediate steps will cause the final solution to be slightly different than if rounding is only done in the final answer.

- **EXAMPLE 3.6.3:** Alex wants to purchase a \$375,000 home. He will need a 20% down payment. His lender has offered him a mortgage with an interest rate of 5.25% for 20 years.
  a. What is the amount of the down payment?
  b. What is the amount of the mortgage?
  c. What is Alex's monthly mortgage payment?
  d. Find the first three lines of Alex's amortization schedule.

  SOLUTION:

  a. The down payment will be \$375,000(0.20) = \$75,000.
  b. The amount of the mortgage will be \$375,000– \$75,000 = \$300,000.
  c. The monthly mortgage installment will be

$$\frac{P\left(\frac{r}{m}\right)}{1-\left(1+\frac{r}{m}\right)^{-mt}} = \frac{300{,}000\left(\frac{0.0525}{12}\right)}{1-\left(1+\frac{0.0525}{12}\right)^{-12*20}} \approx \$2{,}021.54$$

d.

Payment Number	Payment Amount	Interest Charged	Amount Towards Principal	New Principal
				\$300,000
1	\$2,021.54	\$1,312.50	\$709.04	\$299,290.96
2	\$2,021.54	\$1,309.40	\$712.14	\$298,578.82
3	\$2,021.54	\$1,306.29	\$715.25	\$297,863.57

- **YOU TRY IT 3.5.B:** Maura wants to purchase a \$600,000 home. She will be making a 25% down payment. Her lender has offered her a mortgage with an interest rate of 4.75% for 15 years.
  a. What is the amount of the down payment?
  b. What is the amount of the mortgage?
  c. What is Maura's monthly mortgage payment?
  d. Find the first three lines of Maura's amortization schedule.

For mortgages, even though the payments are the same each payment term, the allocation of each payment changes every payment. Notice that the amount applied towards the principal increases as more payments are made. This affects the interest charged on the mortgage, which decreases as more payments are made. So, as you pay down the loan, more money is paid towards the principal. For mortgages, towards the end of the term of the loan, borrowers need to be cautious if they are considering refinancing (getting a new mortgage) because the borrower has already paid most of their interest charged on the original loan. Towards the end of the mortgage, most of the payments are paid towards the principal.

NOTE: For this textbook, all amortization schedules are in terms of monthly payments. Typically, mortgages and automobile loans have monthly payments even though shorter or longer payment terms are possible.

YOU TRY IT 3.6.A SOLUTION:

Payment Number	Payment Amount	Interest Charged	Amount Towards Principal	New Principal
				\$400,000
1	\$2,700	\$2,000	\$700	\$399,300
2	\$2,700	\$1,996.50	\$703.50	\$398,596.50
3	\$2,700	\$1,992.99	\$707.01	\$397,889.49

YOU TRY IT 3.6.B SOLUTION:

a. The down payment will be \$600,000(0.25) = \$150,000.
b. The amount of the mortgage is \$600,000– \$150,000 = \$450,000.
c. The monthly mortgage installment will be

$$\frac{P\left(\frac{r}{m}\right)}{1-\left(1+\frac{r}{m}\right)^{-mt}} = \frac{450{,}000\left(\frac{0.0475}{12}\right)}{1-\left(1+\frac{0.0475}{12}\right)^{-12*15}} \approx \$3{,}500.25$$

d.

Payment Number	Payment Amount	Interest Charged	Amount Towards Principal	New Principal
				\$450,000
1	\$3,500.25	\$1,781.25	\$1719	\$448,281
2	\$3,500.25	\$1,774.45	\$1,725.80	\$446,555.20
3	\$3,500.25	\$1,767.62	\$1,732.63	\$444,822.57

Section 3.6 | Exercises

In Exercises 1 – 8, find the first three lines of the amortization schedule given the conditions of each situation. Each loan has monthly payments.

1. Maksymilian wants to get an Aston Martin which costs $315,000. He will get an 8-year loan at an interest rate of 5.9%. He plans to make $2,000 payments.
2. Monique wants to get an installment loan to purchase kitchen appliances. The loan will be for $25,000. Her lender has offered her an 8% interest rate for 4 years. She will be making $500 payments.
3. Dr. Johnson has student loans totaling $175,000. She has an interest rate of 4.7%. She is making $1,000 payments each month.
4. Suri has a mortgage for $400,000 at an interest rate of 4.75%. Her mortgage payment is $3,000.
5. Julian has a mortgage on his condo for $225,000 at an interest rate of 3.75%. His mortgage payment is $1,050.
6. Marina wants to purchase a Suburu WRX with a purchase price of $29,600. She receives a loan with an interest rate of 3.39% and makes monthly payments of $540 for five years.
7. Mr. Morgan works as a school principal. His school is taking out a $1,000,000 loan for reconstruction. The school gets a special interest rate of 2.1% and a monthly payment of $3,750.
8. The local community college plans to begin a construction project worth $2,500,000. They receive a loan with an interest rate of 3.2% and plan to make **quarterly** payments of $73,500 for ten years. Hint: Quarterly payments would have a time of ¼ of a year.

In Exercises 9 – 16, find each of the following given the conditions of each situation:

a. What is the amount of the down payment?
b. What is the amount of the mortgage?
c. What is monthly mortgage payment?
d. Find the first three lines of the amortization schedule.
e. What is the total interest paid over the life of the loan?

9. Billy wants to purchase a vacation home in Hawaii. He will need to get a loan for $600,000. The lender has offered him a 3.75% interest rate for a 15-year loan with a 25% down payment.

10. Aarav bought his first home. He bought a condominium in Chicago for $425,000. His loan is for 6.3% for 30 years with a 15% down payment.
11. Urszula bought a townhome for $300,000. Her lender offered her a mortgage for 25 years at 5.25% with a 3% down payment.
12. Yessenia and Wayne purchased a duplex home for $325,000. Their mortgage is for 10 years at 7.1%. They will be making a 40% down payment.
13. Henry bought his first car. His car costs $32,000 and he made a 12% down payment. His lender offered him a 5-year loan with a 7.2% interest rate.
14. Amelia and Antoni bought their home for $384,000 and made a 10% down payment. They received a loan with a 2.38% interest rate for 15 years.
15. Elon bought a second home for $1.2 million dollars. He made a 30% down payment. His lender offered him a 20-year loan with a 5.25% interest rate.
16. The local park district plans to renovate all of the playgrounds and sports fields. They have secured a two-year loan for $540,000. The terms of the loan require a 15% down payment in order to receive a special interest rate of 3.75%.

17. Amortization schedules are typically more than three lines. An amortization schedule runs for the full length of the loan. Given below is a complete amortization schedule for a 12-month loan. Yvonne is taking out a home equity loan to have some painting done in her house. She takes out an $8,000 loan for 12 months with an interest rate of 4.8%. Her monthly payments are $684.13.

Payment Number	Payment Amount	Interest Charged	Amount Towards Principal	New Principal
				$8,000
1	$684.13	$32	$652.13	$7,347.87
2	$684.13	$29.40	$654.73	$6,693.14
3	$684.13	$26.78	$657.35	$6,035.79
4	$684.13	$24.15	$659.98	$5,375.81
5	$684.13	$21.51	$662.62	$4,713.19
6	$684.13	$18.86	$665.27	$4,047.92
7	$684.13	$16.20	$667.93	$3,379.99
8	$684.13	$13.52	$670.61	$2,709.38
9	$684.13	$10.84	$673.29	$2,036.09
10	$684.13	$8.15	$675.99	$1,360.10
11	$684.13	$5.45	$678.68	$681.42
12	$684.13	$2.73	$681.40	$-0.02

a. Name three trends displayed in this schedule. For instance, the interest charged decreases over the course of the payments.
b. Why is the final principal value \$-0.02? What could be going on in the table to account for that, and what does that mean about Yvonne's last payment?
c. What is the interest charged over the life of the loan?
d. Suppose Yvonne decides she's ready to pay off her loan after 6 months. That is, at 6 months, rather than writing a check for \$684.13 she writes a check for \$4,047.92. How much will Yvonne save in interest?

Section 3.6 Exercise Solutions

1.

Payment Number	Payment Amount	Interest Charged	Amount Towards Principal	New Principal
				$315,000
1	$2,000	$1,548.75	$451.25	$314,548.75
2	$2,000	$1,546.54	$453.46	$314,095.29
3	$2,000	$1,544.31	$455.69	$313,639.60

2.

Payment Number	Payment Amount	Interest Charged	Amount Towards Principal	New Principal
				$25,000
1	$500	$166.67	$333.33	$24,666.67
2	$500	$164.45	$335.55	$24,331.12
3	$500	$162.21	$337.79	$23,993.33

3.

Payment Number	Payment Amount	Interest Charged	Amount Towards Principal	New Principal
				$175,000
1	$1,000	$685.42	$341.58	$174,685.42
2	$1,000	$684.19	$315.81	$174,369.61
3	$1,000	$682.95	$317.05	$174,052.56

4.

Payment Number	Payment Amount	Interest Charged	Amount Towards Principal	New Principal
				$400,000
1	$3,000	$1,583.34	$1,416.66	$398,583.34
2	$3,000	$1,577.73	$1,422.27	$397,161.07
3	$3,000	$1,572.10	$1,427.90	$395,733.17

5.

Payment Number	Payment Amount	Interest Charged	Amount Towards Principal	New Principal
				$225,000
1	$1,050	$703.13	$346.87	$224,653.13
2	$1,050	$702.05	$347.95	$224,305.18
3	$1,050	$700.96	$349.04	$223,956.14

6.

Payment Number	Payment Amount	Interest Charged	Amount Towards Principal	May
				$29,600
1	$540	$83.62	$456.38	$29,143.62
2	$540	$82.34	$457.66	$28,685.96
3	$540	$81.04	$458.96	$28,227

7.

Payment Number	Payment Amount	Interest Charged	Amount Towards Principal	New Principal
				$1,000,000
1	$3,750	$1750	$2,000	$998,000
2	$3,750	$1746.50	$2,003.50	$995,996.50
3	$3,750	$1,743	$2007	$993,989.50

8.

Payment Number	Payment Amount	Interest Charged	Amount Towards Principal	New Principal
				$2,500,000
1	$73,500	$20,000	$53,500	$2,446,500
2	$73,500	$19,572	$53,928	$2,392,572
3	$73,500	$19,140.58	$54,359.42	$2,338,212.58

9.

a. $150,000
b. $450,000
c. $3272.51
d.

Payment Number	Payment Amount	Interest Charged	Amount Towards Principal	New Principal
				$450,000
1	$3,272.51	$1,406.25	$1866.26	$448,133.74
2	$3,272.51	$1,400.42	$1,872.09	$446,261.65
3	$3,272.51	$1,394.57	$1,877.94	$444,383.71

e. $139,051.80

10.

a. $63,750
b. $361,250
c. $2,236.04
d.

Payment Number	Payment Amount	Interest Charged	Amount Towards Principal	New Principal
				$361,250
1	$2,236.04	$1,896.57	$339.47	$360,910.53
2	$2,236.04	$1,894.79	$341.25	$360,569.28
3	$2,236.04	$1,892.99	$343.05	$360,226.23

e. $443,724.40

11.

a. $9,000
b. $291,000
c. $1,743.82
d.

Payment Number	Payment Amount	Interest Charged	Amount Towards Principal	New Principal
				$291,000
1	$1,743.82	$1,273.13	$470.69	$290,529.31
2	$1,743.82	$1,271.07	$472.75	$290,056.56
3	$1,743.82	$1,269	$474.82	$289,581.74

e. $232,146

12.
   a. $130,000
   b. $195,000
   c. $2,274.18
   d.

Payment Number	Payment Amount	Interest Charged	Amount Towards Principal	New Principal
				$195,000
1	$2,274.18	$1,153.75	$1,120.43	$193,879.57
2	$2,274.18	$1,147.13	$1,127.05	$192,752.52
3	$2,274.18	$1,140.46	$1,133.72	$191,618.80

   e. $77,901.60

13.
   a. $3,840
   b. $28,160
   c. $560.27
   d.

Payment Number	Payment Amount	Interest Charged	Amount Towards Principal	New Principal
				$28,160
1	$560.27	$168.96	$391.31	$27,768.69
2	$560.27	$166.62	$393.65	$27,375.04
3	$560.27	$164.26	$396.01	$26,979.03

   e. $5,456.20

14.
   a. $38,400
   b. $345,600
   c. $2,284.96
   d.

Payment Number	Payment Amount	Interest Charged	Amount Towards Principal	New Principal
				$345,600
1	$2,284.96	$685.44	$1,599.52	$344,000.48
2	$2,284.96	$682.27	$1,602.69	$342,397.79
3	$2,284.96	$679.09	$1,605.87	$340,791.92

   e. $65,692.80

15.

a. $360,000
b. $840,000
c. $5,660.30
d.

Payment Number	Payment Amount	Interest Charged	Amount Towards Principal	New Principal
				$840,000
1	$5,660.30	$3,675	$1,985.30	$838,014.70
2	$5,660.30	$3,666.32	$1,993.98	$836,020.72
3	$5,660.30	$3,657.60	$2,002.70	$834,018.02

e. $518,472

16.

a. $81,000
b. $459,000
c. $19,881.01
d.

Payment Number	Payment Amount	Interest Charged	Amount Towards Principal	New Principal
				$459,000
1	$19,881.01	$1,434.38	$18,446.63	$440,550.37
2	$19,881.01	$1,376.73	$18,504.28	$422,046.09
3	$19,881.01	$1,318.90	$18,562.11	$403,483.98

e. $18,144.24

17.

a. The payment amount stays the same over the life of the loan. The interest charged decreases over the life of the loan. The amount toward principal increases over the life of the loan. The new principal amount decreases over the life of the loan and ends at or under $0.
b. In order to completely pay off the loan within 12 months, Yvonne's monthly payment amount was rounded up. This means that at the end of the term of the loan there was a slight overcompensation of $0.02 (or 2 cents). This means her last monthly payment will be $0.02 smaller than all other monthly payments.
c. $209.59 (Add up all the interest charged for the loan in column 3.)
d. $56.89 (Add up all the interest charged for the loan for payments 7 through 12 in column 3.)

Section 3.7 | Credit Cards

Objectives

- Apply the average daily balance method for calculating credit card interest
- Apply the unpaid balance method for calculating credit card interest

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It may be tempting to acquire a service or an item today and pay for it in the future. Credit card loans allow consumers to do just that! But, in the end, consumers often end up paying more for the service or good than originally anticipated due to interest and fees. Credit card loans involve revolving credit and are known as open-end installment loans. In this section, credit card loans will be compared to other types of loans, advantages and disadvantages of credit cards will be explored, and two methods for calculating credit card interest will be examined.

Open-end Installment Loans

DEFINITION: An **open-end installment loan** is one of revolving credit that allows continual access to money up to a certain limit as long as payments are made on time.

Open-end installment loans differ from fixed installment loans, such as car loans, student loans, and mortgages. With fixed installment loans, a specific amount of money is borrowed and paid back through fixed payments over time. Interest is included in the fixed payment amount and the time it takes to pay off the loan can range from months to years. Open-end installment loans are often more complicated because the amount of the loan can fluctuate as additional charges or payments are made. With open-end installment loans, the same credit can be used repeatedly without a time limit. Also, interest may or may not be charged depending on if the balance is paid in full before the end of the payment cycle. Advantages and disadvantages of credit card loans are listed in Figure 3.7.1.

FIGURE 3.7.1

Advantages and Disadvantages of Credit Card Loans	
Advantages: • Interest is not usually charged on the part of the credit line that is not used • Interest is not usually charged if the balance due is paid by the end of the billing cycle • Allows flexible access to funds • Offers consumer protection services and reward programs (such as airline miles, cash back, etc.) • Convenient to use, especially when reserving hotel rooms, booking rental cars, and shopping online	**Disadvantages:** • Usually have a higher interest rate than other types of loans • May charge annual fees or late payment fees • Risk of fraud • May encourage overspending by cardholders, which can contribute to serious debt • Making only the minimum payment instead of paying the balance in full will cause interest charges

With credit card loans, interest (or the finance charge) is often found using the simple interest formula $I = Prt$, where r is the annual interest rate and t is time in years. Various methods exist for calculating the value of P in this equation. Two common methods that can be used to calculate interest for credit card loans are the average daily balance method and the unpaid balance method. These methods can result in different finance charges for the same credit card activity at the same annual interest rate. Both methods will be explored in this section.

Average Daily Balance Method

The average daily balance method is one of the most common methods for calculating interest for credit card loans. With this method, the loan balance is found by averaging all daily balances for the previous billing period. Then, the interest (finance charge) is found using the formula for simple interest.

Average Daily Balance Method

The **average daily balance** is the sum of unpaid balances for each day in the billing period divided by the number of days in the billing period.

Interest is found using $I = Prt$,

where P is the average daily balance, r is the annual interest rate, and t is time in years (number of days in the month divided by 365).

➢ **EXAMPLE 3.7.1:** Use the average daily balance method to complete the following, rounded to the nearest cent. Assume the annual interest rate is 17.5% of the average daily balance and the billing period is from June 1 – June 30.

Transaction	**Amount**
Previous balance: \$2,500	
June 1: Billing date	
June 5: Payment	\$1,500 credit
June 7: Grocery store purchase	\$124.50 charge
June 15: Coffee shop purchase	\$10.45 charge
June 21: Clothing store return	\$75.50 credit
June 30: End of billing cycle	

a. Calculate the average daily balance.

b. Calculate the interest to be paid on July 1 (the next billing date).

c. Find the balance due on July 1 (including interest).

d. If no additional charges are made using this card, what minimum monthly payment must be made to pay off the remaining balance in 3 years? Use the formula for amortization from Section 3.6.

SOLUTION:

a. To calculate the average daily balance, the sum of unpaid balances for each day in the billing period must be first calculated. This sum is then divided by the number of days in the billing period (here, 30 days). Work is organized in the following table:

Balance	**Days at that balance**	**Number of days**	**Sum of balances for those days**
\$2,500	June 1 – June 4	4	$2{,}500 * 4 = \$\mathbf{10{,}000}$
\$1,000	June 5 – June 6	2	$1{,}000 * 2 = \$\mathbf{2{,}000}$
\$1,124.50	June 7 – June 14	8	$1{,}124.50 * 8 = \$\mathbf{8{,}996}$
\$1,134.95	June 15 – June 20	6	$1{,}134.95 * 6 = \$\mathbf{6{,}809.70}$
\$1,059.45	June 21 – June 30	10	$1{,}059.45 * 10 = \$\mathbf{10{,}594.50}$
	Sum:	**30**	$\$\mathbf{38{,}400.20}$

Starting balance

$2{,}500 - 1{,}500$ payment

$1{,}000 + 124.50$ charge

$1{,}124.50 + 10.45$ charge

$1{,}134.95 - 75.50$ credit

$4 + 2 + 8 + 6 + 10$

$10{,}000 + 2{,}000 + 8{,}996 + 6{,}809.70 + 10{,}594.50$

$$Average\ daily\ balance = \frac{\text{sum of unpaid balances}}{\text{number of days}} = \frac{38{,}400.20}{30} \approx \$1{,}280.01$$

b. To calculate the interest to be paid on July 1, use $I = Prt$. In this calculation, P is the average daily balance, r is the annual interest rate, and t is time in years. The average daily balance was found in part a, and the annual interest rate was given as 17.5%. As a decimal, the annual interest rate is 0.175. Since June has 30 days, t is found by calculating 30/365.

$$I = Prt = 1{,}280.01 * 0.175 * \frac{30}{365} \approx \$18.41$$

This interest will get added to the statement balance at the beginning of the next month. If the statement balance is paid in full (instead of just the minimum payment), then interest will not be charged.

c. To find the balance due at the beginning of the next month (here, on July 1), add the interest calculated in part b to the ending balance in the table created in part a (the last entry in the "Balance" column).

$$\text{Balance due on July } 1 = 1{,}059.45 + 18.41 = \$1{,}077.86$$

d. Using the formula for loan amortization, the monthly payment needed to amortize a loan of $1,077.86 at 17.5% annual interest for 3 years can be calculated as follows. Here, it is assumed that interest is compounded monthly ($m = 12$). Recall that the answer will be rounded up to the nearest cent to ensure that the total balance is paid off.

$$\text{Payment Amount} = \frac{P\left(\frac{r}{m}\right)}{1 - \left(1 + \frac{r}{m}\right)^{-mt}} = \frac{1{,}077.86\left(\frac{0.175}{12}\right)}{1 - \left(1 + \frac{0.175}{12}\right)^{-12\cdot 3}} \approx \$38.70$$

The finance charge in Example 3.7.1 was calculated under the assumption that all purchases were subject to the same interest rate. Some credit cards, however, charge different interest rates for certain categories of spending. For example, there may be different rates for purchases, balance transfers, or cash advances on the same card. In those instances, the average daily balance would have to be calculated separately for each, which would result in multiple finance charges (interest calculations).

NOTE: In this text, when using the average daily balance method, the value of t is found by dividing the number of days in the billing period by 365. However, some institutions find the value of t by dividing the number of days in the billing period by 360. This is called the **Banker's Rule**, because it yields a higher amount of interest on a loan and, thus, benefits the lending institution. Also, final answers are reported using typical rounding rules in this section. Some institutions, however, round up to the nearest cent in these calculations.

The steps for using the average daily balance method for calculating interest are summarized below.

Steps for Using the Average Daily Balance Method

1. Find the unpaid balance for each day in the billing period.
2. Find the sum of all unpaid balances.
3. Divide the sum of all unpaid balances by the number of days in the billing period.
4. Calculate the interest (finance charge) using $I = Prt$, where P is the average daily balance, r is the annual interest rate, and t is time in years (calculated as the number of days in the month divided by 365).

➢ **EXAMPLE 3.7.2:** Use the average daily balance method to find the balance due on November 1, rounded to the nearest cent. Assume the annual interest rate is 22.75% of the average daily balance and the billing period is from October 1 – October 31.

Transaction	**Amount**
Previous balance: \$3,750	
October 1: Billing date	
October 5: Payment	\$1,000 credit
October 10: Clothing purchase	\$64.50 charge
October 11: Grocery store purchase	\$110.15 charge
October 19: Restaurant purchase	\$24.50 charge
October 23: Grocery store purchase	\$75.90 charge
October 31: Entertainment purchase	\$25 charge
October 31: End of billing cycle	

SOLUTION: First, calculate the average daily balance by finding the sum of unpaid balances for each day divided by the number of days in the billing period (here, 31). Work is organized in the following table:

Balance	Days at that balance	Number of days	Sum of balances for those days
$3,750	Oct 1 – Oct 4	4	$3{,}750 * 4 = \$\mathbf{15,000}$
$2,750	Oct 5 – Oct 9	5	$2{,}750 * 5 = \$\mathbf{13,750}$
$2,814.50	Oct 10	1	$2{,}814.50 * 1 = \$\mathbf{2,814.50}$
$2,924.65	Oct 11 – Oct 18	8	$2{,}924.65 * 8 = \$\mathbf{23,397.20}$
$2,949.15	Oct 19 – Oct 22	4	$2{,}949.15 * 4 = \$\mathbf{11,796.60}$
$3,025.05	Oct 23 – Oct 30	8	$3{,}025.05 * 8 = \$\mathbf{24,200.40}$
$3,050.05	Oct 31	1	$3{,}050.05 * 1 = \$\mathbf{3,050.05}$
	Sum:	**31**	$\$\mathbf{94,008.75}$

$$\text{Average daily balance} = \frac{\text{sum of unpaid balances}}{\text{number of days}} = \frac{94{,}008.75}{31} \approx \$3{,}032.54$$

To calculate the interest to be paid on November 1, use $I = Prt$, where P is the average daily balance found above, r is 22.75% as a decimal, and t is 31/365.

$$I = Prt = 3{,}032.54 * 0.2275 * \frac{31}{365} \approx \$58.59$$

To find the balance due on November 1, add the interest to the ending balance in the table created above (the last entry in the "Balance" column).

$$\text{Balance due on November 1} = 3{,}050.05 + 58.59 = \$3{,}108.64$$

❖ **YOU TRY IT 3.7.A:** Use the average daily balance method to complete the following, rounded to the nearest cent. Assume the annual interest rate is 18% of the average daily balance and the billing period is from February 1 – February 28.

Transaction	Amount
Previous balance: \$5,000	
February 1: Billing date	
February 8: Payment	\$500 credit
February 16: Bookstore purchase	\$42.50 charge
February 17: Coffee shop purchase	\$15.15 charge
February 25: Restaurant purchase	\$95.45 charge
February 28: End of billing cycle	

a. Calculate the average daily balance.
b. Calculate the interest to be paid on March 1 (the next billing date).
c. Find the balance due on March 1 (including interest).
d. If no additional charges are made using this card, what minimum monthly payment must be made to pay off the remaining balance in 10 years?

Unpaid Balance Method

A less common method for calculating interest for credit card loans is the unpaid balance method. With this method, interest is based on the unpaid balance from the previous billing period. This is different than the average daily balance method, in which interest is based on the average of all daily balances for the billing period.

Unpaid Balance Method

The **unpaid balance** is the sum of the balance from the previous billing period, finance charges, and purchases, minus any credits made to the account.

Interest is found using $I = Prt$,

where P is the unpaid balance, r is the annual interest rate, and t is time in years (represented as 1/12).

➢ **EXAMPLE 3.7.3:** Assume the annual interest rate on a credit card is 21.5% and the unpaid balance from the previous billing period was \$1,500. This month, purchases were made totaling \$37.50, \$127.15, and \$55.75, and a payment of \$250 was made. Use the unpaid balance method to complete the following, rounded to the nearest cent.

a. Calculate the unpaid balance at the end of this billing period.
b. Calculate the interest to be paid at the beginning of the next billing period.

SOLUTION:

a. First, find the finance charge on the balance from the last billing period using the formula $I = Prt$. In this calculation, P is the previous unpaid balance, r is the annual interest rate, and t is 1/12. Then, find the unpaid balance by finding the sum of the balance from the previous billing period, finance charges from the previous billing period, and new purchases, minus any credits or payments made.

Finance charge on last month's balance: $I = Prt = 1{,}500 * 0.215 * \frac{1}{12} \approx \$26.88$

Unpaid balance: $\text{Previous balance} + \text{finance charge} + \text{purchases} - \text{payment}$

$$= \underbrace{1{,}500}_{\text{Unpaid balance from previous billing cycle}} + \underbrace{26.88}_{\text{Finance charge on balance from previous billing cycle}} + \underbrace{37.50 + 127.15 + 55.75}_{\text{New purchases}} - \underbrace{250}_{\text{Payment (credit)}} = \$1{,}497.28$$

b. To calculate the interest to be paid at the beginning of next billing period, again use the formula $I = Prt$. In this calculation, P is the unpaid balance from this billing period (calculated above), r is the annual interest rate, and t is 1/12.

$$I = Prt = 1{,}497.28 * 0.215 * \frac{1}{12} \approx \$26.83$$

NOTE: If a purchase is made and paid off before the billing date, then interest will not be charged. Some consumers use this to their advantage, because it allows them to build credit responsibly and earn perks (such as cash back rewards or airline miles) without being charged interest.

❖ **YOU TRY IT 3.7.B:** Assume the annual interest rate on a credit card is 16% and the unpaid balance from the previous billing period was $975. This billing cycle, purchases were made totaling $325.50, $27.15, $115.50, and $155. Also, a $34.77 credit from a store return was given and a payment of $50 was made. Use the unpaid balance method to complete the following, rounded to the nearest cent.

a. Calculate the unpaid balance at the end of this billing period.
b. Calculate the interest to be paid at the beginning of the next billing period.

Comparing Average Daily Balance and Unpaid Balance Methods

In this section, two methods for calculating credit card interest were introduced. As shown in the following example, these methods can result in different finance charges for the same credit card activity at the same annual interest rate.

➢ **EXAMPLE 3.7.4:** Use the given information to complete the following, rounded to the nearest cent. Assume the annual interest rate is 15.75% and the billing period is from September 1 – September 30.

Transaction	**Amount**
Previous balance: $500	
September 1: Billing date	
September 3: Payment	$40 credit
September 5: College books	$475 charge
September 12: Fast food purchase	$15.72 charge
September 13: Gas station purchase	$57.15 charge
September 21: Monthly gym membership	$29.99 charge
September 30: End of billing cycle	

a. Calculate the interest to be paid on October 1 (the next billing date) using the average daily balance method.
b. Calculate the interest to be paid on October 1 (the next billing date) using the unpaid balance method.
c. Compare the results from both methods. Which method charges a larger amount of interest in this example?

SOLUTION:

a. First, find the average daily balance. In previous examples, a table was used to organize work. Here, calculations are listed without the use of a table.

Average daily balance:

$$\frac{(500 * 2) + (460 * 2) + (935 * 7) + (950.72 * 1) + (1{,}007.87 * 8) + (1{,}037.86 * 10)}{30}$$

$$= \frac{27{,}857.28}{30} \approx \$928.58$$

Then, calculate the interest to be paid on October 1.

Interest: $I = Prt = 928.58 * 0.1575 * \frac{30}{365} \approx \$12.02$

b. First, find the finance charge on last month's balance:

Finance charge on last month's balance: $I = Prt = 500 * 0.1575 * \frac{1}{12} \approx \$6.56$

Then, find the unpaid balance.

Unpaid balance: $\text{Previous balance} + \text{finance charge} + \text{purchases} - \text{payment}$

$$= 500 + 6.56 + 475 + 15.72 + 57.15 + 29.99 - 40 = \$1{,}044.42$$

Lastly, calculate the interest to be paid on October 1.

Interest: $I = Prt = 1{,}044.42 * 0.1575 * \frac{1}{12} \approx \$13.71$

c. The average daily balance produced an interest charge of \$12.02, whereas the unpaid balance method produced an interest charge of \$13.71. In this situation, the unpaid balance method calculation results in \$1.69 more interest than the average daily balance method.

❖ **YOU TRY IT 3.7.C:** Use the given information to complete the following, rounded to the nearest cent. Assume the annual interest rate is 13.5% and the billing period is from May 1 – May 31.

Transaction	**Amount**
Previous balance: $700	
May 1: Billing date	
May 5: Payment	$50 credit
May 15: Restaurant purchase	$22.75 charge
May 20: Coffee shop purchase	$10.13 charge
May 25: Gas station purchase	$45 charge
May 31: Monthly gym membership	$19.99 charge
May 31: End of billing cycle	

a. Calculate the interest to be paid on June 1 (the next billing date) using the average daily balance method.
b. Calculate the interest to be paid on June 1 (the next billing date) using the unpaid balance method.
c. Compare the results from both methods. Which method charges a larger amount of interest in this example?

Quick Review

Two methods for calculating interest (finance charges) for credit card loans are the average daily balance method and unpaid balance method.

- The **average daily balance** is the sum of unpaid balances for each day in the billing period divided by the number of days in the billing period. **Interest** is found using $I = Prt$, where P is the average daily balance, r is the annual interest rate, and t is time in years (number of days in the month divided by 365).

- The **unpaid balance** is the sum of the balance from the previous billing period, finance charges, and purchases, minus any credits made to the account. **Interest** is found using $I = Prt$, where P is the unpaid balance, r is the annual interest rate, and t is time in years (represented as 1/12).

YOU TRY IT 3.7.A SOLUTION:

c. Work for finding the average daily balance can be summarized in a table:

Balance	Days at that balance	Number of days	Sum of balances for those days
\$5,000	Feb 1 – Feb 7	7	$5{,}000 * 7 = \$\mathbf{35,000}$
\$4,500	Feb 8 – Feb 15	8	$4{,}500 * 8 = \$\mathbf{36,000}$
\$4,542.50	Feb 16	1	$4{,}542.50 * 1 = \$\mathbf{4,542.50}$
\$4,557.65	Feb 17 – Feb 24	8	$4{,}557.65 * 8 = \$\mathbf{36,461.20}$
\$4,653.10	Feb 25 – Feb 28	4	$4{,}653.10 * 4 = \$\mathbf{18,612.40}$
	Sum:	**28**	$\$\mathbf{130,616.10}$

$$\text{Average daily balance} = \frac{130{,}616.10}{28} \approx \$4{,}664.86$$

d. $I = Prt = 4{,}664.86 * 0.18 * \frac{28}{365} \approx \$64.41$

e. $\text{Balance due on March 1} = 4{,}653.10 + 64.41 = \$4{,}717.51$

f. Using the formula for loan amortization, the monthly payment needed to amortize a loan of \$4,717.51 at 18% annual interest for 10 years is \$85.01.

YOU TRY IT 3.7.B SOLUTION:

a. Finance charge on last month's balance: $I = Prt = 975 * 0.16 * \frac{1}{12} \approx \$13$

Unpaid balance: $975 + 13 + 325.50 + 27.15 + 115.50 + 155 - 34.77 - 50$
$= \$1{,}526.38$

b. $I = Prt = 1{,}526.38 * 0.16 * \frac{1}{12} \approx \$20.35$

YOU TRY IT 3.7.C SOLUTION:

a. Average daily balance:

$$\frac{(700*4)+(650*10)+(672.75*5)+(682.88*5)+(727.88*6)+(747.87*1)}{31} = \frac{21{,}193.30}{31} \approx \$683.65$$

Interest to be paid on June 1: $I = Prt = 683.65 * 0.135 * \frac{31}{365} \approx \$7.84$

b. Finance charge on last month's balance: $I = Prt = 700 * 0.135 * \frac{1}{12} \approx \$7.88$

Unpaid balance: $700 + 7.88 + 22.75 + 10.13 + 45 + 19.99 - 50 = \$755.75$

Interest to be paid on June 1: $I = Prt = 755.75 * 0.135 * \frac{1}{12} \approx \$8.50$

c. In this situation, the unpaid balance method calculation results in $0.66 more interest than the average daily balance method.

Section 3.7 | Exercises

In Exercises 1 – 4, calculate the average daily balance, rounded to the nearest cent, given the account activity for each month.

1. **Beginning balance:** \$567.00
   Charges: \$37.50 on June 10, \$150.45 on June 15, \$29.57 on June 23
   Credits: \$40 payment on June 5, \$25.67 merchandise return on June 20
   Billing period: June 1 – June 30

2. **Beginning balance:** \$3,014.35
   Charges: \$117.92 on May 4, \$27.51 on May 11, \$79.99 on May 15, \$5.61 on May 23, \$101 on May 29
   Credits: \$250.00 payment on May 17
   Billing period: May 1 – May 31

3. **Beginning balance:** \$1,437.50
   Charges: \$29.99 on December 7, \$235.75 on December 18, \$56.90 on December 20, \$127.35 on December 23, \$75 on December 29
   Credits: \$100 payment on December 9
   Billing period: December 1 – December 31

4. **Beginning balance:** \$0
   Charges: \$31.20 on January 3, \$49.95 on January 5, \$112.72 on January 21, \$5.86 on January 22, \$500.00 on January 30
   Credits: \$300 payment on January 31
   Billing period: January 1 – January 31

In Exercises 5 – 10, complete the following, rounded to the nearest cent.

5. Assume the annual interest rate is 13.5% of the average daily balance and the billing period is from March 1 – March 31.
   a. Calculate the average daily balance.
   b. Calculate the interest to be paid at the beginning of the next month.
   c. Calculate the balance due at the beginning of the next month.

Transaction	Amount
Previous balance: \$500	
March 1: Billing date	
March 9: Payment	\$30 credit
March 10: Gym membership	\$19.99 charge
March 13: Coffee shop purchase	\$12.31 charge
March 21: Food purchase	\$28.21 charge
March 31: End of billing cycle	

6. Assume the annual interest rate is 11.75% of the average daily balance and the billing period is from July 1 – July 31.
   a. Calculate the average daily balance.
   b. Calculate the interest to be paid at the beginning of the next month.
   c. Calculate the balance due at the beginning of the next month.

Transaction	Amount
Previous balance: \$735.84	
July 1: Billing date	
July 4: Carnival tickets	\$60.00 charge
July 15: Payment	\$100.00 credit
July 19: Bookstore purchase	\$127.89 charge
July 25: Gym membership	\$99.00 charge
July 31: End of billing cycle	

7. Assume the annual interest rate is 12.75% of the average daily balance and the billing period is from November 1 – November 30.
    a. Calculate the average daily balance.
    b. Calculate the interest to be paid at the beginning of the next month.
    c. Calculate the balance due at the beginning of the next month.

Transaction	Amount
Previous balance: $1,500	
November 1: Billing date	
November 5: Movie theatre purchase	$31.75 charge
November 8: Payment	$35 credit
November 13: Gas station purchase	$54.31 charge
November 14: Fast food purchase	$16.77 charge
November 22: Sporting event	$75.82 charge
November 30: End of billing cycle	

8. Assume the annual interest rate is 14.99% of the average daily balance and the billing period is from August 1 – August 31.
    a. Calculate the average daily balance. Note that the previous balance is negative, meaning that there is a credit on the account at the start of this billing period.
    b. Calculate the interest to be paid at the beginning of the next month.
    c. Calculate the balance due at the beginning of the next month.

Transaction	Amount
Previous balance: $—634.12	
August 1: Billing date	
August 5: Tuition charge	$931.75 charge
August 8: Payment	$35 credit
August 9: Archery lesson	$70.00 charge
August 22: Venmo to your friend for lunch	$11.45 charge
August 31: End of billing cycle	

9. Assume the annual interest rate is 14.25% of the average daily balance and the billing period is from October 1 – October 31.
   a. Calculate the average daily balance.
   b. Calculate the interest to be paid at the beginning of the next month.
   c. Calculate the balance due at the beginning of the next month.

Transaction	**Amount**
Previous balance: $5,106.73	
October 1: Billing date	
October 7: Gas station purchase	$57.00 charge
October 10: Payment	$350.00 credit
October 11: Return	$15.93 credit
October 17: Costume purchase	$84.15 charge
October 27: Candy purchase	$51.84 charge
October 31: End of billing cycle	

10. Assume the annual interest rate is 22.30% of the average daily balance and the billing period is from March 1 – March 31.
    a. Calculate the average daily balance.
    b. Calculate the interest to be paid at the beginning of the next month.
    c. Calculate the balance due at the beginning of the next month.

Transaction	**Amount**
Previous balance: $634.12	
March 1: Billing date	
March 5: Payment	$500 credit
March 8: Payment	$134.12 credit
March 9: Target purchase	$72.54 charge
March 12: Coffee purchase	$8.23 charge
March 22: Portillo’s purchase	$11.45 charge
March 31: End of billing cycle	

In Exercises 11 – 14, use the average daily balance method to complete the following rounded to the nearest cent.

11. Assume the annual interest rate is 13% of the average daily balance and the billing period is from February 1 – February 28.
    a. Calculate the balance due at the beginning of the next month.
    b. Use the loan amortization formula to find the minimum monthly payment that must be made to pay off this balance in 3 years if no additional charges are made.

Transaction	**Amount**
Previous balance: $750.05	
February 1: Billing date	
February 7: Payment	$50 credit
February 8: Coffee shop purchase	$15.38 charge
February 19: Gas station purchase	$54.31 charge
February 22: Pet store purchase	$67.65 charge
February 27: Food court purchase	$17.47 charge
February 28: End of billing cycle	

12. Assume the annual interest rate is 10.75% of the average daily balance and the billing period is from July 1 – July 31.
    a. Calculate the balance due at the beginning of the next month. (Hint: combine both transactions on January 12.)
    b. Use the loan amortization formula to find the minimum monthly payment that must be made to pay off this balance in 5 years if no additional charges are made.

Transaction	**Amount**
Previous balance: $1,204.18	
January 1: Billing date	
January 12: Payment	$1,000 credit
January 12: Video game purchase	$75.43 charge
January 23: Gas station purchase	$44.56 charge
January 25: Coffee purchase	$7.34 charge
January 27: Food court purchase	$17.47 charge
January 31: End of billing cycle	

13. Assume the annual interest rate is 16.99% of the average daily balance and the billing period is from July 1 – July 31.
    a. Calculate the balance due at the beginning of the next month. (Hint: combine both charges on July 5.)
    b. Use the loan amortization formula to find the minimum monthly payment that must be made to pay off this balance in 3 years if no additional charges are made.

Transaction	Amount
Previous balance: $2,250.71	
July 1: Billing date	
July 5: Hotel reservation	$211.25 charge
July 5: Rental car reservation	$113.56 charge
July 7: Payment	$100 credit
July 17: Gas station purchase	$61.35 charge
July 18: Restaurant purchase	$76.50 charge
July 19: Concert tickets	$237.99 charge
July 31: End of billing cycle	

14. Assume the annual interest rate is 9.75% of the average daily balance and the billing period is from September 1 – September 30.
    a. Calculate the balance due at the beginning of the next month. (Hint: combine both charges on September 14.)
    b. Use the loan amortization formula to find the minimum monthly payment that must be made to pay off this balance in 4 years if no additional charges are made.

Transaction	Amount
Previous balance: $515	
September 1: Billing date	
September 10: Candy purchase	$24.12 charge
September 14: Flower purchase	$98.79 charge
September 14: Restaurant purchase	$123.07 charge
September 20: Payment	$225.00 credit
September 28: Restaurant purchase	$76.50 charge
September 29: Coffee purchase	$7.64 charge
September 30: End of billing cycle	

In Exercises 15 – 22, use the unpaid balance method to calculate the unpaid balance at the end of the billing period and the interest to be paid at the beginning of the next billing period. Round to the nearest cent. Last month's unpaid balance, the annual interest rate, and the account activity for the month are given.

15. **Unpaid balance from previous billing period:** \$675.43
    Annual interest rate: 15.8%
    Credits: \$35 payment
    Charges: \$34.95 food purchase, \$76.95 clothing purchase, \$37.50 gas purchase

16. **Unpaid balance from previous billing period:** \$324.15
    Annual interest rate: 12.35%
    Credits: \$45 payment
    Charges: \$35 gas purchase, \$67.88 shoe purchase, \$156.50 bookstore purchase

17. **Unpaid balance from previous billing period:** \$3,455.70
    Annual interest rate: 14.5%
    Credits: \$150 payment, \$57.65 credit
    Charges: \$219.59 grocery purchase, \$124.35 dental appointment, \$60 festival tickets, \$45.67 food purchase, \$19.99 gym membership

18. **Unpaid balance from previous billing period:** \$2,112.37
    Annual interest rate: 13.25%
    Credits: \$250 payment, \$175.65 credit
    Charges: \$187.65 veterinarian appointment, \$121.32 store purchase, \$13.45 coffee shop purchase, \$15.49 streaming service, \$8.99 music purchase

19. **Unpaid balance from previous billing period:** \$1,718.23
    Annual interest rate: 11.75%
    Credits: \$135.00 payment, \$27.65 credit
    Charges: \$9.80 coffee shop purchase, \$37.61 bookstore purchase, \$20.11 fast food purchase, \$135.00 hair salon purchase, \$53.00 nail salon purchase

20. **Unpaid balance from previous billing period:** \$4,512.57
    Annual interest rate: 17%
    Credits: \$300.00 payment
    Charges: \$303.50 sport tickets, \$85.00 concert tickets, \$45.71 car-ride service, \$37.59 concession stand purchase, \$59.56 car-ride service

21. **Unpaid balance from previous billing period:** \$0
    Annual interest rate: 22.1%
    Credits: \$135 payment
    Charges: \$9.34 Venmo to friend, \$375.00 textbook purchase, \$45 oil change

22. **Unpaid balance from previous billing period:** \$92.37
    Annual interest rate: 14.2%
    Credits: \$72.34 clothing return, \$9.34 Venmo from friend
    Charges: \$23.72 craft store purchase, \$99.90 monthly gym membership, \$9.99 Netflix bill

In Exercises 23 – 26, use the given information to complete the following, rounded to the nearest cent.

23. Assume the annual interest rate is 16% and the billing period is from April 1 – April 30.
    a. Calculate the interest to be paid on May 1 using the average daily balance method.
    b. Calculate the interest to be paid on May 1 using the unpaid balance method.
    c. Compare the results from both methods. Which method charges a larger amount of interest in this example?

Transaction	**Amount**
Previous balance: \$987.65	
April 1: Billing date	
April 5: Payment	\$65 credit
April 10: Doctor co-pay	\$30 charge
April 17: Haircut	\$50 charge
April 21: Merchandise return	\$89.25 credit
April 22: Grocery store purchase	\$191.57 charge
April 30: End of billing cycle	

24. Assume the annual interest rate is 14.25% and the billing period is from August 1 – August 31. (Table on next page.)
    a. Calculate the interest to be paid on September 1 using the average daily balance method.
    b. Calculate the interest to be paid on September 1 using the unpaid balance method.
    c. Compare the results from both methods. Which method charges a larger amount of interest in this example?

Transaction	Amount
Previous balance: $1,578.12	
August 1: Billing date	
August 3: Airline tickets	$350.72 charge
August 5: Hotel purchase	$908.11 charge
August 7: Restaurant purchase	$111.51 charge
August 8: Amusement tickets	$83.50 charge
August 15: Payment	$500.00 credit
August 23: Return	$23.65 credit
August 31: End of billing cycle	

25. Assume the annual interest rate is 15.4% and the billing period is from May 1 – May 31.
    a. Calculate the interest to be paid on June 1 using the average daily balance method.
    b. Calculate the interest to be paid on June 1 using the unpaid balance method.
    c. Compare the results from both methods. Which method charges a larger amount of interest in this example?

    Beginning balance: $3,125.15
    Charges: $37.50 on May 10, $150.45 on May 16, $29.57 on May 18, $221.16 on May 20, $451.36 on May 23, $29.99 on May 23
    Credits: $150 payment on May 11
    Billing period: May 1 – May 31

26. Assume the annual interest rate is 23.9% and the billing period is from June 1 – June 30.
    a. Calculate the interest to be paid on July 1 using the average daily balance method.
    b. Calculate the interest to be paid on July 1 using the unpaid balance method.
    c. Compare the results from both methods. Which method charges a larger amount of interest in this example?

    Beginning balance: $943.23
    Charges: $43.78 on June 5, $82.71 on June 12, $5.63 on June 18
    Credits: $200 payment on June 22
    Billing period: June 1 – June 30

Section 3.7 | Exercise Solutions

1. 637.30
2. $3,073.79
3. $1,560.22
4. $137.16
5. a. $509.48
   b. $5.84
   c. $536.35
6. a. $811.18
   b. $8.10
   c. $930.83
7. a. $1,565.52
   b. $16.41
   c. $1,660.06
8. a. $205.94
   b. $2.62
   c. $346.70
9. a. $4,942.60
   b. $59.82
   c. $4,993.61
10. a. $157.62
    b. $2.99
    c. $95.21
11. a. $862.44
    b. $29.06
12. a. $354.69
    b. $7.67
13. a. $2,889.12
    b. $102.99
14. a. $624.79
    b. $15.77
15. Unpaid balance: $798.72
    Interest to be paid: $10.52
16. Unpaid balance: $541.87
    Interest to be paid: $5.58

17. Unpaid balance: $3,759.41
    Interest to be paid: $45.43
18. Unpaid balance: $2,056.94
    Interest to be paid: $22.71
19. Unpaid balance: $1,827.92
    Interest to be paid: $17.90
20. Unpaid balance: $4,807.86
    Interest to be paid: $68.11
21. Unpaid balance: $294.34
    Interest to be paid: $5.42
22. Unpaid balance: $145.39
    Interest to be paid: $1.72
23. a. $13.20
    b. $14.91
    c. The unpaid balance method results in $1.71 more interest charged.
24. a. $31.11
    b. $30.01
    c. The average daily balance method results in $1.10 more interest charged.
25. a. $44.03
    b. $50.50
    c. The unpaid balance method results in $6.47 more interest charged.
26. a. $19.17
    b. $17.81
    c. The average daily balance method results in $1.36 more interest charged.

Chapter 4

Probability

Section 4.A Dice and Playing Cards

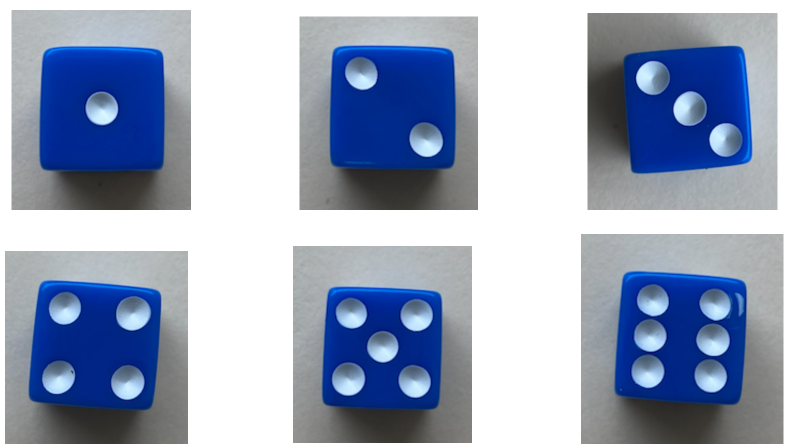
Prerequisite Content

Dice and playing cards are often used to teach probability. In this section, basic information about a six-sided die and a standard 52-card deck will be provided.

Dice

For this textbook, a six-sided die (plural: dice) is a cube that has dots representing 1, 2, 3, 4, 5, or 6 on one of its faces. In Figure 4.A.1 an image showing each face of a six-sided die is given. Each of the six faces of the die are assumed to have the same likelihood of being rolled.

FIGURE 4.A.1 – All six sides of a standard die



- **EXAMPLE 4.A.1:** On a six-sided die, what even numbers can be rolled?

  SOLUTION: The numbers 2, 4, and 6 can be rolled.

➢ **EXAMPLE 4.A.2:** On a six-sided die, what numbers can be rolled that are greater than 3?

SOLUTION: The numbers 4, 5, and 6 can be rolled.



➢ **EXAMPLE 4.A.3:** On a six-sided die, what numbers are less than 4 and also odd?

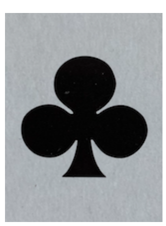
SOLUTION: The numbers 1 and 3.


Playing Cards

For this textbook, a standard 52-card playing deck has four suits. The suits are clubs, diamonds, hearts, and spades as shown in Figure 4.A.2 below.

FIGURE 4.A.2 – Each of the four suits (diamond, club, heart, and spade, respectively) in a standard 52-card playing deck


Diamond

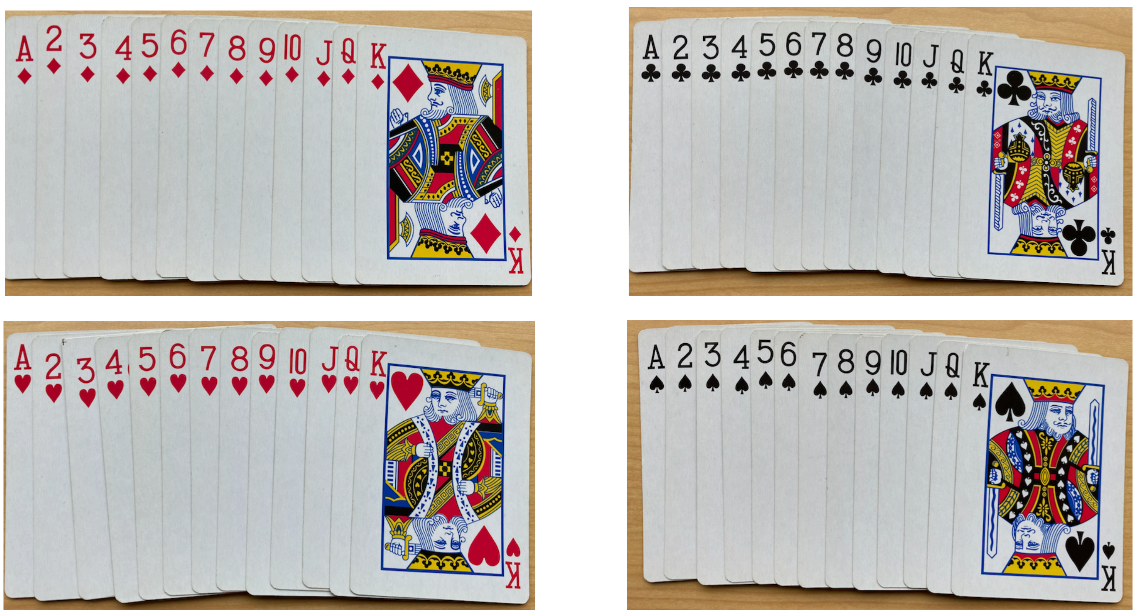

Club


Heart


Spade

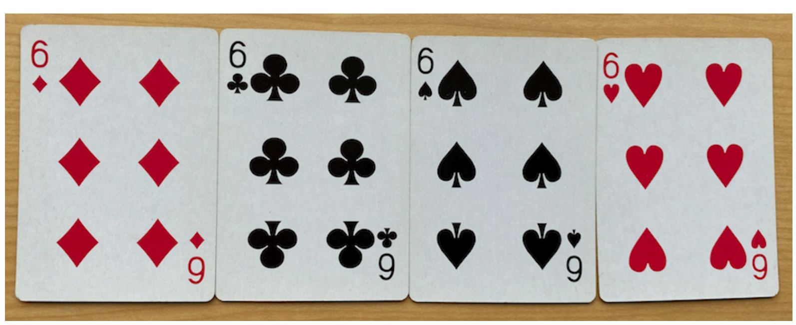
Clubs and spades are black cards. Diamonds and hearts are red cards. There are 13 cards in each suit. The cards in each suit are A (ace), 2, 3, 4, 5, 6, 7, 8, 9, 10, J (jack), Q (queen), and K (king). Jacks, queens, and kings are called face cards. Each of the cards in a 52-card deck is shown in Figure 4.A.3 below.

FIGURE 4.A.3 – All 52 cards in a standard playing deck



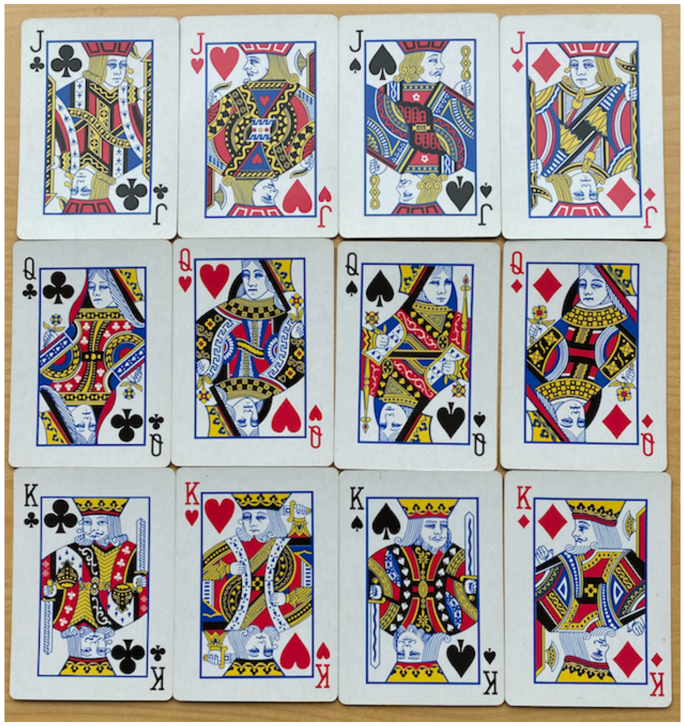
➢ **EXAMPLE 4.A.4:** In a standard 52-card playing deck, how many 6s are there?

SOLUTION: Four. Each of the four suits has a 6.



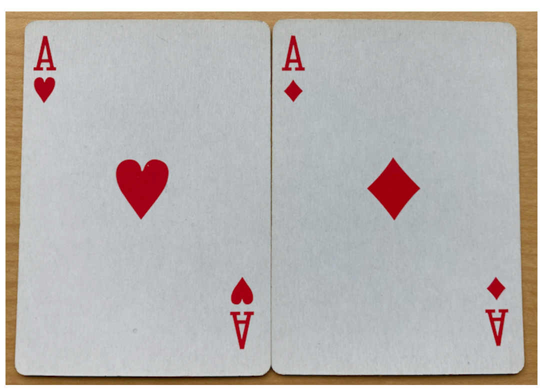
➢ **EXAMPLE 4.A.5:** In a standard 52-card playing deck, how many face cards are there?

SOLUTION: Twelve. There are three face cards per suit and four suits per deck.



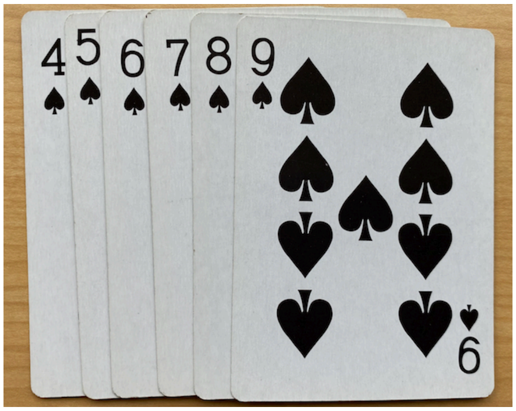
➢ **EXAMPLE 4.A.6:** In a standard 52-card playing deck, how many red aces are there?

SOLUTION: Two. There is an ace of hearts and an ace of diamonds.



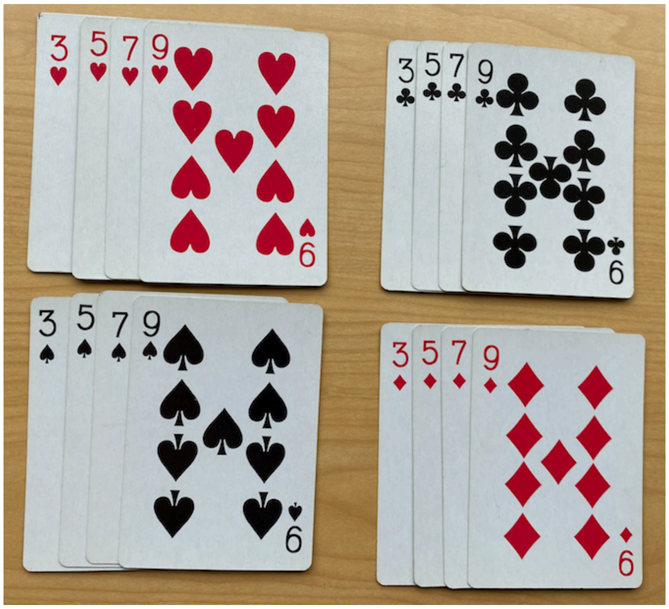
➢ **EXAMPLE 4.A.7:** In a standard 52-card playing deck, how many cards are spades greater than 3 and less than 10?

SOLUTION: Six. The cards are the 4, 5, 6, 7, 8, and 9 of spades. Notice that jack, queen, and king of spades are not considered cards greater than 3 or less than 10. Jacks, queens, and kings are face cards.



➢ **EXAMPLE 4.A.8:** In a standard 52-card playing deck, how many odd cards are greater than 2? For this question do not include the aces or the face cards (jack, queen, and king).

SOLUTION: Sixteen. The odd cards greater than 2 are 3, 5, 7, and 9 from each of the four suits.



Section 4.1 | Fundamental Counting Principle & Tree Diagrams

Objectives

- Solve problems using the Fundamental Counting Principle
- Solve problems using tree diagrams

Suppose Alice wants to fly to Honolulu from New York City on short notice. The only flight options have two stops: one in the Midwest and one in California. Alice has two options for the stops in the Midwest: Chicago or St. Louis. Alice has three options for stops in California: Sacramento, Los Angeles, or San Diego. How many different flight options does Alice have to get from New York to Honolulu?

For Alice, it may be beneficial to list out all of her options. Alice has six options. The flight options from New York (NYC) via the Midwest and California to Honolulu (HNL) are as follows:

NYC → **Chicago to Sacramento** → HNL
NYC → **Chicago to Los Angeles** → HNL
NYC → **Chicago to San Diego** → HNL

NYC → **St. Louis to Sacramento** → HNL
NYC → **St. Louis to Los Angeles** → HNL
NYC → **St. Louis to San Diego** → HNL

Writing out the different flight options may become cumbersome if Alice had more options; however, the different number of choices can be found by calculating $2 \cdot 3 = 6$. Here, 2 represents the two stop options for the Midwest (Chicago or St. Louis) and 3 represents the three stop options for California (Sacramento, Los Angeles, or San Diego). This calculation is an example of the Fundamental Counting Principle.

Fundamental Counting Principle

The **Fundamental Counting Principle** can be used to calculate the number of ways in which a series of events can occur. The number of ways a series of events can occur is calculated by multiplying the number of ways each individual event can occur.

- **EXAMPLE 4.1.1:** Manu would like to take three courses during the spring semester: one math class, one English class, and one geography class. There are five options he can take for math, seven options for English, and four options for geography. How many different course schedules could Manu have assuming there are no time conflicts among the sections?

  SOLUTION: Manu has $5 \cdot 7 \cdot 4 = 140$ options. Here, 5 represents the five different options for math classes, 7 represents the seven different options for English classes, and 4 represents the four different options for geography classes.

- **EXAMPLE 4.1.2:** Rosemary needs to choose an outfit. She has four skirts, nine shirts, five pairs of shoes, and three purses to choose from. How many outfit options does she have?

  SOLUTION: Rosemary has $4 \cdot 9 \cdot 5 \cdot 3 = 540$ options.

- **EXAMPLE 4.1.3:** Suppose a person wants to order a pizza. The local pizza place offers four crust options, five sauce options, three cheese options, and seven topping choices. How many different one-topping pizzas could a person order?

  SOLUTION: A person can have $4 \cdot 5 \cdot 3 \cdot 7 = 420$ options.

- **YOU TRY IT 4.1.A:** How many ice cream cone options are there if there are two cone types, 15 ice cream flavors, and nine topping choices assuming only one ice cream flavor and one topping is chosen?

Problems can become more complicated if more options are to be considered. As the following problems illustrate, the Fundamental Counting Principle still applies.

- **EXAMPLE 4.1.4:** Suppose an identification number has the format # # # - # # - # # # #. The first digit cannot be zero or one. The rest of the digits have no restrictions. It is possible for digits to be repeated. How many identification numbers are possible?

  SOLUTION: There are 800,000,000 possible identification numbers. The first digit has eight possible options (2 through 9) and the remaining eight digits have 10 possible options (0 through 9). So, there are $8 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 = 8 \cdot 10^8 = 800{,}000{,}000$ possible identification numbers.

➢ **EXAMPLE 4.1.5:** George is taking a multiple – choice exam. He knows it is in his best interest to answer each question. Each question has only one answer. If there are four choices for each of the first 10 questions and three choices for the next five questions, in how many ways can the questions be answered?

SOLUTION: There are $4^{10} \cdot 3^5 = 1{,}048{,}576 \cdot 243 = 254{,}803{,}968$ ways to answer the questions on the exam. Since the first 10 questions have four choices each, the number of ways to answer the first ten questions is calculated as $4 \cdot 4 \cdot 4 \cdot 4 \cdot 4 \cdot 4 \cdot 4 \cdot 4 \cdot 4 \cdot 4 = 4^{10} = 1{,}048{,}576$. Since the following five questions have three choices each, the number of ways to answer these questions is calculated as $3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 = 3^5 = 243$. Multiplying $1{,}048{,}576$ and 243 together results in $254{,}803{,}968$ possible ways the exam questions can be answered.

❖ **YOU TRY IT 4.1.B:** Suppose US telephone numbers have the format # - # # # - # # # - # # # # where the first digit is always 1. The second digit is never 0 or 1. There are no restrictions on the remaining digits. How many telephone numbers are possible?

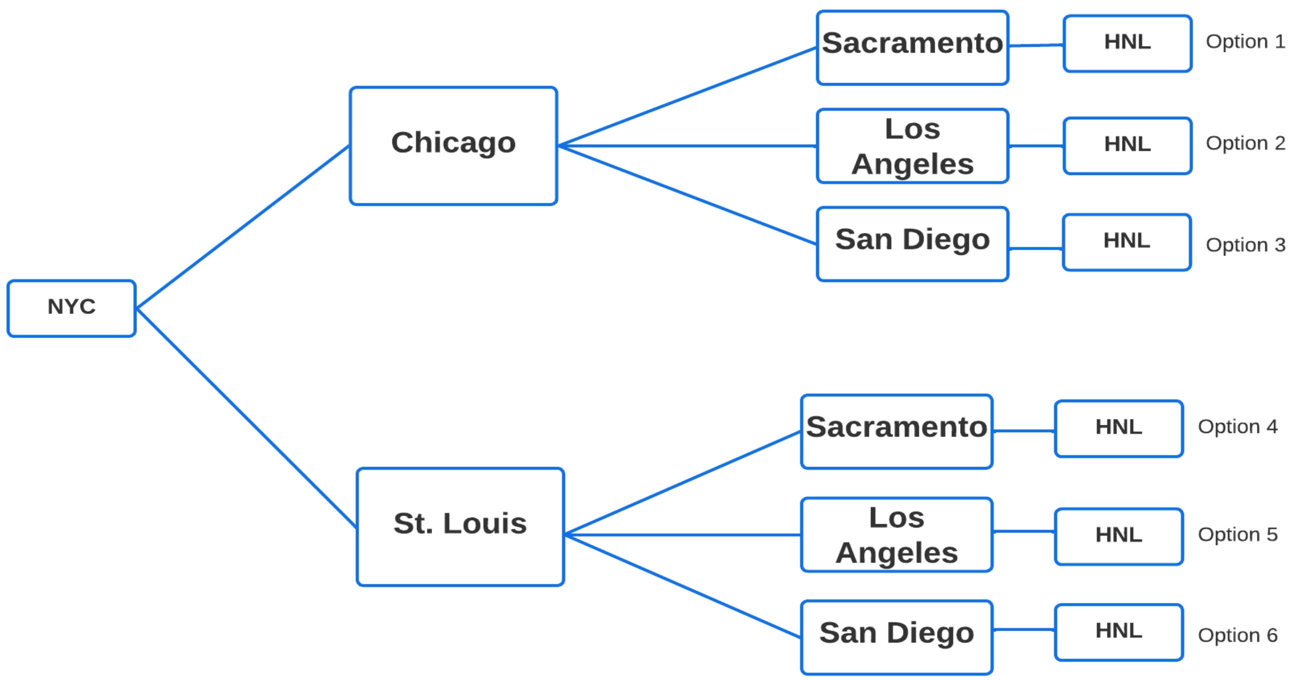
➢ **EXAMPLE 4.1.6:** In how many ways can 11 students line up if the first two students are the two shortest (any order) and the last two students are the two tallest (any order)?

SOLUTION: Since the first two students to line up are the two shortest there are $2 \cdot 1 = 2$ ways they can line up. Here, since there are only two students considered the shortest, there are two choices for the first spot and one choice for the second spot. Similarly, for the last two students, there are $2 \cdot 1 = 2$ ways they can line up. Again, there are two choices for the second to last spot and one choice for the last spot for the two students considered the tallest. At this point, the work should be: $2 \cdot 1 \cdot \_ \cdot \_ \cdot \_ \cdot \_ \cdot \_ \cdot \_ \cdot \_ \cdot 2 \cdot 1$. For the students lined up in the middle, there are $7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5{,}040$ ways they can be lined up. Notice, for the students in the middle, the two shortest and two tallest (four students) would not be included. There are $11 - 4 = 7$ students left to line up in the middle. Therefore, seven students are available to fill the third spot in the line up. This would leave six students who could fill the fourth spot. This continues until there are no more students to line up in between the shortest and tallest students. So, the number of ways 11 students can line up if the first two students are the two shortest and the last two students are the two tallest is $2 \cdot 1 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \cdot 2 \cdot 1 = 20{,}160$ ways.

Tree Diagrams

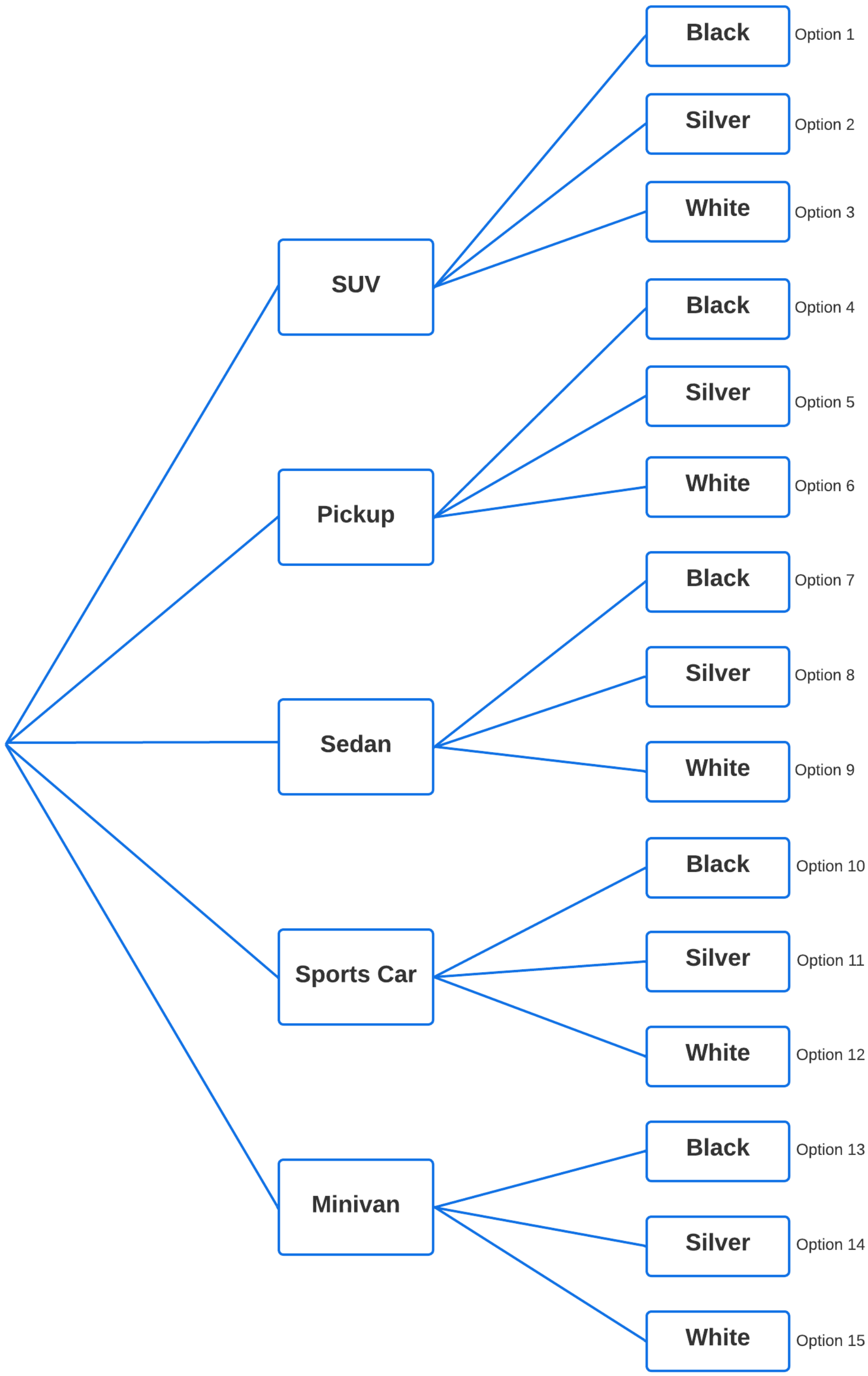
DEFINITION: A **tree diagram** is a visual representation that shows different ways in which events can occur.

Using the opening example where Alice has to choose from a number of flight options to fly from New York (NYC) to Honolulu (HNL), a tree diagram can be used to visualize the problem and solution. Each of the branches of the tree diagram lists one of Alice's options.



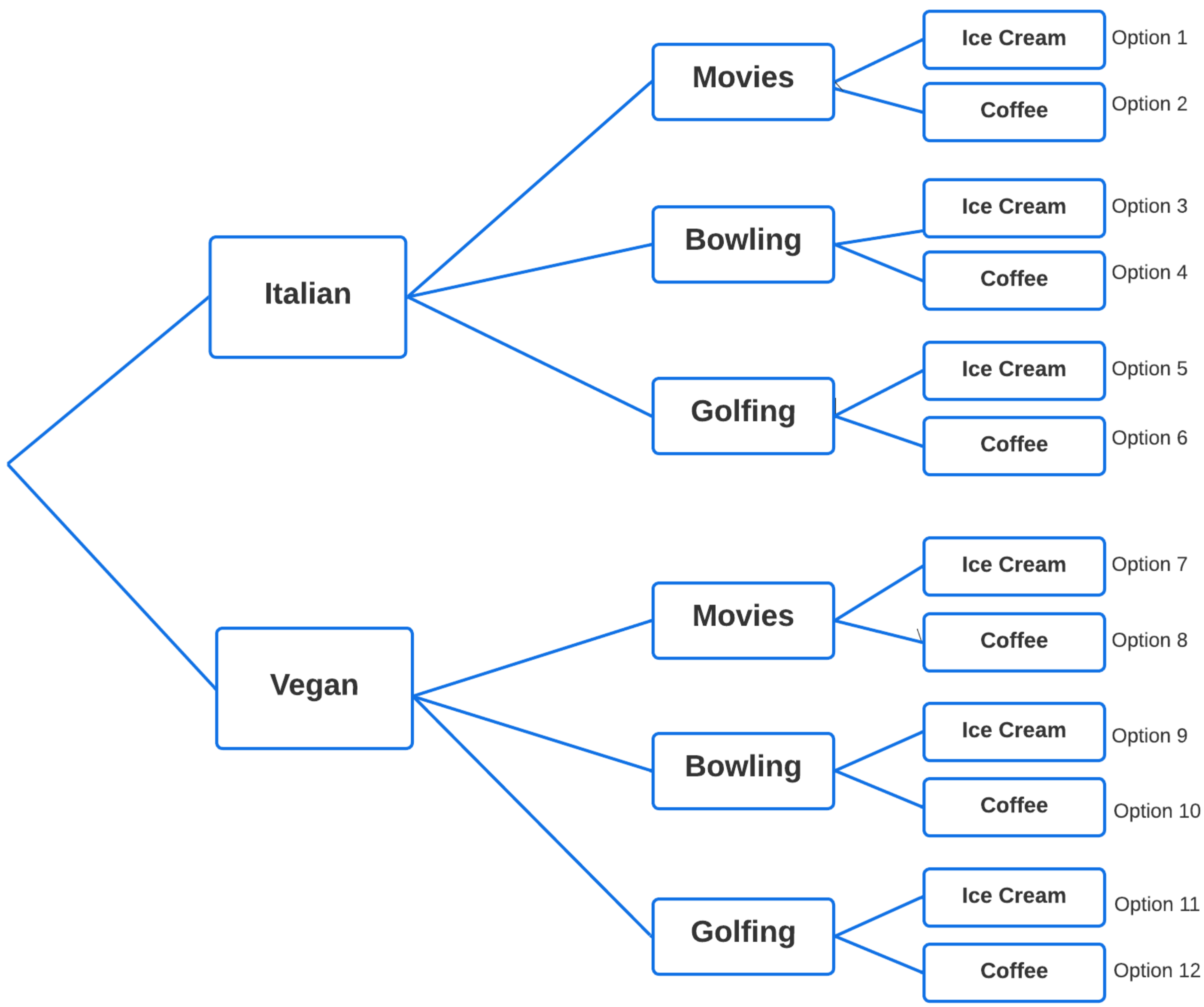
➢ **EXAMPLE 4.1.7:** Kamla would like to buy a new vehicle. Kamla will be choosing from the following vehicle options: SUV, pickup, sedan, sports car, or minivan. Kamla also has the following color choices: black, silver, or white. Create a tree diagram illustrating Kamla's choices for buying a vehicle and also find the number of options Kamla has.

SOLUTION: Kamla has $5 \cdot 3 = 15$ options.



➢ **EXAMPLE 4.1.8:** Lisa is deciding what to do Sunday night. She would like to go out to dinner at either an Italian or vegan restaurant. Thereafter, she would like to go to the movies, go bowling, or go golfing. She would like to end her evening by going to an ice cream shop or a coffee shop. Create a tree diagram illustrating Lisa's options and find how many different options she has for Sunday night.

SOLUTION: Lisa has 12 options for Sunday night.



❖ **YOU TRY IT 4.1.C:** Jorge has three different routes to get to work. After work, there are two different routes to get to the gym. From the gym to get home, there is only one route. Create a tree diagram illustrating the different options Jorge has to go to work then to the gym and then back home. Also, find the number of routes possible.

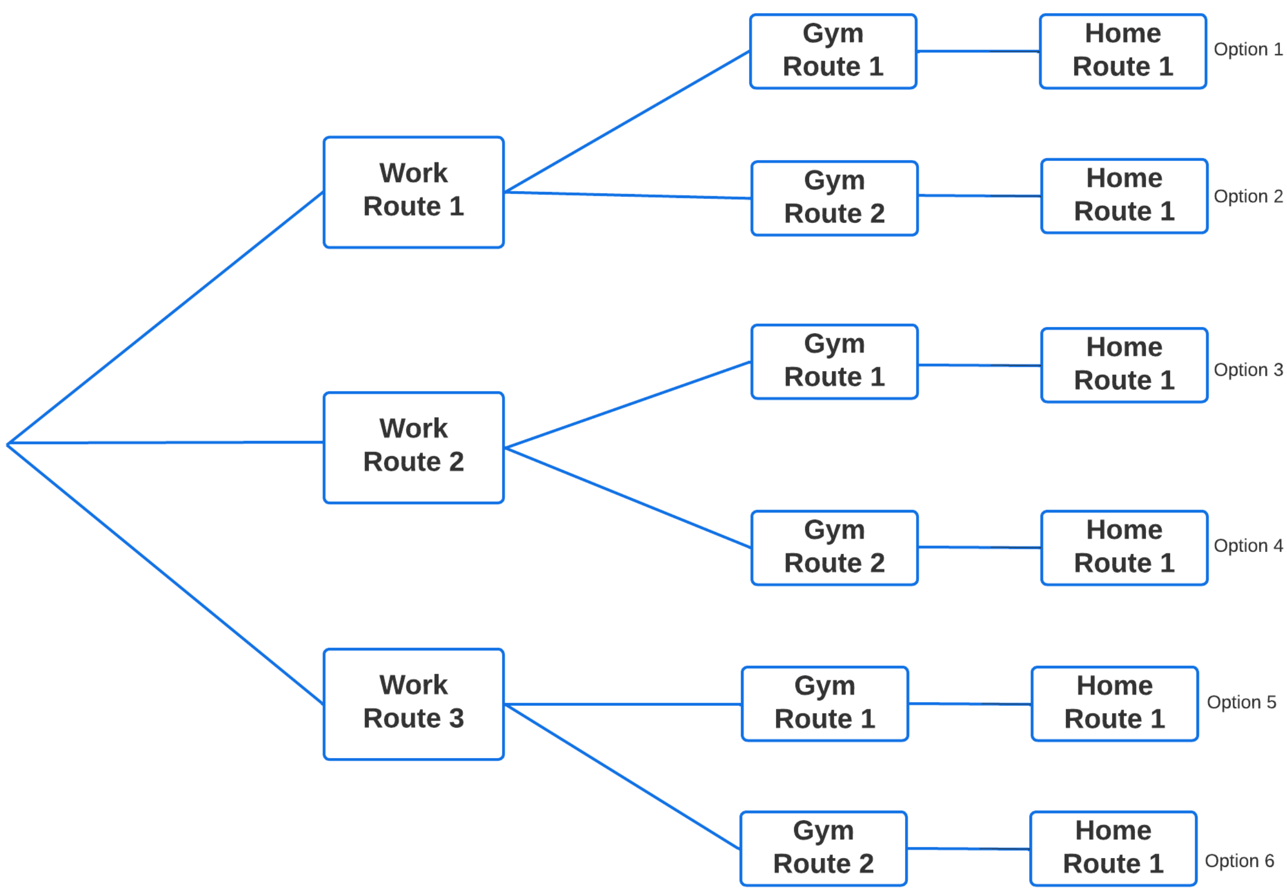
Quick Review

- The **Fundamental Counting Principle** can be used to calculate the number of ways in which a series of events can occur. This is calculated by multiplying the number of ways each individual event can occur.
- A **tree diagram** is a visual representation that shows different ways in which events can occur.

YOU TRY IT 4.1.A SOLUTION: There are $2 \cdot 15 \cdot 9 = 270$ ice cream cone options.

YOU TRY IT 4.1.B SOLUTION: There are $1 \cdot 8 \cdot 10^9 = 8{,}000{,}000{,}000$ telephone numbers possible.

YOU TRY IT 4.1.C SOLUTION: There are $3 \cdot 2 \cdot 1 = 6$ route options.



Section 4.1 | Exercises

In Exercises 1 – 22, use the Fundamental Counting Principle to solve.

1. How many outcomes are possible when flipping a quarter five times?

2. There are 15 people on a committee in which four members must be chosen to serve as president, vice president, treasurer, and secretary. How many ways can this be done assuming only one person can serve each position.

3. How many dinner options are available to Eddie if he can choose from six appetizers, four salads, 23 entrees, and five desserts? Assume Eddie can only choose one option from each category.

4. Elizabeth plans to start winter break with a weekend television binge. She is interested in five crime shows, three home improvement shows, six cooking shows, and two holiday shows. In how many ways can Elizabeth complete her television binge session if she plans on choosing one show from each category?

5. Ondro is completing assignments for final exam week. He needs to complete three history questions, 10 math problems, two English essays, and four science labs. If he chooses to pick one assignment from each course to complete today, in how many ways can Ondro complete his assignments?

6. A local pizzeria offers the following types of crust: stuffed, hand-tossed thin, crispy extra thin, and gluten free. It offers the following toppings: barbeque chicken, black olive, green pepper, ham, mushroom, onion, pepperoni, pineapple, sausage, spinach, and tomato. Pizzas can be ordered with the following amounts of sauce: none, regular, or extra. Pizzas can also be ordered with the following amounts of cheese: none, regular, or extra. Size options include small, medium, large, and extra large. How many one-topping pizzas are possible?

7. Antoinette has 11 shirt options, six pant options, three shoe options, and four purse options. How many different outfit options does Antoinette have?

8. In how many ways can eight speakers be lined up at an event from a group of 20 volunteers assuming that one volunteer has already been promised the first spot?

9. A preschool class has eight students. In how many ways can the class line up if the birthday student is first in line and the music helper is last in line?

10. Suppose a person has a collection of 10 science fiction books, eight historical fiction books, 11 mysteries, and six non-fiction books. In how many ways can they rank their top eight from this collection if they know that a science fiction book will be ranked first and a mystery will be ranked last?

11. Ten students go in a group to a basketball game at their college. Five of the students are wearing orange shirts and five of the students are wearing blue shirts. In how many ways can these students line up if they want to alternate shirt colors, starting with orange?

12. A four-digit lock is randomly assigned a combination. Assuming that no digit can be repeated in the combination, how many possible combinations are even and greater than 9000?

13. On a quiz, if there are five choices for each of the first 10 questions and four choices for the next six questions, in how many ways can the questions be answered?

14. In how many ways can 10 campers and two camp counselors be lined up if the line must begin and end with a camp counselor?

15. In the past, Illinois license plates had three letters followed by three numbers. How many license plates options are there? How many options are possible if letters and numbers cannot be repeated?

16. An elementary school has a school assembly for a local physicist. Each grade (kindergarten, first grade, second grade, and third grade) sent four students to sit in the front row of the auditorium. In how many ways can these 16 students sit in the front row if they must sit in a pattern by grade: kindergarten, first grade, second grade, and third grade.

17. A group of children and adults are attending the local school play. Find a general formula to show the number of ways the children and adults can sit in the audience if children and adults must alternate while seated. Use variable c to represent the number of children and variable a to represent the number of adults. Assume there are the same number of children and adults.

18. Refer to problem 16 above where students must sit in the pattern kindergarten, first grade, second grade, and third grade. Find a general formula to show the number of ways students can sit if there are K kindergarten students, F first grade students, S second grade students, and T third grade students.

19. Refer to problem 17 above where children and adults alternate sitting in an audience. Assume the number of children is the same as the number of adults. Find the minimum number of children and adults needed to create at least 1 billion ways to alternate sitting child and adult.

20. The local fitness club is hosting a 5K running event. How many runners are needed for there to be at least 5,000 ways for the competitors to finish? How many runners are needed for there to be at least 1 billion ways for the competitors to finish? Assume there are no ties.

21. Suppose an exam consists of a combination of four-option multiple choice questions and true-false questions. If the exam consists of the same number of multiple choice and true-false questions, how many of each type of question must the exam contain for there to be at least a billion ways in which the questions can be answered?

22. Phone numbers in the U.S. consist of a three-digit area code followed by a seven-digit number. According to the North American Numbering Plan Administrator (NANPA), phone numbers in Glen Ellyn have an area code of either 331 or 630. If the seven-digit number cannot begin with 0 or with 9-1-1, how many phone numbers are possible in Glen Ellyn? Note that digits can be repeated. (Hint: Calculate the number of possible phone numbers in which the seven-digit number begins with 0. Then, calculate the number of possible phone numbers in which the seven-digit number begins with 9-1-1. Lastly, subtract these values from total number of possible seven-digit phone numbers.)

In Exercises 23 – 28, create a tree diagram and then use the Fundamental Counting Principle to find the total number of options.

23. A coffee shop sells hot chocolate with three flavor shots (peppermint, chili pepper, or caramel) and two toppings (whipped cream or marshmallows). If only one flavor shot and one topping are chosen for the hot chocolate, how many different options are there?

24. Russ is going on vacation to Hawaii. He can go to one of the following islands: Oahu, Kauai, Maui, Big Island, or Molokai. As soon as Russ is on an island the first thing he will do is go to the beach, to a luau, or on a hike. How many different options does Russ have given these three choices for the first thing he will do once he reaches a Hawaiian island?

25. Miriam is playing a game with her brother. She asks him to choose one of two boxes. Inside each box, there are four envelopes of cash. How many options does Miriam's brother have to choose his prize money?

26. Nola plans to enjoy music and have a snack for the evening. She can't decide between listening to classical music or jazz music. Next, she plans to enjoy a dessert but hasn't decided between crème brulee, apple pie, or chocolate lava cake. Finally, she plans to drink coffee, tea, or milk. How many options does Nola have to enjoy her evening?

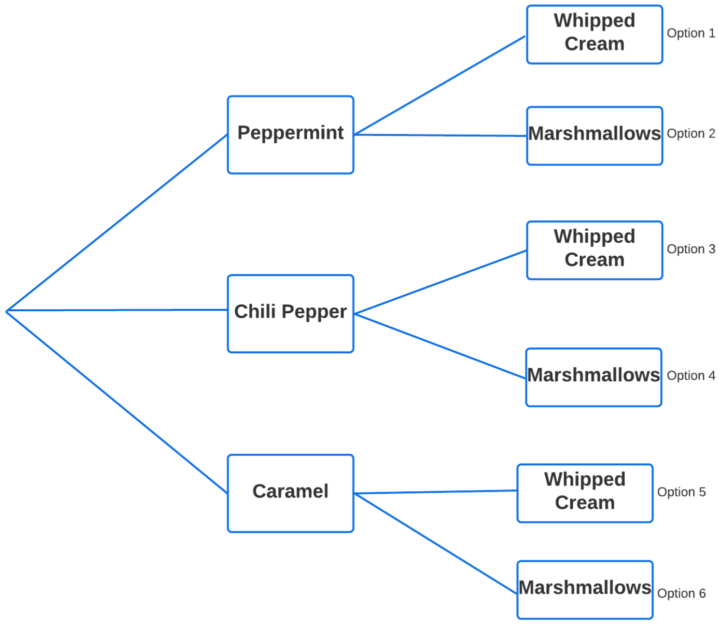
27. When planning her course schedule, Addie knows she will take General Education Math. She also plans to choose between Composition II and Speech. She also plans to select Biology I, Chemistry I, or Introduction to Astronomy. Lastly, she will select Introduction to Humanities or World History. How many different possible schedules can Addie build given these options?

28. Nate is deciding on which extracurricular activities to participate in. He can opt to play the saxophone or trumpet. He would also like to join the math club, student newspaper, debate team, or student council. Finally, he can run track or play volleyball. How many options does he have given these choices?

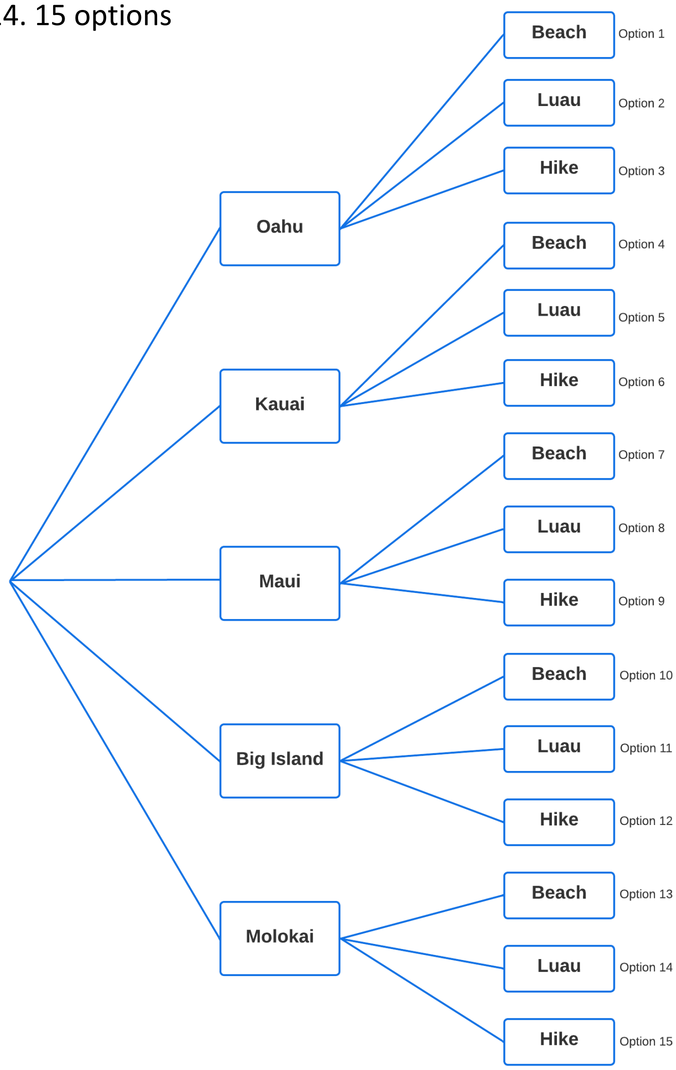
Section 4.1 Exercise Solutions

1. 32
2. 32,760
3. 2,760
4. 180
5. 240
6. 1,584
7. 792
8. 253,955,520
9. 720
10. 87,719,385,600
11. 14,400
12. 280
13. 40,000,000,000
14. 7,257,600
15. 17,576,000; 11,232,000
16. 331,776
17. $(c)(a)(c-1)(a-1)(c-2)(a-2) \ldots (3)(3)(2)(2)(1)(1)$
18. $(K)(F)(S)(T)(K-1)(F-1)(S-1)(T-1)(K-2) \ldots (2)(2)(2)(2)(1)(1)(1)(1)$
19. 8 children and 8 adults
20. 7 competitors; 13 competitors
21. 10 of each type of question
22. $2 \cdot 10^7 - 2 \cdot 1 \cdot 10^6 - 2 \cdot 1^3 \cdot 10^4 =$ 17,980,000

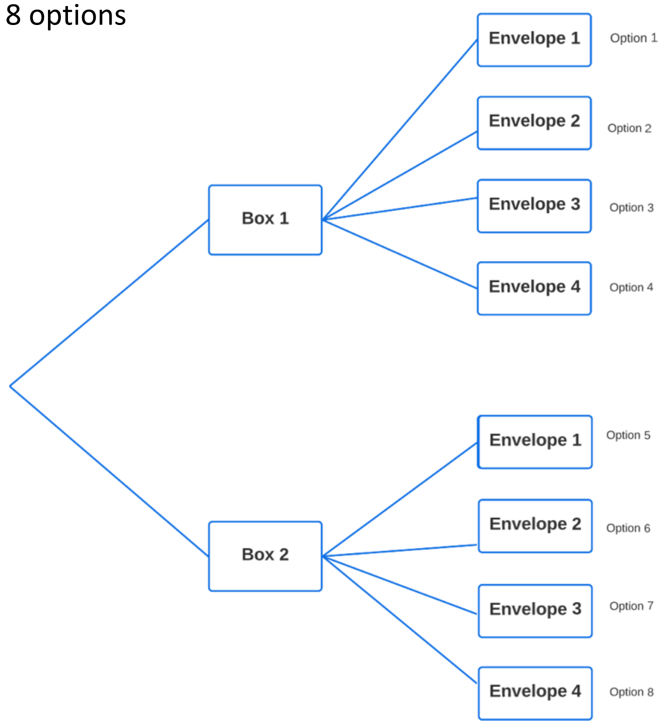
23. 6 options



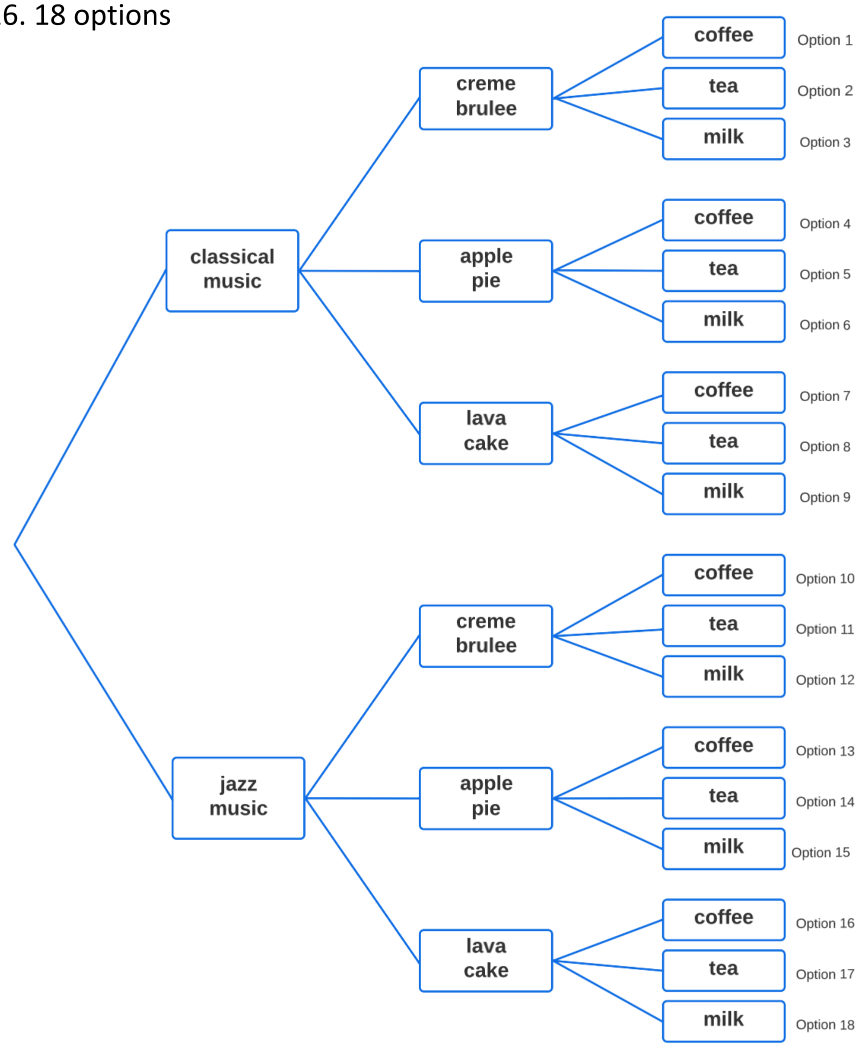
24. 15 options



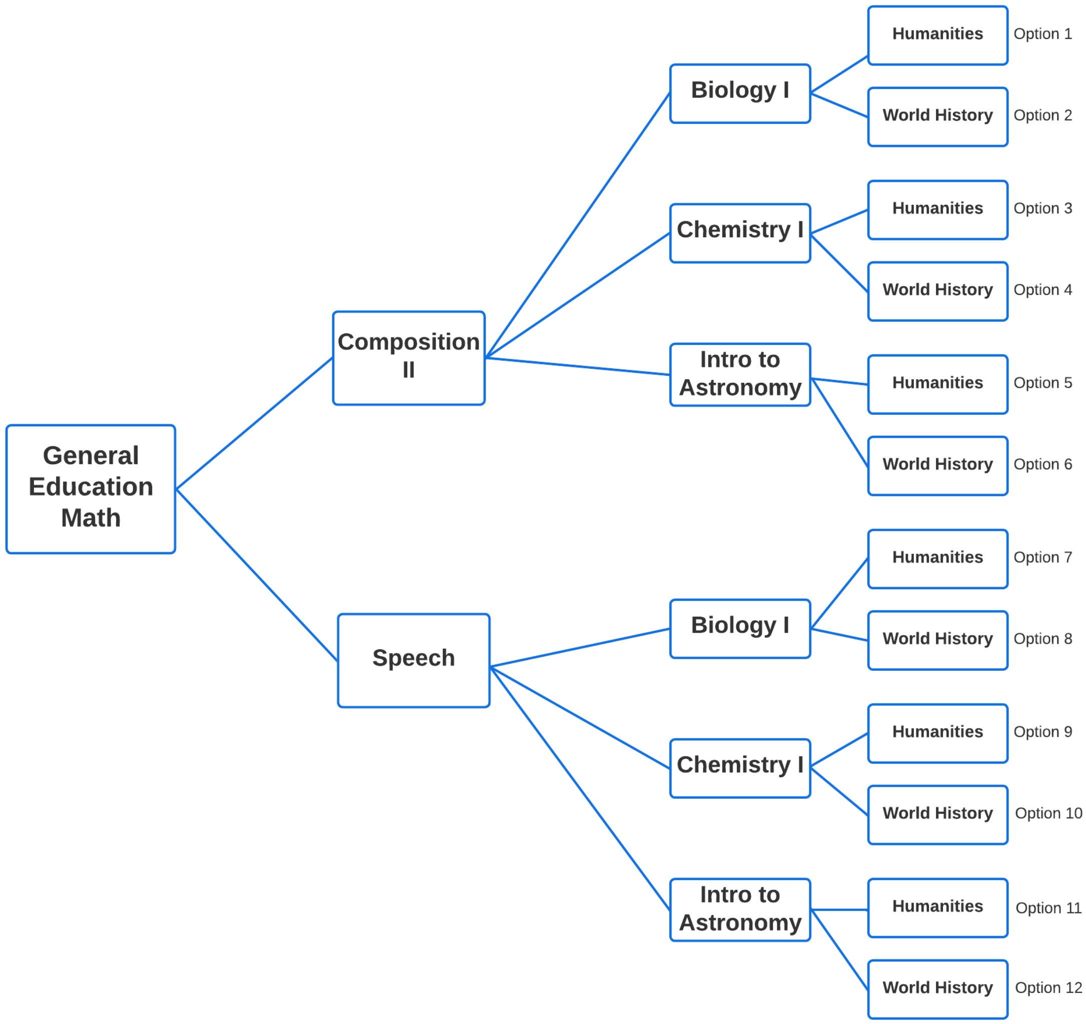
25. 8 options



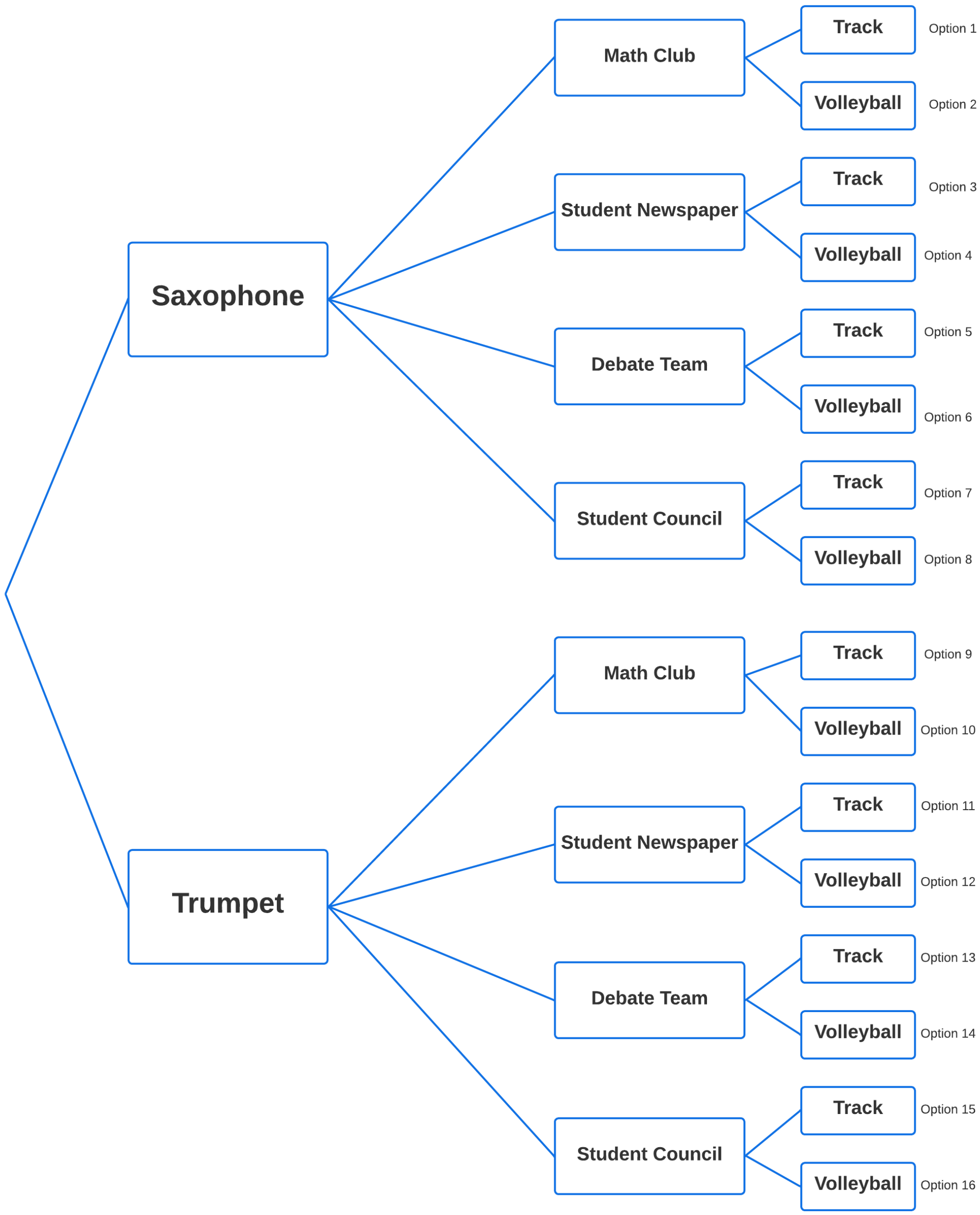
26. 18 options



27. 12 options



28. 16 options



Section 4.2 | Permutations & Combinations

Objectives

- Calculate factorials
- Calculate permutations
- Calculate combinations
- Solve problems involving permutations and combinations

There are many times when it is helpful or even necessary to count the number of ways in which something can occur. For example, a softball coach might want to determine the number of batting lineups that can be created. Organizers of a race may want to calculate the number of ways contestants can cross the finish line. A person might want to calculate their chance of winning the lottery. Or the situation might be as simple as a person trying to decide how many different pizzas can be ordered given a list of toppings from which to choose. Permutations and combinations are methods that can be used to count in situations like these.

Factorial

Suppose a person is having a dinner party and is expecting six guests. In how many ways can the guests arrive? There are six possibilities for the first person to arrive. Once the first person arrives, there are five remaining possibilities for the second person to arrive. Once the second person arrives, there are four remaining possibilities for the third person to arrive. This continues until there is only one remaining possibility for the last person to arrive. By the Fundamental Counting Principle, these values are multiplied to calculate the total number of ways in which the guests can arrive.

$$6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720$$

Thus, there are 720 ways in which six guests can arrive at a dinner party, assuming no guests arrive simultaneously.

The product $6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$ can be written in a more concise way.

DEFINITION: If n is a positive integer, then $\boldsymbol{n}$ **factorial** is the product of all integers from n down to 1. Symbolically, $n!$ is used to represent this product.

Thus, in the dinner party example given, $6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$.

Factorial

If n is a positive integer, then $\boldsymbol{n}$ **factorial** is given by:

$$n! = n \cdot (n-1) \cdot (n-2) \cdots 3 \cdot 2 \cdot 1$$

As a special case, $0! = 1$.

- **EXAMPLE 4.2.1:** In how many ways can nine speakers be lined up at an event assuming that one speaker has already been promised the first slot?

  SOLUTION: There is only one possibility for the first slot since it has already been promised to a specific person. Out of the nine speakers, this leaves eight possibilities for filling the second slot. Once the second slot is filled, then seven possibilities remain for filling the third slot. This process continues until possibilities for all nine slots have been considered. By the Fundamental Counting Principle, the product of the possibilities for each of the nine slots gives the total number of possible arrangements.

  $$1 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 1 \cdot 8! = 40{,}320$$

  There are 40,320 ways the speakers can be arranged.

- **YOU TRY IT 4.2.A:** In how many ways can eight students be lined up to perform at a music recital, assuming that one student has been promised the first spot and another student has been promised the last spot?

The counting situations presented thus far have common characteristics. They all involve counting the number of ways in which people can be arranged, where no person can occupy more than one spot in the arrangement and where order matters. Such an ordered arrangement is called a permutation.

Permutations

Suppose a softball coach wants to know the number of nine-person batting lineups that can be created from a team of 15 players. This type of problem can be solved using the Fundamental Counting Principle by multiplying the number of options for each spot of the lineup.

$$15 \cdot 14 \cdot 13 \cdot 12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 = 1{,}816{,}214{,}400$$

Thus, there are 1,816,214,400 possible nine-player batting lineups from a group of 15 players. Note that the order of arrangement matters in a batting lineup and no player can be repeated in the lineup. This is an example of a permutation.

DEFINITION: A **permutation** is a way in which a set number of items can be arranged. In a permutation, no item is used more than once and the order of arrangement matters.

A formula for permutations will be developed using the batting lineup situation. As previously determined, the number of nine-person batting lineups from a group of 15 players is calculated using the Fundamental Counting Principle. This calculation can be written in the following equivalent ways:

$$15 \cdot 14 \cdot 13 \cdot 12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 = \frac{15 \cdot 14 \cdot 13 \cdot 12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = \frac{15!}{6!} = \frac{15!}{(15-9)!}$$

There are 15 players, and nine are being selected for the lineup. These values appear in the last step of the calculation. The numerator of 15! is the factorial of the total number of players. The denominator of $(15 - 9)!$ is the factorial of the difference between the total number of players and the number of players selected for the lineup. In other words, the denominator is the factorial of the number of players that will not be selected for the lineup. The number of permutations of nine players from a group of 15 is written symbolically as $P(15, 9)$ or ${}_{15}P_9$.

In general, the number of permutations of r items from a collection of n items is symbolized as $P(n, r)$ or nPr. The batting lineup example can be generalized to write the formula for calculating the number of permutations of r items from a collection of n items.

Permutation Formula

The number of **permutations** of r items from a collection of n items can be calculated as follows:

$$P(n, r) = \frac{n!}{(n-r)!}$$

➢ **EXAMPLE 4.2.2:** Use the permutation formula to calculate the following:

f. $P(10,4)$
g. $P(7,7)$
h. $P(30,10)$

SOLUTION:

e. $P(10,4) = \frac{10!}{(10-4)!} = \frac{10!}{6!} = \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 10 \cdot 9 \cdot 8 \cdot 7 = 5{,}040$

f. $P(7,7) = \frac{7!}{(7-7)!} = \frac{7!}{0!} = \frac{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{1} = 5{,}040$

g. $P(30,10) = \frac{30!}{(30-10)!} = \frac{30!}{20!} = \frac{30 \cdot 29 \cdot 28 \cdot 27 \cdot 26 \cdot 25 \cdot 24 \cdot 23 \cdot 22 \cdot 21 \cdot 20 \cdot 19 \cdot 18 \cdots 1}{20 \cdot 19 \cdot 18 \cdots 1}$

$$= 30 \cdot 29 \cdot 28 \cdot 27 \cdot 26 \cdot 25 \cdot 24 \cdot 23 \cdot 22 \cdot 21 = 109{,}027{,}350{,}432{,}000$$

Since this solution is so large, a calculator may display the answer using scientific notation. Additionally, the exact value may not be displayed as calculators often round to a certain place value. For example, a calculator may display the answer in scientific notation as 1.090273504 E 14. This is the same as $1.090273504 \cdot 10^{14}$ or 109,027,350,400,000.

NOTE: Factorial notation can be used to simplify calculations. In part c of Example 4.2.2, the calculation could be written as follows:

$$P(30,10) = \frac{30!}{(30\ -\ 10)!} = \frac{30!}{20!} = \frac{30 \cdot 29 \cdot 28 \cdot 27 \cdot 26 \cdot 25 \cdot 24 \cdot 23 \cdot 22 \cdot 21 \cdot 20!}{20!}$$

At this point, 20! (the product of all the integers from 20 down to 1) in the numerator and 20! in the denominator simplify to 1.

CAUTION! In the calculation above, $\frac{30!}{20!}$ does NOT simplify to $\frac{3!}{2!}$ because the products in each factorial must be written before simplifying. Factorials must match exactly to be simplified, such as $\frac{20!}{20!} = 1$.

❖ **YOU TRY IT 4.2.B:** Use the permutation formula to calculate the following:

e. $P(16, 9)$
f. $P(13, 13)$

➢ **EXAMPLE 4.2.3:** The Indianapolis 500 is a race that typically includes 33 cars. In how many ways can the first five finishers come in, assuming there are no ties?

SOLUTION: This is an example of a permutation because it involves an ordered arrangement in which no car can finish in more than one place and the order in which cars finish the race matters. In other words, there is a difference between first place, second place, and so on. The formula for $P(33, 5)$ can be used to calculate the number of ways the first five finishers can come in.

$$P(33,5) = \frac{33!}{(33-5)!} = \frac{33!}{28!} = \frac{33 \cdot 32 \cdot 31 \cdot 30 \cdot 29 \cdot 28!}{28!} = 33 \cdot 32 \cdot 31 \cdot 30 \cdot 29 = 28{,}480{,}320$$

Thus, there are 28,480,320 different ways that the first five finishers can cross the finish line from a group of 33 cars, assuming there are no ties.

This problem can also be solved using the Fundamental Counting Principal by multiplying the number of options for each of the finishers. This gives the same calculation performed when using the formula for permutations:

$$33 \cdot 32 \cdot 31 \cdot 30 \cdot 29 = 28{,}480{,}320$$

➢ **EXAMPLE 4.2.4:** Suppose an art collector would like to hang artwork in a gallery. How many ways are there to arrange eight pieces of art from a collection of 20 pieces, assuming the order of arrangement matters?

SOLUTION: This is an example of a permutation because it involves an ordered arrangement in which no piece of art can occupy more than one place at the same time. The formula for $P(20, 8)$ can be used to calculate the number of ways the artwork can be arranged.

$$P(20,8) = \frac{20!}{(20-8)!} = \frac{20!}{12!} = 5{,}079{,}110{,}400$$

Thus, there are over 5 billion ways to arrange eight art pieces from a collection of 20.

❖ **YOU TRY IT 4.2.C:** Suppose a person would like to create a playlist to listen to while lifting weights. Assuming the order of arrangement of songs makes a difference to this person, how many seven-song playlists can be created from a group of 25 songs?

Combinations

Suppose a person plays a game in which they must guess the arrangement of three numbers taken from the group of numbers 1, 2, 3, and 4. How many different arrangements are possible? This is an example of a permutation of four items taken three at a time. It can be solved using the formula for permutations.

$$P(4,3) = \frac{4!}{(4-3)!} = \frac{4!}{1!} = 4 \cdot 3 \cdot 2 \cdot 1 = 24$$

These 24 permutations can be represented visually as follows:

$1, 2, 3$	$1, 2, 4$	$1, 3, 4$	$2, 3, 4$
$1, 3, 2$	$1, 4, 2$	$1, 4, 3$	$2, 4, 3$
$2, 1, 3$	$2, 1, 4$	$3, 1, 4$	$3, 2, 4$
$2, 3, 1$	$2, 4, 1$	$3, 4, 1$	$3, 4, 2$
$3, 1, 2$	$4, 1, 2$	$4, 1, 3$	$4, 2, 3$
$3, 2, 1$	$4, 2, 1$	$4, 3, 1$	$4, 3, 2$

Now suppose that to win the game, a person must only identify the correct three numbers without regard to order. In other words, the grouping 1, 2, 3 is the same as 1, 3, 2, etc. It would be easier to win this game because a person would only need to select the correct three numbers without worrying about how they are ordered. There are only four possibilities, represented by the four columns in the list of numbers above. This is an example of a combination.

DEFINITION: A **combination** is a way in which a set number of items can be chosen from a group. In a combination, no item is used more than once in the grouping and the order of the items is not important.

A formula for combinations will be developed using the example of selecting three numbers from the group of numbers 1, 2, 3, and 4. Begin with the 24 permutations of these four numbers taken three at a time. Since each column involves the same group of items, there are four possible combinations.

<u>Combination #1</u>	<u>Combination #2</u>	<u>Combination #3</u>	<u>Combination #4</u>
$1, 2, 3$	$1, 2, 4$	$1, 3, 4$	$2, 3, 4$
$1, 3, 2$	$1, 4, 2$	$1, 4, 3$	$2, 4, 3$
$2, 1, 3$	$2, 1, 4$	$3, 1, 4$	$3, 2, 4$
$2, 3, 1$	$2, 4, 1$	$3, 4, 1$	$3, 4, 2$
$3, 1, 2$	$4, 1, 2$	$4, 1, 3$	$4, 2, 3$
$3, 2, 1$	$4, 2, 1$	$4, 3, 1$	$4, 3, 2$

Each column contains six possibilities that represent one single combination. These six possibilities consist of the possible arrangements of three numbers, which can be calculated as $3! = 3 \cdot 2 \cdot 1 = 6$.

Thus, the calculation to find the number of combinations, 4, based on the number of permutations, 24, can be completed as follows:

$$4\ combinations = \frac{24\ permutations}{6\ arrangements\ per\ column} = \frac{24}{3!} = \frac{P(4,3)}{3!} = \frac{\frac{4!}{(4-3)!}}{3!} = \frac{4!}{(4-3)! \cdot 3!}$$

When order is not important, the number of ways to group three items taken from a collection of four items is a combination. The number of combinations of three numbers from a group of four is written symbolically as $C(4,3)$ or ${}_4C_3$.

In general, the number of combinations of r items taken from a collection of n items is symbolized as $C(n,r)$ or nCr. The example of selecting groups of three numbers from a list of four numbers can be generalized to write the formula for calculating the number of combinations of r items from a collection of n items.

Combination Formula

The number of **combinations** of r items taken from a collection of n items can be calculated as follows:

$$C(n,r) = \frac{n!}{(n-r)! \cdot r!}$$

➢ **EXAMPLE 4.2.5:** Use the combination formula to calculate the following.

a. $C(10,4)$
b. $C(7,7)$
c. $C(30,10)$

SOLUTION:

a. $C(10,4) = \frac{10!}{(10-4)!\cdot 4!} = \frac{10!}{6!\cdot 4!} = \frac{10\cdot 9\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}{(6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1)\cdot(4\cdot 3\cdot 2\cdot 1)} = \frac{10\cdot 9\cdot 8\cdot 7}{4\cdot 3\cdot 2\cdot 1} = \frac{5{,}040}{24} = 210$

CAUTION! In the calculation above, $6!\cdot 4!$ does NOT simplify to $24!$ because $6!\cdot 4! = 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1\cdot 4\cdot 3\cdot 2\cdot 1$.

b. $C(7,7) = \frac{7!}{(7-7)!\cdot 7!} = \frac{7!}{0!\cdot 7!} = \frac{7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}{(1)\cdot(7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1)} = 1$

c. $C(30,10) = \frac{30!}{(30-10)!\cdot 10!} = \frac{30!}{20!\cdot 10!} = \frac{30\cdot 29\cdot 28\cdot 27\cdot 26\cdot 25\cdot 24\cdot 23\cdot 22\cdot 21\cdot 20!}{20!\cdot 10!}$

$$= \frac{30\cdot 29\cdot 28\cdot 27\cdot 26\cdot 25\cdot 24\cdot 23\cdot 22\cdot 21}{10!} = \frac{109{,}027{,}350{,}432{,}000}{3{,}628{,}800} = 30{,}045{,}015$$

Here, factorial notation is used to simplify calculations. The 20! (product of all the integers from 20 down to 1) in the numerator and the 20! in the denominator simplify to 1.

NOTE: Typically, the number of combinations for a given situation will be less than the number of permutations, because the order of arrangement does not matter with combinations. This is demonstrated in Examples 4.2.2 and 4.2.5.

❖ **YOU TRY IT 4.2.D:** Use the combination formula to calculate the following:

a. $C(36,10)$
b. $C(13,1)$
c. $C(8,0)$

➢ **EXAMPLE 4.2.6:** To play the Lucky Day Lotto® in Illinois in 2023, a person must select five numbers from the numbers 1 – 45. Numbers cannot be repeated, and the order of selection does not matter. In how many ways can numbers be selected for this lotto?

SOLUTION: This is an example of a combination because numbers cannot be repeated and order does not matter. The formula for $C(45, 5)$ can be used to calculate the number of ways to select five numbers from the numbers 1 – 45.

$$C(45,5) = \frac{45!}{(45-5)! \cdot 5!} = \frac{45!}{40! \cdot 5!} = \frac{45 \cdot 44 \cdot 43 \cdot 42 \cdot 41 \cdot 40!}{40! \cdot 5!}$$

$$= \frac{45 \cdot 44 \cdot 43 \cdot 42 \cdot 41}{5!} = \frac{146{,}611{,}080}{120} = 1{,}221{,}759$$

Thus, there are 1,221,759 different ways that numbers can be selected for this lotto.

➢ **EXAMPLE 4.2.7:** A local pizzeria offers the following toppings: sausage, pepperoni, ham, bacon, ground beef, chicken, mushroom, green pepper, black olives, green olives, onion, pineapple, green chilis, anchovy, tomato, roasted red pepper, and spinach. How many different three-topping pizzas are possible?

SOLUTION: This is an example of a combination because we are assuming that toppings are not repeated and the order in which the toppings are selected does not matter. The formula for $C(17, 3)$ can be used to calculate number of possible three-topping pizzas from a list of 17 toppings.

$$C(17,3) = \frac{17!}{(17-3)! \cdot 3!} = \frac{17!}{14! \cdot 3!} = \frac{17 \cdot 16 \cdot 15 \cdot 14!}{14! \cdot 3!} = \frac{17 \cdot 16 \cdot 15}{3!} = \frac{4{,}080}{6} = 680$$

Thus, there are 680 possible three-topping pizzas.

❖ **YOU TRY IT 4.2.E:** Suppose 30 students try out for a high school swim team, but there are only 15 available spots on the team. How many groups of 15 students are possible

Problem Solving with Permutations and Combinations

It is important to know the difference between a permutation and a combination when problem solving. Most times it will not be specified if a permutation or combination applies in a given scenario. Once it is determined if a problem involves a permutation or combination, then the appropriate formula can be applied to solve the problem.

➢ **EXAMPLE 4.2.8:** Determine if each of the following involves a permutation or a combination. Then, use the appropriate formula to solve.

a. There are 25 students interested in joining student council, but there are only six openings. How many groups of students are possible?

b. A student club needs to elect six officers – a president, a vice president, a treasurer, a secretary, a communications chair, and a membership chair. If the club has 25 members from which to choose for these positions, in how many ways can the positions be filled?

SOLUTION:

a. This is a combination because there is no stated difference in the six openings. Thus, the order in which students are selected does not matter.

$$C(25,6) = \frac{25!}{(25-6)! \cdot 6!} = \frac{25!}{19! \cdot 6!} = 177{,}100$$

There are 177,100 ways to select six students for the student council.

b. This is a permutation because each of the six positions is different. Thus, the order in which students are assigned to the positions matters.

$$P(25,6) = \frac{25!}{(25-6)!} = \frac{25!}{19!} = 127{,}512{,}000$$

There are 127,512,000 ways to elect six students for the student club positions.

❖ **YOU TRY IT 4.2.F:** Determine if each of the following involves a permutation or a combination. Then, use the appropriate formula to solve.

a. How many ways are there to arrange eight bands in a lineup for a music festival from a group of 16 bands?

b. A person has 15 band t-shirts and would like pack five of them for vacation. How many different collections of five t-shirts can they bring along?

Permutations and Combinations with the Fundamental Counting Principle

Suppose a committee consisting of five students and five teachers must be formed from a group of 30 students and eight teachers. How many committees can be formed?

First, the number of combinations of five students from a group of 30 should be calculated.

$$C(30,5) = \frac{30!}{(30-5)!\cdot 5!} = \frac{30!}{25!\cdot 5!} = 142{,}506$$

Then, the number of combinations of five teachers from a group of eight should be calculated.

$$C(8,5) = \frac{8!}{(8-5)!\cdot 5!} = \frac{8!}{3!\cdot 5!} = 56$$

There are 142,506 ways to select students and 56 ways to select teachers for this committee. The Fundamental Counting Principle can be used to find the total number of possible committees as follows:

$$C(30,5)\cdot C(8,5) = (142{,}506)(56) = 7{,}980{,}336$$

Thus, there are 7,980,336 student-teacher committees that can be formed.

- **EXAMPLE 4.2.9:** Ten people are running a race. If two people tie for first place (cross the finish line at the same time), in how many ways can the 10 runners finish?

  SOLUTION: First, find the number of possible groups of finishers to share the first-place spot. This is a combination because these individuals finish at the same time and order does not need to be considered. If there are 10 runners, the possibilities for two to finish first can be found using the formula for $C(10,2)$.

  $$C(10,2) = \frac{10!}{(10-2)!\cdot 2!} = \frac{10!}{8!\cdot 2!} = 45$$

  Next, find the number of ways the remaining eight runners can finish. This is a permutation because the order in which these runners cross the finish line matters. The number of ways in which the remaining runners can finish can be found using the formula for $P(8,8)$.

  $$P(8,8) = \frac{8!}{(8-8)!} = \frac{8!}{0!} = 40{,}320$$

Finally, the Fundamental Counting Principle can be used to find the number of ways these 10 runners can finish the race.

$$C(10,2) \cdot P(8,8) = (45)(40{,}320) = 1{,}814{,}400$$

Thus, there are 1,814,400 ways in which the runners can finish the race given that two finishers tie for first place.

❖ **YOU TRY IT 4.2.G:** A committee consisting of five teachers and 15 parents must be formed from a group of 10 teachers and 25 parents. In how many ways can this committee be formed?

Quick Review

- If n is a positive integer, then $\boldsymbol{n}$ **factorial**, written $n!$, is the product of all integers from n down to 1. As a special case, $0! = 1$.

$$n! = n \cdot (n-1) \cdot (n-2) \cdots 3 \cdot 2 \cdot 1$$

- A **permutation** is a way in which a set number of items can be arranged. In a permutation, no item is used more than once in the arrangement and the order of the items is important.

  The number of permutations of r items taken from a collection of n items is found by:

$$P(n,r) = \frac{n!}{(n-r)!}$$

- A **combination** is a way in which a set number of items can be chosen from a group. In a combination, no item is used more than once in the grouping and the order of the items is not important.

  The number of combinations of r items taken from a collection of n items is found by:

$$C(n,r) = \frac{n!}{(n-r)! \cdot r!}$$

YOU TRY IT 4.2.A SOLUTION: 720

YOU TRY IT 4.2.B SOLUTION:

- **g.** 4,151,347,200
- **h.** 6,227,020,800

YOU TRY IT 4.2.C SOLUTION: 2,422,728,000

YOU TRY IT 4.2.D SOLUTION:

- **d.** 254,186,856
- **e.** 13
- **f.** 1

YOU TRY IT 4.2.E SOLUTION: 155,117,520

YOU TRY IT 4.2.F SOLUTION:

- **a.** Permutation; 518,918,400
- **b.** Combination; 3,003

YOU TRY IT 4.2.G SOLUTION: (252)(3,268,760) = 823,727,520

Section 4.2 | Exercises

In Exercises 1 – 4, calculate the factorial.

1. $5!$
2. $7!$
3. $0!$
4. $10!$

In Exercises 5 – 12, use the Fundamental Counting Principle to solve.

5. In how many ways can a group of 13 students be lined up to have their pictures taken?
6. In how many ways can a group of seven boys and six girls line up, if they must alternate boy, girl, boy, etc.?
7. In a communications class, 12 students are set to give individual speeches. In how many ways can the speeches be delivered, given that one student has requested to go first and another student has requested to go last?
8. At a banquet, there are eight main entrees, five side dishes, and three deserts. Find the number of ways the food can be lined on the table if the host wants to place all main entrees first, followed by the side dishes, and then ending with the deserts.
9. Suppose workers at a pet shelter would like to post pictures of eight cats and eight dogs that are up for adoption. In how many ways can the pictures be lined up on the website if they are to alternate between dog and cat, beginning with a dog's picture?
10. Suppose a group of 10 juniors and 12 seniors are running a race. In how many ways can the first five finishers cross the finish line if they alternate between senior and junior, with a senior finishing first?
11. A baseball coach needs to fill a nine-person batting line up from a group of 15 players. Of the three players who have a batting average above 0.300, the coach would like to select one to bat first. Of the two players who have hit more than 30 home runs, the coach would like to select one to bat fourth. The pitcher will bat last. How many batting lineups are possible under these circumstances?
12. An amusement park has 11 roller coasters and three water rides. On a hot day, Sandro wants to ride all the rides. In how many ways can he ride the amusement park rides if he wants to begin with the water rides.

In Exercises 13 – 18, calculate using the permutation formula.

13. $P(12, 6)$
14. $P(14, 9)$
15. $P(15, 15)$
16. $P(7, 7)$
17. $P(73, 5)$
18. $P(37, 4)$

In Exercises 19 – 26, use the permutation formula to solve.

19. How many ways are there to arrange 10 books on a shelf from a collection of 20 books?
20. In how many ways can the first five turtles from a group of 15 cross the finish line in a race, assuming that there are no ties?
21. A student club has the following open positions: president, vice president, secretary, treasurer, social media director, and events coordinator. In how many ways can students be elected to these positions if 14 students are interested in serving?
22. In how many ways can four letters from the word "fraction" be arranged?
23. In the mathematics department, there are openings for division chair and discipline chair. If there are 18 candidates for the positions, in how many ways can division and discipline chairs be selected?
24. Three bands will perform at a concert from a list of 15 possible bands. One band will be selected as the opener, one band as the headliner, and one band as the encore. In how many ways can bands be selected as the opener, the headliner, and the encore?
25. The local fair has 12 rides. In how many ways can Samuel select and rank his top five favorite rides?
26. The drama club is putting on a spring play. If there are seven distinct roles for the play, in how many ways can the 23 members of the drama club be cast in the play?

In Exercises 27 – 32, calculate using the combination formula.

27. $C(12, 6)$
28. $C(14, 9)$
29. $C(15, 15)$
30. $C(7, 7)$
31. $C(73, 5)$
32. $C(37, 4)$

Exercises 33 – 40, use the combination formula to solve.

33. How many five-card poker hands are there if cards are dealt from a standard 52-card deck?
34. In a neighborhood with 198 residents, five need to be elected as board members for the homeowner's association. In how many ways can this board be formed?
35. From a list of 25 historical figures, a student is asked to choose four to write about in a paper. In how many ways can a student select four of these historical figures?

36. An ice cream shop offers 32 flavors of ice cream. A person would like to order a bowl containing three scoops of different flavors. How many possibilities are there?
37. Min can take four novels with him on vacation. He has 17 favorite novels he wants to choose from. In how many ways can he select the four novels to take on vacation?
38. On a doctoral qualifying exam, a master's student must answer five of the ten questions asked. In how many ways can the master's student answer the five questions?
39. There are 32 students in choir and a small group is being sent to sing the national anthem at a basketball game. In how many ways can seven choir members be selected to sing at the game?
40. Kitzia is going on a beach vacation and plans to take four bathing suits. She has 10 bathing suits in her drawer. In how many ways can Kitzia pick four bathing suits?

In Exercises 41 – 48, determine if the problem involves a permutation or a combination. Then, use the appropriate formula to solve.

41. In how many ways can a person select and rank their five favorite songs from a list of 25 songs?
42. An investor would like to build a portfolio consisting of 10 stocks from a group of 30 stocks that offer a high growth potential. In how many ways can the investor create the portfolio?
43. Fifty raffle tickets have been purchased, and three of these tickets will be chosen at random. All three winners will receive $100. In how many ways can the winning tickets be selected?
44. A comedy club is hosting an event where nine comedians will perform. There are 20 comedians who have expressed interest in performing. In how many ways can the club set the line up?
45. For the upcoming academic year, there are four mathematics positions open at a local college. If there are 93 applicants, in how many ways can four applicants be chosen?
46. How many five-number passwords can be made with the digits 1, 2, 3, 4, 5, 6, 7, 8 if a digit can only be used once?
47. The science club is awarding three scholarships: one for $1,000, one for $500, and one for $250. If there are 20 applicants, in how many ways can the scholarships be awarded?
48. The STEM club is awarding three $500 scholarships. If there were 17 applicants, in how many ways can the club award the scholarships?

In Exercises 49 – 52, solve using the appropriate formula.

49. In a state lottery, six numbers must be chosen from the numbers 1 – 50. How many different selections are possible? If each lottery ticket takes five seconds to print at a gas station, how many days would it take to print all possible tickets?

50. An interior decorator would like to arrange four photos on a gallery wall in a specific order and has 16 photos from which to choose. In how many ways can the four photos be arranged? Suppose the designer would like to see each arrangement before deciding which to use. If it takes five minutes to hang the photos in each arrangement, how many days would it take to hang all possibilities?
51. Beckham has 12 toy cars. He built a ramp to watch them speed down the stairs. In how many ways can he line up eight cars to send down the ramp? If it takes 10 seconds to send a group of eight cars down the ramp, how many days would it take for Beckham to send all possible arrangements of cars down the ramp?
52. If there are three chemistry positions open at a local college and 38 applicants, in how many ways can three applicants be chosen? If each group of three interviews takes two hours, how many days would it take to interview all possible groups of three applicants?

In Exercises 53 – 60, solve using the appropriate formula(s).

53. To play the Powerball®, five numbers must be selected from the numbers 1 – 69. Then, one additional number must be selected from the numbers 1 – 26. In how many ways are there to select numbers for the Powerball?
54. In a card game using a standard 52-card deck, five cards are selected. In how many ways can three red cards and two black cards be selected?
55. When choosing a six-character password, the first three characters must be letters selected from A through Z and the last three characters must be digits selected from 0 – 9. No letter or digit can be repeated. How many passwords are possible?
56. The cross-country team consists of 34 runners and four coaches. At the end-of-year banquet, the coaches present awards for the team MVP, the most improved, the best team spirit, the natural leader, and most determined. If one coach is selected to read and present the awards, in how many ways can the cross-country team awards be presented?
57. Suppose there were 20 runners in a race. In how many ways could first, second, and third place be awarded if two runners tied for first place, two runners tied for third place, and only one runner came in second place?
58. Boden is looking at colleges and has created a list of six large colleges, seven medium sized colleges and four small colleges that he wants to learn more about. He plans to start by conducting an internet comparison of 10 schools: three large, five medium, and two small schools. Since Boden is compiling notes on the schools, the order in which he selects the schools does not matter. In how many ways can Boden select colleges for his internet search?

59. A student newspaper has five open positions that must be filled from a group of 15 teachers and 30 students. The open positions include two managing editors, a copy editor, a photographer, and a columnist. The managing editor positions must be occupied by teachers. The other three positions must be occupied by students. In how many ways can these positions be filled?
60. Ten students are competing in a race at a track meet. In how many ways can the runners finish if the last two cross the finish line at the same time (tie for last place)?

In Exercises 61 – 64, a new formula is provided for calculating permutations with repeated items.

When calculating permutations, it is assumed that no item is used more than once in the arrangement and the order of the items is important. If a group of items has some that are identical, it is still possible to count permutations but the formula must be adjusted to account for these repeated items.

For example, there are six permutations of the letters in the word "cat". They are: cat, cta, act, atc, tca, tac. This can be found using the permutation formula for $P(3,3)$, the number of permutations of three letters taken from this group of three. However, the number of permutations of the letters in the word "see" is not six, even though this word also contains three letters. Instead, there are only three permutations. They are: see, ese, ees. Since this word contains a repeated letter (the letter e), there are fewer permutations.

The number of permutations of n items if a items are identical, b items are identical, c items are identical, and so on, can be calculated using the following formula:

$$\frac{n!}{a! \cdot b! \cdot c! \cdots}$$

The word "see" contains three letters ($n = 3$) with the letter e appearing twice ($a = 2$). Thus, the number of permutations of the letters in the word "see" can be confirmed using this formula as follows:

$$\frac{3!}{2!} = 6$$

61. In how many ways can the letters in the word "mathematics" be arranged?
62. In how many ways can the letters in the word "calculus" be arranged?
63. In how many ways can the digits in the number 46,414,658 be arranged?
64. The speed of light is 186,282 miles per second. In how many ways can the digits in the number 186,282 be arranged?

In Exercises 65 – 70, Pascal's triangle is used to calculate combinations.

As seen in the Section 1.2 Exercises, Pascal's triangle is a pattern of numbers in which each number is the sum of the two numbers immediately above it to the right and to the left. A portion of Pascal's triangle appears below. Additional lines can be added to the triangle by continuing the pattern. The first line is known as "row 0", the second line is known as "row 1", and so on.

1 row 0
1 1 row 1
1 2 1 row 2
1 3 3 1 row 3
1 4 6 4 1 row 4
1 5 10 10 5 1 row 5

In addition to being used to count subsets, Pascal's triangle can also be used to calculate combinations. For example, the value of $C(5,3)$ appears in 4th entry of the 5th row of Pascal's triangle. Thus, $C(5,3) = 10$. In general, the value of $C(n,r)$ appears in the $(r+1)^{th}$ entry of the n^{th} row of Pascal's triangle.

65. What are the entries of row 6 of Pascal's triangle? Use this row to calculate $C(6,4)$.
66. What are the entries of row 7 of Pascal's triangle? Use this row to calculate $C(7,5)$.
67. What is the 3rd entry in the 4th row of Pascal's triangle? What combination, $C(n,r)$, does this position of Pascal's triangle represent?
68. What is the 5th entry in the 6th row of Pascal's triangle? What combination, $C(n,r)$, does this position of Pascal's triangle represent?
69. There are seven cities you would like to visit on a road trip, but only have time to visit three. How many different selections of three cities can you make from the original seven on your list? Hint: First, find row 7 of Pascal's triangle.
70. Your gym offers eight specialty exercise classes you would like to take. If you can only take four specialty classes this week, how many different specialty exercise classes can you take this week (in no particular order)? Hint: First, find row 8 of Pascal's triangle.

Concept Review

71. Explain the difference between a permutation and a combination.
72. Give an example of a permutation in daily life.
73. Give an example of a combination in daily life.
74. Write a word problem involving a permutation.
75. Write a word problem involving a combination.

Section 4.2 Exercise Solutions

1. 120
2. 5,040
3. 1
4. 3,628,800
5. 6,227,020,800
6. 3,628,800
7. 3,628,800
8. 29,030,400
9. 1,625,702,400
10. 118,800
11. 3,991,680
12. 239,500,800
13. 665,280
14. 726,485,760
15. 1,307,674,368,000
16. 5,040
17. 1,802,440,080
18. 1,585,080
19. 670,442,572,800
20. 360,360
21. 2,162,160
22. 1,680
23. 306
24. 2,730
25. 95,040
26. 1,235,591,280
27. 924
28. 2,002
29. 1
30. 1
31. 15,020,334
32. 66,045
33. 2,598,960
34. 2,410,141,734
35. 12,650
36. 4,960
37. 2,380
38. 252
39. 3,365,856
40. 210
41. Permutation; 6,375,600
42. Combination; 30,045,015
43. Combination; 19,600
44. Permutation; 60,949,324,800
45. Combination; 2,919,735
46. Permutation; 6,720
47. Permutation; 6,840
48. Combination; 680
49. 15,890,700 combinations; 920 days
50. 43,680 permutations; 152 days
51. 19,958,400 permutations; 2,310 days
52. 8,436 combinations; 703 days
53. 292,201,338
54. 845,000
55. 11,232,000
56. 133,562,880
57. 465,120
58. 2,520
59. 2,557,800
60. 1,814,400
61. 4,989,600
62. 5,040
63. 3,360
64. 180
65. 1, 6, 15, 20, 15, 6, 1; $C(6, 4)$ = 15 (the 5th entry of this row)

66. 1, 7, 21, 35, 35, 21, 7, 1;
$C(7,5)$ = 21
(the 6^{th} entry of this row)
67. 6; $C(4,2)$
68. 15; $C(6,4)$
69. 35
70. 70
71. Both permutations and combinations involve items taken from a collection. However, a permutation is an ordered arrangement of items, whereas with a combination order does not matter.
72. Answers will vary
73. Answers will vary
74. Answers will vary
75. Answers will vary

Section 4.3 | Probability

Objectives

- Identify possible outcomes of an experiment
- Determine the number of outcomes in a sample space
- Determine an equally likely sample space
- Compute theoretical probabilities
- Compute empirical probabilities

What is the possibility of rain today? What are the chances of getting a seat on the next flight when waiting on standby? What is the likelihood that the Chicago Cubs will win the World Series? How likely is it for a person driving a red car to be in an accident?

Probability is described as the mathematics of chance. In each of the scenarios above, the term probability could have been used instead of possibility, chance, or likelihood. Probability is a value that reflects the likelihood that an event will take place.

This section begins with several definitions used throughout this chapter and then continues with calculating probabilities.

Experiment, Outcome, and Sample Space

Suppose a person flips a coin. What results are possible? What is the possible result if one card is chosen from a standard deck of cards? What sums are possible when rolling two dice?

DEFINITION: An **experiment** is the action taken to collect data.

Rolling a die, flipping a coin, or choosing a card from a standard 52-card deck are commonly used examples of experiments. Other experiment examples include shooting a basketball, spinning a spinner, flipping a coin, or choosing a marble from a bag.

DEFINITION: An **outcome** is a result from an experiment.

If an experiment is flipping a coin, one possible outcome is that the coin lands on heads. Or if an experiment is selecting a card from a standard 52-card deck, one possible outcome is drawing a king.

DEFINITION: The set of all possible outcomes from an experiment is called the **sample space**.

The sample space for flipping one coin is $S = \{\text{Heads}, \text{Tails}\}$. The sample space for rolling one die is $S = \{1, 2, 3, 4, 5, 6\}$.

When flipping a quarter and a penny, one possible sample space describes the outcomes: $S_1 = \{\text{two heads}, \text{two tails}, \text{one head and one tail}\}$. There is only one way to have the outcome be two heads: the quarter must land on heads and the penny must also land on heads. Similarly, there is only one way to have the outcome be two tails: the quarter must land on tails and the penny must also land on tails. However, the last outcome from the sample space of one head and one tail could occur in two distinct ways: heads on the quarter and tails on the penny or tails on the quarter and heads on the penny. Having the sample space stated in this way, there are more ways to achieve the one head and one tail outcome. Therefore, the outcomes of the sample space $S_1 = \{\text{two heads}, \text{two tails}, \text{one head and one tail}\}$ are not equally likely to occur.

NOTE: The sample space for flipping two coins is the same even if the two coins have the same value. One coin would be considered coin one and recorded as the first result while the other coin would be considered coin two and have the result recorded second.

DEFINITION: A sample space in which each outcome has the same chance of occurring is an **equally likely sample space**.

To write an equally likely sample space for flipping two coins, list the outcomes from each coin in a specific order. List the outcome of the first coin flip followed by the outcome of the second coin flip such as $S_2 = \{\text{HH}, \ \text{HT}, \ \text{TH}, \ \text{TT}\}$ where H represents heads and T represents tails.

DEFINITION: An **event** is a subset of outcomes from the sample space of an experiment.

Using the experiment of rolling one die, an event could be rolling a two which represents one outcome from the sample space. Another event could be rolling a prime number which represents three outcomes from the sample space. At minimum, an event can have zero outcomes such as rolling a number greater than 10 on a single six-sided die. At maximum, an event can include all outcomes of the sample space such as rolling a number less than 10 on a single sided die.

- **EXAMPLE 4.3.1:** Given the following scenario, state the experiment, an equally likely sample space, and the outcome.

  A single die is rolled and the result is a five.

SOLUTION: The experiment is rolling a die. An equally likely sample space is $S = \{1, 2, 3, 4, 5, 6\}$. The outcome is rolling a five.

➢ **EXAMPLE 4.3.2:** Given the following scenario, state the experiment, an equally likely sample space, and the outcome.

A card is drawn from a standard 52-card deck and the result is the four of clubs.
SOLUTION: The experiment is picking a card from a standard 52-card deck. One equally likely sample space is $S = \{\mathrm{A}, 2, 3, 4, 5, 6, 7, 8, 9, 10, \mathrm{J}, \mathrm{Q}, \mathrm{K}\}$ where each element in this sample space represents four cards, one from each suit (heart, diamond, spade, and club). Another possible equally likely sample space would be to list each of the 52 cards individually indicating both the card and suit. The outcome is choosing the four of clubs.

❖ **YOU TRY IT 4.3.A:** Given the following scenario, state the experiment, an equally likely sample space, and the outcome.

A bag of marbles contains eight marbles: two red marbles, one orange marble, one yellow marble, three green marbles, and one blue marble. Jessica reaches in the bag and pulls out a green marble.

Outcomes and Visual Representations

An array or a tree diagram can be used as a visual representation of the sample space. The following array in Figure 4.3.1 and tree diagram in Figure 4.3.2 display the possible outcomes for the experiment of rolling a single die and then flipping a coin.

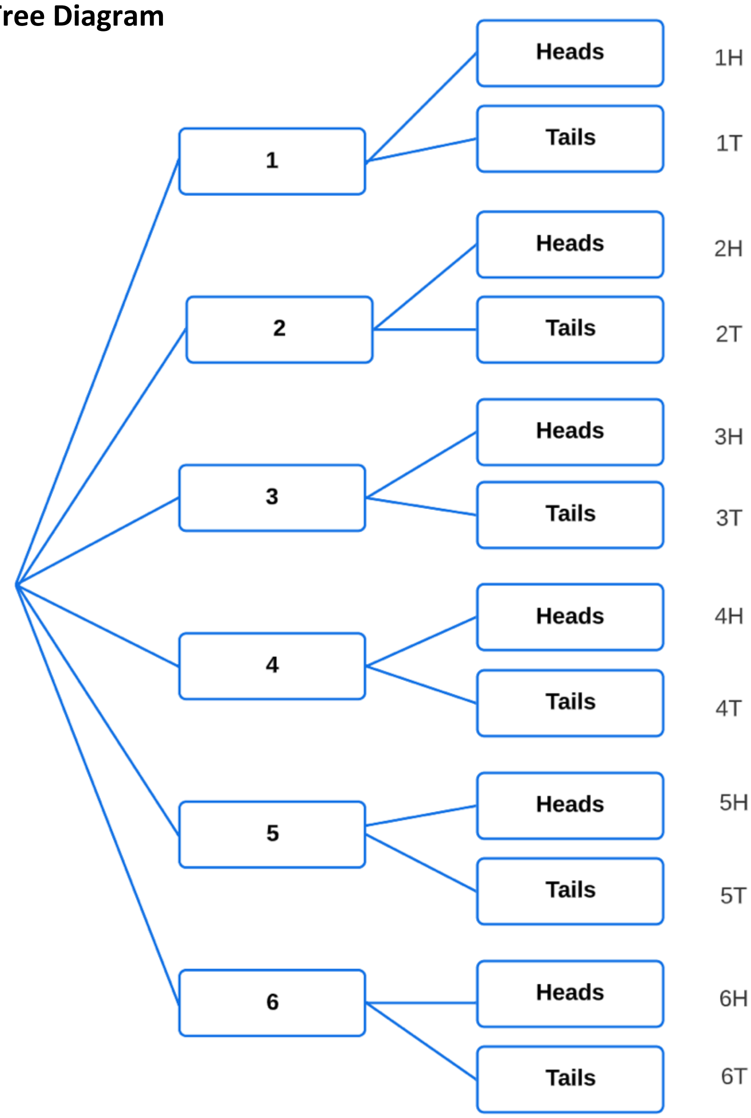
FIGURE 4.3.1

Array

		First Roll a Die					
		1	**2**	**3**	**4**	**5**	**6**
Second Flip a Coin	**H**	1, H	2, H	3, H	4, H	5, H	6, H
	T	1, T	2, T	3, T	4, T	5, T	6, T

FIGURE 4.3.2

Tree Diagram



In both visual representations, it can be shown that the sample space when rolling one die and flipping one coin consists of the following 12 outcomes shown below.

$$S = \{1H, 1T, 2H, 2T, 3H, 3T, 4H, 4T, 5H, 5T, 6H, 6T\}$$

The Fundamental Counting Principle can be used to determine the number of outcomes in the sample space. Rolling a six-sided die has six possible outcomes, while flipping a coin has two possible outcomes. For the experiment of first rolling a die and then flipping a coin, there are $6 \cdot 2 = 12$ total outcomes which supports the information shown in both the array and tree diagram.

➢ **EXAMPLE 4.3.3:** Given the experiment, determine the number of outcomes possible.

A coin is flipped three times.

SOLUTION: Flipping a coin has two outcomes. Therefore, flipping a coin three times has $2 \cdot 2 \cdot 2 = 2^3 = 8$ total outcomes.

➢ **EXAMPLE 4.3.4:** Given the experiment, determine the number of outcomes possible.

A card is chosen from a standard 52-card deck. The card is set aside (not replaced) and a second card is chosen from the same deck.

SOLUTION: Choosing a card from a standard deck has 52 outcomes. If the first card is not replaced back into the deck, when choosing the second card there are only 51 outcomes remaining. There are $52 \cdot 51 = 2{,}652$ total outcomes.

➢ **EXAMPLE 4.3.5:** Given the experiment, determine the number of outcomes possible.

Rolling a die two times.

SOLUTION: Rolling one die has six outcomes. Therefore, there are $6 \cdot 6 = 6^2 = 36$ total outcomes from rolling a die two times. Note in Example 4.3.4, one card was left out of the deck (not replaced) which affected the sample space for the next time a card was selected. In this case, all six outcomes are possible with each roll. The sample space does not change because all six outcomes are always visible on the die.

Rolling two dice produces 36 outcomes in the sample space. The outcomes could be organized using an array as shown in Figure 4.3.3. Notice that the outcome of $(3,5)$ represents a three rolled on the first die and a five rolled on the second die while $(5,3)$ represents a five rolled on the first die and a three rolled on the second die. Thus, they represent two distinct outcomes in the equally likely sample space.

FIGURE 4.3.3

		First Roll					
		1	**2**	**3**	**4**	**5**	**6**
Second Roll	**1**	(1, 1)	(2, 1)	(3, 1)	(4, 1)	(5, 1)	(6, 1)
	2	(1, 2)	(2, 2)	(3, 2)	(4, 2)	(5, 2)	(6, 2)
	3	(1, 3)	(2, 3)	(3, 3)	(4, 3)	(5, 3)	(6, 3)
	4	(1, 4)	(2, 4)	(3, 4)	(4, 4)	(5, 4)	(6, 4)
	5	(1, 5)	(2, 5)	(3, 5)	(4, 5)	(5, 5)	(6, 5)
	6	(1, 6)	(2, 6)	(3, 6)	(4, 6)	(5, 6)	(6, 6)

❖ **YOU TRY IT 4.3.B:** Given the experiment, determine the number of outcomes possible.

A single six-sided die is rolled and then one card is chosen from a standard 52-card deck.

Probability

What are the chances that an alarm doesn't sound and a student misses their first class? How likely is it that all the authors of this book are female and vegetarian? What are the chances of being chosen for the lead in the school play? Probability is described as the mathematics of chance and uses the concepts of event, outcome, and sample space introduced earlier in this section to create a value from zero to one to determine how likely it is that an event will occur.

DEFINITION: **Probability** is a value that reflects the likelihood that an event will take place.

The probability of event E occurring is symbolized as $P(E)$ and is calculated by creating a ratio that divides the number of outcomes from the event of interest by the total number of outcomes in the sample space.

Because probability is calculated as a ratio of the number of outcomes of an event compared to the number of total possible outcomes, it is a value between zero and one, inclusive. A probability can be represented as a fraction, decimal, or percent. A probability represented as a percent will always be between 0% and 100%, inclusive. A probability of zero indicates that an event cannot occur. For example, the probability of rolling a number greater than 10 on a single standard six-sided die is zero. On the other extreme, a probability of one indicates that an event is certain to occur. The probability of choosing a black card from the 13 spade cards is one because every spade card that could be chosen is also a black card. The closer a probability is to zero, the more unlikely the event is to occur. The closer a probability is to one, the more likely the event is to occur.

Probability

The **probability** of event E is the ratio that divides the number of outcomes in event E by the number of total possible outcomes in the sample space.

$$P(E) = \frac{\text{number of outcomes in the event of interest}}{\text{number of total outcomes in the sample space}} = \frac{n(E)}{n(S)}$$

Probability is a value between zero and one, inclusive.

$$0 \leq P(E) \leq 1$$

NOTE: In Sections 4.1 and 4.2, the Fundamental Counting Principle, permutations, and combinations were used to calculate the number of ways an event or series of events could occur. The number of ways an event or series of events could happen results in a counting number. In contrast, probability is a ratio that has a value between zero and one, inclusive.

Theoretical Probability

Theoretical probability is calculated using information from an equally likely sample space. Although an experiment is not conducted, theoretical probability is based on the idea that if an experiment were to be conducted and repeated over many trials, the results would be evenly distributed among the outcomes from an equally likely sample space.

DEFINITION: Probability using theoretical outcomes from an equally likely sample space is called **theoretical probability**.

In the equally likely sample space $\mathrm{S} = \{\mathrm{H}, \mathrm{T}\}$, there is one outcome where the coin lands on heads while the sample space contains two total outcomes. So, if a coin is flipped one time the probability of landing on heads is calculated as follows.

$$P(\text{heads}) = \frac{1 \text{ heads outcome in the equally likely sample space}}{2 \text{ total outcomes in the equally likely sample space}} = \frac{1}{2}$$

➢ **EXAMPLE 4.3.6:** Find the following theoretical probabilities when rolling a single die.

a. $P(\text{rolling a four})$
b. $P(\text{rolling a number less than ten})$

SOLUTION:

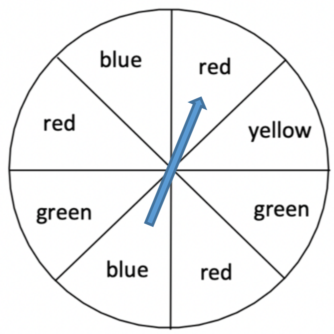
a. There is one possible result when rolling a four: 4.

$$P(\text{rolling a four}) = \frac{\text{1 outcome that is a four}}{\text{6 outcomes in the sample space}} = \frac{1}{6}$$

b. All six possible results are less than ten: 1, 2, 3, 4, 5, or 6.

$$P(\text{rolling a number less than ten}) = \frac{\text{6 outcomes less than ten}}{\text{6 outcomes in the sample space}} = \frac{6}{6} = 1$$

➢ **EXAMPLE 4.3.7:** If the pointer given below is spun one time, find the following probabilities.



a. $P(\text{blue})$
b. $P(\text{green})$
c. $P(\text{red})$
d. $P(\text{yellow})$

SOLUTION:

a. $P(\text{blue}) = \frac{2}{8} = \frac{1}{4}$

b. $P(\text{green}) = \frac{2}{8} = \frac{1}{4}$

c. $P(\text{red}) = \frac{3}{8}$

d. $P(\text{yellow}) = \frac{1}{8}$

The sum of all probabilities from the outcomes in a sample space is 1, which represents 100% of the results possible. For example, when flipping one coin, the possible results are heads or tails. The sum of these probabilities is $\mathrm{P(heads) + P(tails)} = \frac{1}{2} + \frac{1}{2} = \frac{2}{2} = 1$. As another example, when choosing one card from a standard 52-card deck, one way to list the sample space would be $\mathrm{S = \{hearts, clubs, diamonds, spades\}}$. The sum of probabilities of being dealt each suit is $\mathrm{P(heart) + P(club) + P(diamond) + P(spade)} = \frac{13}{52} + \frac{13}{52} + \frac{13}{52} + \frac{13}{52} = \frac{52}{52} = 1$.

In Example 4.3.7, the spinner is separated into eight sections with the colors blue, green, red, and yellow represented. Again, the sum of the probabilities from the possible outcomes is $\mathrm{P(blue)} + \mathrm{P(green)} + \mathrm{P(red)} + \mathrm{P(yellow)} = \frac{2}{8} + \frac{2}{8} + \frac{3}{8} + \frac{1}{8} = \frac{8}{8} = 1.$

Sum of Probabilities of All Outcomes

If a sample space is separated into outcomes A, B, C, etc., then the sum of the probabilities of all possible outcomes is 1 which can be represented as

$$P(A) + P(B) + P(C) + \cdots = 1$$

➢ **EXAMPLE 4.3.8:** Find the following probabilities when choosing one card from a standard 52-card deck.

a. $P(\text{ace of spades})$
b. $P(\text{red king})$
c. $P(\text{club})$
d. $P(\text{jack or ten})$

SOLUTION:

a. $P(\text{ace of spades}) = \frac{\text{1 ace of spades}}{\text{52 outcomes in the sample space}} = \frac{1}{52}$

b. $P(\text{red king}) = \frac{\text{2 red kings}}{\text{52 outcomes in the sample space}} = \frac{2}{52} = \frac{1}{26}$

c. $P(\text{club}) = \frac{\text{13 clubs}}{\text{52 outcomes in the sample space}} = \frac{13}{52} = \frac{1}{4}$

d. There are eight possible results: four jacks and four tens.
$P(\text{jack or ten}) = \frac{\text{8 cards that are a jack or a ten}}{\text{52 outcomes in the sample space}} = \frac{8}{52} = \frac{2}{13}$

❖ **YOU TRY IT 4.3.C:** Find the following probabilities when choosing one bagel from a bagel pack that includes four plain bagels, three cinnamon bagels, two cranberry bagels, two cheese bagels, and one everything bagel:

a. $P(\text{cheese bagel})$
b. $P(\text{cinnamon or cranberry bagel})$
c. $P\ (\text{bagel that is not plain})$

➢ **EXAMPLE 4.3.9:** Find the following probabilities when rolling two dice. It might be helpful to reference Figure 4.3.3 to see all possible outcomes for rolling two dice.

a. $P(\text{sum of five})$
b. $P(\text{sum less than six})$
c. $P(\text{at least one die with the number one})$
d. $P(\text{an even number on any die})$
e. $P(\text{sum less than two})$

SOLUTION:

a. The outcomes that fit the description are $(1,4), (4,1), (2,3)$, and $(3,2)$, so $P(\text{sum of five}) = \frac{4}{36} = \frac{1}{9}$.

b. The outcomes that fit the description are $(1,1), (1,2), (1,3), (1,4), (2,1), (2,2), (2,3), (3,1), (3,2)$, and $(4,1)$, so $P(\text{sum less than six}) = \frac{10}{36} = \frac{5}{18}$.

c. The outcomes that fit the description are $(1,1), (1,2), (1,3), (1,4), (1,5), (1,6), (2,1), (3,1), (4,1), (5,1)$ and $(6,1)$, so $P(\text{at least one die with the number one}) = \frac{11}{36}$.

d. The outcomes that fit the description are $(1,2), (1,4), (1,6), (2,1), (2,2), (2,3), (2,4), (2,5), (2,6), (3,2), (3,4), (3,6),$ $(4,1), (4,2), (4,3), (4,4), (4,5), (4,6), (5,2), (5,4), (5,6),$ $(6,1), (6,2), (6,3), (6,4), (6,5)$ and $(6,6)$, so $P(\text{even number on any die}) = \frac{27}{36} = \frac{3}{4}$

e. There are no outcomes that fit the description, so $P(\text{sum less than two}) = \frac{0}{36} = 0$.

❖ **YOU TRY IT 4.3.D:** Find the following probabilities when rolling a die and choosing a marble from a bag containing one red marble, one blue marble, and one green marble. It might be helpful to make a tree diagram or an array to list all possible outcomes.

a. $P(\text{rolling a three and then choosing a red marble})$
b. $P(\text{rolling an even number then choosing a blue marble})$
c. $P(\text{rolling a five and then choosing a red or blue marble})$
d. P $(\text{rolling an odd number and choosing a blue or green marble})$

Empirical Probability

At times, theoretical probability doesn't mirror real life events. For example, in studying the effects of a new medicine or medical procedure, the results are not equally distributed between improvement and no improvement. Similarly, in education settings, test scores are not equally distributed among possible grade categories such as A, B, C, D, or F. In basketball a player might not make half of the free throws, and in baseball a player might not get a hit half of the time.

When a doctor says that a particular medicine has an 90% chance of improving a patient's symptoms, what does this mean? A 90% chance of improvement indicates that if there were 100 people with similar conditions and symptoms, improvement was observed 90 times. This results in the ratio $\frac{90}{100} = \frac{9}{10}$ which is also equivalent to 90%.

In the 2022 baseball season, Jeff McNeil had the highest batting average of 0.326[12]. What does this mean? Coaches and fans would like to know, what is the probability that he will get a hit? During that season, McNeil had 533 times at bat and got a hit 174 times. So, the probability that McNeil got a hit is given by the following ratio:

$$P(\text{hit}) = \frac{174 \text{ hits}}{533 \text{ total times at bat}} \approx 0.326 \text{ or } 32.6\%$$

In February 2023, Stephen Curry had the highest free throw percentage in NBA history[13]. Out of 206 attempts, he made 190 free throws. What is the probability that he made the basket when he shot from the free throw line during that season?

$$P(\text{making a free throw}) = \frac{190 \text{ free throws made}}{206 \text{ free throws attempts}} = \frac{95}{103} \approx 0.922 \text{ or } 92.2\%$$

DEFINITION: Probability using data collected or observed is called **empirical probability**.

Unlike theoretical probability where outcomes are assumed to be distributed evenly to all equally likely outcomes, empirical probability is given by a ratio of outcomes observed in the event of interest compared to the total number of observations in the sample space as shown in the two previous sports examples.

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[12] *Jeff McNeil stats, height, weight, position, rookie status & more*. Baseball. (n.d.). Retrieved February 8, 2023, from https://www.baseball-reference.com/players/m/mcneije01.shtml

[13] Schievenin, Z. (2021, November 1). *Best NBA free throw shooters in history*. Best NBA Free Throw Shooters in history. Retrieved February 8, 2023, from https://www.dunkest.com/en/nba/news/31740/best-nba-free-throw-shooters

➢ **EXAMPLE 4.3.10:** Find the following probabilities.

a. Samir flipped a coin 50 times. The coin landed on heads 22 times. Based on empirical data, find $P(\text{heads})$ and $P(\text{tails})$.

b. Christy and Beckham played six card games. Beckham won five times. Based on empirical data, if they played another card game, find $P(\text{Beckham wins})$.

SOLUTION:

a. $P(\text{heads}) = \frac{22}{50} = \frac{11}{25}$ and $P(\text{tails}) = \frac{28}{50} = \frac{14}{25}$

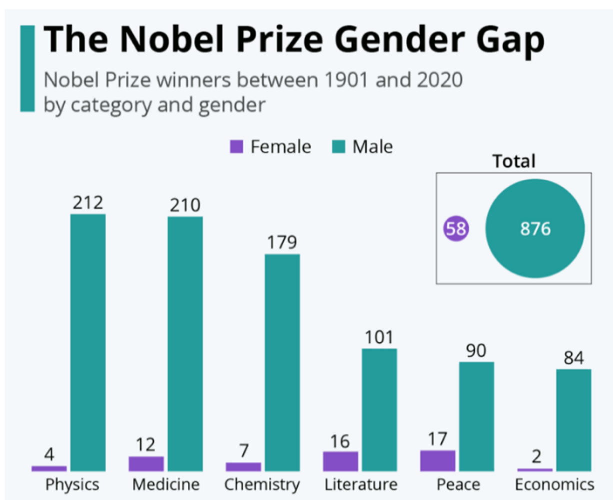
b. $P(\text{Beckham wins}) = \frac{5}{6}$

❖ **YOU TRY IT 4.3.E:** According to a study[14] of 101,397 high school drivers, 38,531 drivers admitted to texting while driving within the last month. If a high school driver was chosen at random, what is the probability that the driver texted while driving in the last month? Round the solution to the nearest hundredths place.

➢ **EXAMPLE 4.3.11:** Use the information in the Statista[15] infographic to find the following probabilities.

a. When randomly choosing a person's name who has earned a Nobel Prize between 1901 and 2020, what is the probability that person was female?

b. When randomly choosing a person's name who has earned a Nobel Prize between 1901 and 2020, what is the probability that person earned their award in peace?



[14] Li, L., Shults, R., Andridge, R., Yellman, M., Xiang, H., & Zhu, M. (2018). Texting/Emailing While Driving Among High School Students in 35 States, United States, 2015. *Journal of Adolescent Health*, *63*(6), 701–708.

[15] Richter, F. (2020, October 13). *Infographic: The nobel prize gender gap*. Statista Infographics. Retrieved February 8, 2023, from https://www.statista.com/chart/2805/nobel-prize-winners-by-gender/

SOLUTION:

a. $P(\text{female}) = \frac{58}{934} = \frac{29}{467} \approx 0.062$ or 6.2%

b. $P(\text{peace}) = \frac{107}{934} \approx 0.115$ or 11.5%

➢ **EXAMPLE 4.3.12:** The CDC conducted research on college students and insomnia[16]. A table summarizing a portion of the data collected from 330 college students is provided.

Survey Results of College Students and Insomnia				
		No Insomnia	**Insomnia**	**Total**
Level of Education	**Undergraduate**	149	57	206
	Professional or graduate	94	30	124
Employment	**No**	132	46	178
	Yes	111	41	152
Caffeine Consumption	**< Six cups per day**	221	77	298*
	≥ Six cups per day	16	4	20
Physical Activity	**< Two days per week**	30	18	48
	≥ Two days per week	213	69	282
ADHD	**No**	222	54	276*
	Yes	21	31	52

a. If a survey response is chosen at random, what is the probability the person who completed the form was an undergraduate student?

b. If a survey response is chosen at random, what is the probability the person who completed the form has ADHD? *Notice the total number of responses for ADHD is 328 not 330.

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[16] Mbous, Y. P. V. (2022). Psychosocial Correlates of Insomnia Among College Students. *Preventing Chronic Disease*, *19*.

* Not all totals have a sum of 330 participants.

SOLUTION:

a. $P(\text{undergraduate}) = \frac{206}{330} = \frac{103}{165} \approx 0.624$ or 62.4%

b. $P(\text{ADHD}) = \frac{52}{328} = \frac{13}{82} \approx 0.159$ or 15.9%

❖ **YOU TRY IT 4.3.F:** A local animal rescue conducted a census of their animals. Use the information from the table to answer the following questions.

Animal Rescue Census			
Animal	**Male**	**Female**	**Total**
Bird	5	11	**16**
Opossum	1	3	**4**
Rabbit	12	9	**21**
Racoon	3	5	**8**
Skunk	2	0	**2**
Squirrel	11	14	**25**
Total	**34**	**42**	**76**

a. If an animal is chosen at random, find the probability that the animal is a male.
b. If an animal is chosen at random, find the probability that the animal is a squirrel.
c. If an animal is chosen at random, find the probability that the animal is a male rabbit.
d. If an animal is chosen at random, find the probability that the animal is a female skunk.

Quick Review

- An **experiment** is the action taken to collect data.
- An **outcome** is a result from an experiment.
- The set of all possible outcomes from an experiment is called the **sample space**.
- A sample space in which each outcome has the same chance of occurring is an **equally likely sample space**.
- An **event** is a subset of outcomes from the sample space of an experiment.
- **Probability** is a value that reflects the likelihood that an event will take place.
- The probability of event E is calculated as follows:

$$P(E) = \frac{\text{number of outcomes in the event of interest}}{\text{number of total outcomes in the sample space}} = \frac{n(E)}{n(S)}$$

Quick Review Continued

- Probability is a value between zero and one, inclusive $0 \leq P(E) \leq 1$.
- The sum of all possible probabilities in a sample space is 1.
- **Theoretical probability** is calculated using information from an equally likely sample space.
- **Empirical probability** is calculated using data collected or observed.

YOU TRY IT 4.3.A SOLUTION: The experiment is picking a marble from the bag. An equally likely sample space is $S = \{\text{red}, \text{red}, \text{orange}, \text{yellow}, \text{green}, \text{green}, \text{green}, \text{blue}\}$ or $S = \{\text{R}, \text{R}, \text{O}, \text{Y}, \text{G}, \text{G}, \text{G}, \text{B}\}$ in abbreviated form. The outcome is a green marble was chosen. Note the sample space $S = \{\text{R}, \text{O}, \text{Y}, \text{G}, \text{B}\}$ is not equally likely, and therefore not part of the solution for this question.

YOU TRY IT 4.3.B SOLUTION: 312 outcomes

YOU TRY IT 4.3.C SOLUTION:

a. $\frac{1}{6}$ **b.** $\frac{5}{12}$ **c.** $\frac{2}{3}$

YOU TRY IT 4.3.D SOLUTION: The possible outcomes that fit the experiment are:
$(1, \text{R}), (1, \text{B}), (1, \text{G}), (2, \text{R}), (2, \text{B}), (2, \text{G}), (3, \text{R}), (3, \text{B}), (3, \text{G})$,
$(4, \text{R}), (4, \text{B}), (4, \text{G}), (5, \text{R}), (5, \text{B}), (5, \text{G}), (6, \text{R}), (6, \text{B})$, and $(6, \text{G})$

a. $\frac{1}{18}$ **b.** $\frac{1}{6}$ **c.** $\frac{1}{9}$ **d.** $\frac{1}{3}$

YOU TRY IT 4.3.E SOLUTION: $\frac{38{,}531}{101{,}397} \approx 0.38 \text{ or } 38\%$

YOU TRY IT 4.3.F SOLUTION:

a. $\frac{17}{38}$ **b.** $\frac{25}{76}$ **c.** $\frac{3}{19}$ **d.** 0

Section 4.3 | Exercises

In Exercises 1 – 6, given the following scenarios, state the experiment, an equally likely sample space and the outcome.

1. A single die is rolled and the result is a four.
2. Two coins are flipped and the result is two tails.
3. A spinner with equally-sized regions numbered 1 – 8 is spun and the result is five.
4. A card is drawn from a standard 52-card deck and the result is the jack of hearts.
5. A contestant in a game show has the option to select a prize hidden behind one of three doors and selects door #2.
6. A scientist opens a book about the planets in our solar system. The book opens to a page on Saturn.

In Exercises 7 – 15, given the experiment, determine the number of outcomes possible.

7. A coin is flipped and then a die is rolled.
8. Four coins are flipped.
9. Two coins are flipped and then a ten-sided die is rolled.
10. Three six-sided dice are rolled.
11. Two different states from the United States of America are chosen.
12. Two six-sided dice are rolled and then one card is chosen from a standard 52-card deck.
13. Three photos are selected from a student's 1^{st} grade – 12^{th} grade school photos (without replacement).
14. Three cards chosen one at a time from a standard 52–card deck (without replacement).
15. A spinner with 10 equally sized regions is spun four times in a row.

In Exercises 16 – 24, a single six-sided die is rolled. Find the following theoretical probabilities.

16. The probability of rolling an odd number.
17. The probability of rolling an even number greater than four.
18. The probability of rolling an odd number greater than two.
19. The probability of rolling a prime number greater than four.
20. The probability of rolling a number less than 10.
21. The probability of rolling a number less than one.
22. The probability of rolling a number that is a perfect square.
23. The probability of rolling a number that is between 0 and 7.
24. The probability of rolling a prime number less than 2 (Note: one is not a prime number).

In Exercises 25 – 33, two six-sided die are rolled. Figure 4.3.3 shows the 36 equally likely outcomes. Find the following theoretical probabilities.

25. The probability of rolling a sum of four.
26. The probability of rolling a sum of nine.
27. The probability of rolling a sum greater than ten.
28. The probability of rolling a sum greater than two and less than seven.
29. The probability of rolling at least one five.
30. The probability of rolling at least one odd number.
31. The probability of rolling at most one even number.
32. The probability of rolling different numbers on each die.
33. The probability of rolling a prime number first and an even number second.

In Exercises 34 – 42, a single card is selected from a standard 52-card deck. Find the following theoretical probabilities. Remember that aces and face cards are not considered even or odd.

34. The probability of selecting the ace of spades.
35. The probability of selecting a green five.
36. The probability of selecting an even number.
37. The probability of selecting a queen.
38. The probability of selecting a face card.
39. The probability of selecting a red seven.
40. The probability of selecting a black five.
41. The probability of selecting a heart.
42. The probability of selecting a face card that is a diamond.

In Exercises 43 – 48, a snack mix was made by mixing 3 cups cheddar flavored Goldfish®, 2 cups pretzel Goldfish, ¼ cup pizza flavored Goldfish, ½ cup parmesan flavored Goldfish, and ¼ cup sour cream and onion flavored Goldfish. If a single Goldfish is selected at random from the snack mix, find the following probabilities.

43. The probability of selecting a cheddar flavored Goldfish.
44. The probability of selecting a pizza flavored Goldfish.
45. The probability of selecting a parmesan flavored Goldfish.
46. The probability of selecting a pretzel Goldfish or cheddar Goldfish.
47. The probability of selecting a goldfish that is not a pretzel Goldfish.
48. The probability of selecting a goldfish that is not a parmesan flavored Goldfish.

In Exercises 49 – 52, in a recent survey students reported the number of pets in their home. Use the information from the table to find the following probabilities.

Number of Pets in the Home					
0 pets	1 pet	2 pets	3 pets	4 or more pets	Total
117	318	402	56	29	922

49. If a student is chosen at random, find the probability the student has 0 pets.
50. If a student is chosen at random, find the probability the student has 2 pets.
51. If a student is chosen at random, find the probability the student has 3 or more pets.
52. If a student is chosen at random, find the probability the student has at least one pet.

In Exercises 53 – 62, the table below summarizes the number of US high school participants in leading sports programs for girls[17] and boys[18] during the 2021 – 2022 school year. Use the table provided to find the following probabilities rounded to the nearest thousandth.

Sports Program	High School Girls Participants	High School Boys Participants	Total Participants
Basketball	370,466	521,616	892,082
Track and Field	456,697	569,262	1,025,959
Cross Country	191,323	231,387	422,710
Soccer	374,773	436,465	811,238
Baseball or Softball	340,923	481,004	821,927
Tennis	176,185	145,858	322,043
Swimming and Diving	149,751	123,208	272,959
Total	**2,060,118**	**2,508,800**	**4,568,918**

53. If a participant is selected at random, find the probability they participated in soccer.
54. If a participant is selected at random, find the probability they participated in tennis.
55. If a participant is selected at random, find the probability they are a high school girl.
56. If a participant is selected at random, find the probability they are a high school boy.

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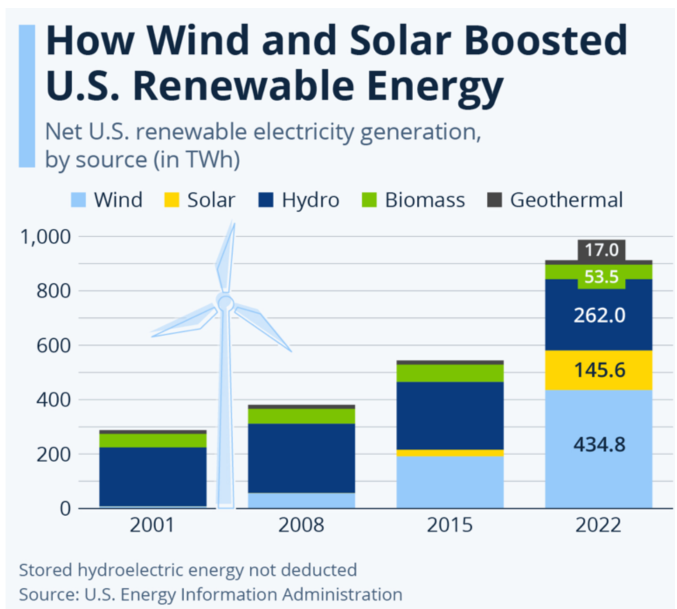
[17] Published by Statista Research Department, & 8, D. (2022, December 8). *Top high school sports programs for girls in U.S.* Statista. Retrieved April 4, 2023, from https://www.statista.com/statistics/197272/us-high-schools-with-athletic-programs-for-girls-2010/

[18] Published by Statista Research Department, & 8, D. (2022, December 8). *Top high school sports programs for boys in U.S.* Statista. Retrieved April 4, 2023, from https://www.statista.com/statistics/197251/us-high-schools-with-athletics-programs-for-boys/

57. If a participant is selected at random, find the probability they were participating in girls' basketball.
58. If a participant is selected at random, find the probability they were participating in boys' cross country.
59. If a participant is selected at random, find the probability that they were not participating in basketball.
60. If a participant is selected at random, find the probability they were participating in swimming and diving.
61. If a participant is selected at random, find the probability they were participating in cross country or tennis.
62. If a participant is selected at random, find the probability they were not participating in track and field.

In Exercises 63 – 67, use the information from the infographic[19] to find the following probabilities. Round the answer to the nearest thousandth.



63. If a renewable energy source was chosen at random in the US during 2022, find the probability that the renewable energy is geothermal power.
64. If a renewable energy source was chosen at random in the US during 2022, find the probability that the renewable energy is solar power.
65. If a renewable energy source was chosen at random in the US during 2022, find the probability that the renewable energy is wind or solar power.
66. If a renewable energy source was chosen at random in the US during 2022, find the probability that the renewable energy is not geothermal power.
67. If a renewable energy source was chosen at random in the US during 2022, find the probability that the renewable energy is not wind energy.

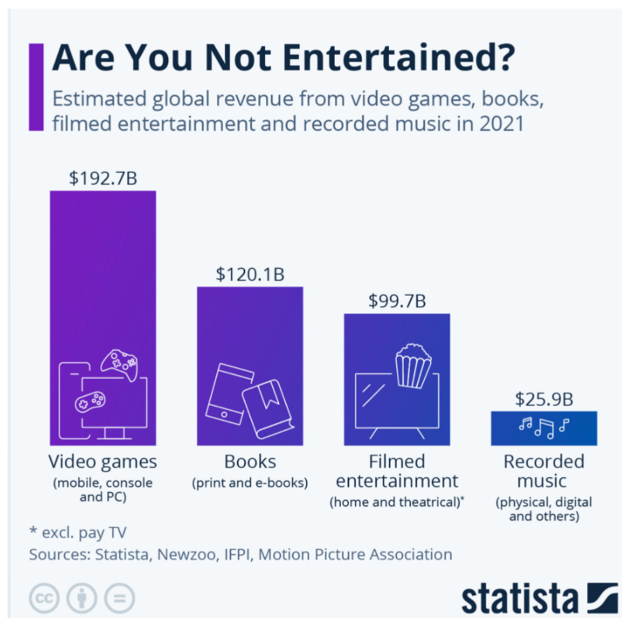
[19] Buchholz, K., & Richter, F. (2023, March 28). *Infographic: How wind and solar boosted U.S. Renewable Electricity*. Statista Infographics. https://www.statista.com/chart/17616/shares-of-us-renewable-electricity-generation-by-source/

In Exercises 68 – 70, use the information from the infographic[20] to find the following probabilities.

68. Determine the probability that a randomly selected reigning World Cup champion would win the world cup again.
69. Determine the probability that a randomly selected reigning World Cup champion would finish their appearance in the group stage.
70. Determine the probability that a randomly selected reigning World Cup champion participates again.

In Exercises 71 – 74, use the information from the infographic[21] to find the following probabilities. Round the answer to the nearest thousandth.

71. If a person spent money on entertainment categories from the 2021 infographic, find the probability that it was a video game.
72. If a person spent money on entertainment categories from the 2021 infographic, find the probability that it was recorded music.
73. If a person spent money on entertainment categories from the 2021 infographic, find the probability that it was a book or film.
74. If a person spent money on entertainment categories from the 2021 infographic, find the probability that it was not a film.



[20] Richter, F. (2022, December 12). *Infographic: Defending a World Cup title is a rare achievement*. Statista Infographics. Retrieved February 8, 2023, from https://www.statista.com/chart/28927/reigning-champions-at-the-fifa-world-cup/

[21] Richter, F. (2022, December 12). *Infographic: Are you not entertained?*. Statista Infographics. https://www.statista.com/chart/22392/global-revenue-of-selected-entertainment-industry-sectors/

Section 4.3 | Exercise Solutions

1. Experiment: rolling a die
   $S = \{1, 2, 3, 4, 5, 6\}$
   Outcome: roll a four
2. Experiment: flip two coins
   $S = \{\mathrm{HH}, \mathrm{HT}, \mathrm{TH}, \mathrm{TT}\}$
   Outcome: TT
3. Experiment: spin a spinner
   $S = \{1, 2, 3, 4, 5, 6, 7, 8\}$
   Outcome: spin a five
4. Experiment: pick a card from standard 52-card deck
   One possible sample space is
   $S = \{A, 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K\}$
   Outcome: jack of hearts
5. Experiment: select a door
   $S = \{\text{door \#1}, \text{door \#2}, \text{door \#3}\}$
   Outcome: door #2
6. Experiment: open a book to a page about a planet
   $S = \{\text{Mercury}, \text{Venus}, \text{Earth}, \text{Mars}, \text{Jupiter}, \text{Saturn}, \text{Uranus}, \text{Neptune}\}$
   Outcome: open to a page on Saturn
7. 12
8. 16
9. 40
10. 216
11. 2,450
12. 1,872
13. 1,320
14. 132,600
15. 10,000
16. $\frac{1}{2}$
17. $\frac{1}{6}$
18. $\frac{1}{3}$
19. $\frac{1}{6}$
20. 1
21. 0
22. $\frac{1}{3}$
23. 1
24. 0
25. $\frac{1}{12}$
26. $\frac{1}{9}$
27. $\frac{1}{12}$
28. $\frac{7}{18}$
29. $\frac{11}{36}$
30. $\frac{3}{4}$
31. $\frac{3}{4}$
32. $\frac{5}{6}$
33. $\frac{1}{4}$
34. $\frac{1}{52}$
35. 0
36. $\frac{5}{13}$
37. $\frac{1}{13}$

38. $\frac{3}{13}$

39. $\frac{1}{26}$

40. $\frac{1}{26}$

41. $\frac{1}{4}$

42. $\frac{3}{52}$

43. $\frac{1}{2}$

44. $\frac{1}{24}$

45. $\frac{1}{12}$

46. $\frac{5}{6}$

47. $\frac{2}{3}$

48. $\frac{11}{12}$

49. $\frac{117}{922} \approx 0.127$

50. $\frac{201}{461} \approx 0.436$

51. $\frac{85}{922} \approx 0.092$

52. $\frac{805}{922} \approx 0.873$

53. $\frac{811{,}238}{4{,}568{,}918} \approx 0.178$

54. $\frac{322{,}043}{4{,}568{,}918} \approx 0.070$

55. $\frac{2{,}060{,}118}{4{,}568{,}918} \approx 0.451$

56. $\frac{2{,}508{,}800}{4{,}568{,}918} \approx 0.549$

57. $\frac{370{,}466}{4{,}568{,}918} \approx 0.081$

58. $\frac{231{,}387}{4{,}568{,}918} \approx 0.051$

59. $\frac{3{,}676{,}836}{4{,}568{,}918} \approx 0.805$

60. $\frac{272{,}958}{4{,}568{,}918} \approx 0.060$

61. $\frac{744{,}753}{4{,}568{,}918} \approx 0.163$

62. $\frac{3{,}542{,}959}{4{,}568{,}918} \approx 0.775$

63. $\frac{17.0}{912.9} = \frac{170}{9{,}129} \approx 0.019$

64. $\frac{145.6}{912.9} = \frac{1{,}456}{9{,}129} \approx 0.159$

65. $\frac{580.4}{912.9} = \frac{5{,}804}{9{,}129} \approx 0.636$

66. $\frac{895.8}{912.9} = \frac{8959}{9129} \approx 0.981$

67. $\frac{478.1}{912.9} = \frac{4{,}781}{9{,}129} \approx 0.524$

68. $\frac{1}{10}$

69. $\frac{2}{5}$

70. $\frac{19}{20}$

71. $\frac{192.7}{438.4} = \frac{1{,}927}{4{,}384} \approx 0.440$

72. $\frac{25.9}{438.4} = \frac{259}{4{,}384} \approx 0.059$

73. $\frac{219.8}{438.4} = \frac{1{,}099}{2{,}192} \approx 0.501$

74. $\frac{338.7}{438.4} = \frac{3{,}387}{4{,}384} \approx 0.773$

Section 4.4 Probability with Complements, Unions, & Mutually Exclusive Events

Objectives

- Compute the probabilities of the complements of events
- Compute the probabilities of the unions of events
- Solve problems involving mutually exclusive events

Suppose the probability of losing weight on the newest fad diet is 7%. What would the probability of NOT losing weight on the newest fad diet be? Recall from Section 4.3, the sum of all probabilities from the outcomes in a sample space is one. Thus, the sum of the probability of an event happening and the probability of the same event not happening is 100% or 1. So, the probability of not losing weight on the newest fad diet would be $100\% - 7\% = 93\%$ or $1 - 0.07 = 0.93$. This is an example of calculating the probability of a complement.

Probability of Complements

Recall from Section 2.3, complements were discussed in terms of set theory. The complement of set A consists of all elements that are in the universal set but not in set A. Similarly, in probability, the complement of event A consists of all outcomes from the sample space that are not part of event A.

DEFINITION: Given event A, the **complement** of event A consists of the outcomes from the sample space that are not part of event A.

Calculating Probabilities of Complements

Given event A, the probability of the complement of A, notated as $P(\overline{A})$ can be calculated in the following three ways:

1. $P(\overline{A}) = \frac{n(\overline{A})}{n(S)}$ where S is the sample space.
2. $P(\overline{A}) = 1 - \frac{n(A)}{n(S)}$ where S is the sample space.
3. $P(\overline{A}) = 1 - P(A)$

➢ **EXAMPLE 4.4.1:** Using a standard 52-card deck,

a. find the probability of not choosing a face card.
b. find the probability of not selecting a black 3, 4, 5, or 6.

SOLUTION:

a. Since there are 12 face cards, the number of ways of selecting a non-face card is $52 - 12 = 40$. So, the probability of not choosing a face card is $1 - \frac{12}{52} = \frac{40}{52} = \frac{10}{13}$.

b. There are two black 3s, two black 4s, two black 5s, and two black 6s in a standard 52-card deck. So, there are eight ways of selecting a black 3, 4, 5, or 6. The number of ways of not selecting a black 3, 4, 5, or 6 is $52 - 8 = 44$. Therefore, the probability of not selecting a black 3, 4, 5, or 6 is $1 - \frac{8}{52} = \frac{44}{52} = \frac{11}{13}$.

➢ **EXAMPLE 4.4.2:** Using a six-sided die,

a. find the probability of not rolling an even number.
b. find the probability of not rolling an odd number greater than 2.

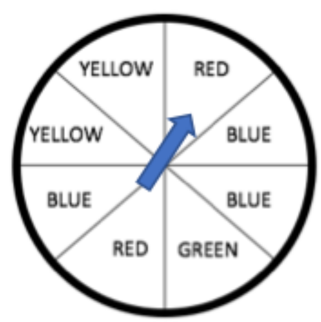
SOLUTION:

a. Since there are three even numbers (2, 4, and 6), the number of ways of rolling a number that is not even is $6 - 3 = 3$. So, the probability is $1 - \frac{3}{6} = \frac{3}{6} = \frac{1}{2}$.

b. Since there are two odd numbers greater than 2 (3 and 5), the number of ways of rolling a number that is not an odd number greater than 2 is $6 - 2 = 4$. So, the probability is $1 - \frac{2}{6} = \frac{4}{6} = \frac{2}{3}$.

➢ **EXAMPLE 4.4.3:** Using the given spinner,

a. find the probability of not landing on yellow.
b. find the probability of not landing on blue.



SOLUTION:

a. Since two sections of the eight sections of the spinner are yellow, the number of ways of not spinning yellow is $8 - 2 = 6$. So, the probability is $1 - \frac{2}{8} = \frac{6}{8} = \frac{3}{4}$.

b. Since three of the eight sections of the spinner are blue, the number of ways of not landing on blue is $8 - 3 = 5$. So, the probability is $1 - \frac{3}{8} = \frac{5}{8}$.

➢ EXAMPLE 4.4.4:

a. If the probability of going to the market today is 0.15, what is the probability of not going to the market?
b. Suppose the probability that pigs can't fly is 0.99998. What is the probability that pigs can fly?

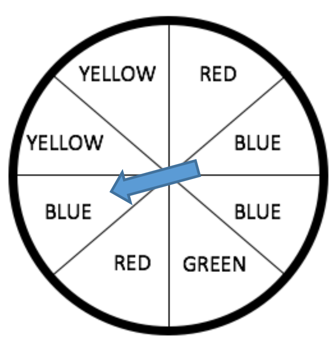
SOLUTION:

a. Since the probability of going to the market is 0.15, the probability of not going to the market is $1 - 0.15 = 0.85$.

b. Since the probability that pigs can't fly is 0.99998, the probability that pigs can fly is $1 - 0.99998 = 0.00002$.

❖ YOU TRY IT 4.4.A:

a. Using a standard 52-card deck, find the probability of not selecting a card that is 2, 4, 8, or 10.
b. Using a six-sided die, find the probability of not rolling 5.
c. Given the spinner below, find the probability of not landing on green.
d. The probability that a person does not win the lottery is 0.999999685, what is the probability that a person wins the lottery?



- **EXAMPLE 4.4.5:**

a. Using a standard 52-card deck, find the probability of not selecting a purple card.
b. Using a six-sided die, find the probability of not rolling a number less than 9.

SOLUTION:

a. Since there are no purple cards in a standard 52-card deck (only red and black cards), the probability of not selecting a purple card is $1 - \frac{0}{52} = 1$.

b. The only values on a standard six-sided die are 1, 2, 3, 4, 5, and, 6. All of the numbers on the die are less than 9. So, the probability of not rolling a number less than 9 is $1 - \frac{6}{6} = 0$.

NOTE: The sum of all probabilities from the outcomes in a sample space is one. Thus, the sum of the probabilities of an event and its complement will always be 1 or 100%.

Probability of Unions

Recall from Section 2.3, the union of sets A and B consists of the elements in set A or set B or in both sets A and B. Similarly, in probability, the union of events A and B contains the outcomes from event A or event B or events A and B from the sample space. Recall that the intersection of A and B must be subtracted so that outcomes are not counted more than one time. This leads to the following definition.

DEFINITION: The **union** of two events A and B consists of the outcomes of event A, event B, or both events A and B. As a reminder, this is the inclusive definition of *or*.

Calculating the Probability of Unions

Given events A and B, the probability of the union of A and B, notated as $P(A \cup B)$, is the probability of outcomes from event A or event B or events A and B in the sample space and is given by:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

➢ EXAMPLE 4.4.6:

a. What is the probability of rolling an even number or a number greater than 4 on a standard six-sided die?
b. What is the probability of selecting a red card or an ace from a standard 52-card deck?

SOLUTION:

a. The probability of rolling an even number (2, 4, or 6) is $\frac{3}{6}$. The probability of rolling a number greater than 4 (5 or 6) is $\frac{2}{6}$. Notice that rolling a 6 is listed twice. Therefore, the intersection of even outcomes and outcomes greater than 4 must be subtracted to avoid being counted twice. So, the probability of rolling an even number or a number greater than 4 is $\frac{3}{6}+\frac{2}{6}-\frac{1}{6}=\frac{4}{6}=\frac{2}{3}$.

b. The probability of selecting a red card is $\frac{26}{52}$ because half of the cards in a standard 52-card deck are red. The probability of selecting an ace is $\frac{4}{52}$. Since two of the four aces are red (ace of hearts and ace of diamonds), they are listed twice. Therefore, the intersection of the set of red cards and the set of aces must be subtracted to avoid being counted twice. So, the probability of selecting a red card or an ace is $\frac{26}{52}+\frac{4}{52}-\frac{2}{52}=\frac{28}{52}=\frac{7}{13}$.

❖ YOU TRY IT 4.4.B:

a. What is the probability of rolling an even number or a number less than 5 on a six-sided die?
b. What is the probability of selecting a king or a black face card from a standard 52-card deck?

➢ **EXAMPLE 4.4.7:** The following table lists the number of factory workers by age and employment status (part-time or full-time) in a town. If one person is randomly selected from the group,

a. find the probability that the person is part-time or between 36 – 40 years old.
b. find the probability that the person is above 65 or full-time.
c. find the probability that the person is not part-time or not between 16 – 20 years old.

Age	Part-time	Full-time	Total
16 – 20	27	24	51
21 – 25	47	43	90
26 – 30	33	49	82
31 – 35	31	53	84
36 – 40	36	45	81
41 – 45	25	47	72
46 – 50	19	46	65
51 – 55	15	40	55
56 – 60	10	35	45
61 – 65	10	30	40
Above 65	27	22	49
Total	280	434	714

SOLUTION:

a. The probability of selecting a part-time person is $\frac{280}{714}$. The probability of selecting a person between 36 – 40 years old is $\frac{81}{714}$. The probability of selecting a part-time person ***and*** a person between 36 – 40 years old is $\frac{36}{714}$. Notice that the intersection of the two events is counted twice (circled in red in the following table). So, the probability of selecting a part-time person or a person between 36 – 40 years old is $\frac{280}{714} + \frac{81}{714} - \frac{36}{714} = \frac{325}{714}$.

Age	Part-time	Full-time	Total
16 – 20	27	24	51
21 – 25	47	43	90
26 – 30	33	49	82
31 – 35	31	53	84
36 – 40	36	45	81
41 – 45	25	47	72
46 – 50	19	46	65
51 – 55	15	40	55
56 – 60	10	35	45
61 – 65	10	30	40
Above 65	27	22	49
Total	280	434	714

b. The probability of selecting a person above 65 is $\frac{49}{714}$. The probability of selecting a full-time person is $\frac{434}{714}$. There are 22 full-time people who are above 65 (circled in red in the following table) that have been counted twice. Thus, the probability of selecting a person above 65 or a full-time person is $\frac{49}{714}+\frac{434}{714}-\frac{22}{714}=\frac{461}{714}$.

Age	Part-time	Full-time	Total
16 – 20	27	24	51
21 – 25	47	43	90
26 – 30	33	49	82
31 – 35	31	53	84
36 – 40	36	45	81
41 – 45	25	47	72
46 – 50	19	46	65
51 – 55	15	40	55
56 – 60	10	35	45
61 – 65	10	30	40
Above 65	27	22	49
Total	280	434	714

c. The probability of selecting a person who is not part-time (thus full-time) is $\frac{434}{714}$. The probability of selecting a person who is not 16 – 20 years old is $1-\frac{51}{714}=\frac{663}{714}$. There are 410 people who are not part-time (thus full-time) and who are not 16 – 20 years old (circled in red in the following table) that have been counted twice. Thus, the probability of selecting a person who is not part-time or is not 16 – 20 years old is $\frac{434}{714}+\frac{663}{714}-\frac{410}{714}=\frac{687}{714}=\frac{229}{238}$.

Age	Part-time	Full-time	Total
16 – 20	27	24	51
21 – 25	47	43	90
26 – 30	33	49	82
31 – 35	31	53	84
36 – 40	36	45	81
41 – 45	25	47	72
46 – 50	19	46	65
51 – 55	15	40	55
56 – 60	10	35	45
61 – 65	10	30	40
Above 65	27	22	49
Total	280	434	714

❖ **YOU TRY IT 4.4.C:** Using the table from Example 4.4.7, if one person is randomly selected from the group,

a. what is the probability of selecting a full-time person or a person 16 – 20 years old?
b. what is the probability of selecting a person 51 – 55 years old or a part-time person?

Mutually Exclusive Events

Suppose there are two disjoint events, meaning the events have no outcomes in common. Then the probability of their union would only include the outcomes of event A and the outcomes of event B. The number of outcomes where event A ***and*** event B occur simultaneously is zero, since the events have no outcomes in common. The probability of unions of disjoint events are a special example of the probability of unions stated earlier. This leads to the following definition.

DEFINITION: **Mutually exclusive events** are events that have no common outcomes.

Mutually exclusive events cannot happen simultaneously. If A and B are mutually exclusive events, then the probability of their union is the probability of A plus the probability of B since the probability of their intersection is zero.

Calculating the Probability of Mutually Exclusive Events

Given mutually exclusive events A and B, the probability of $A \cup B$ is given by:

$$P(A \cup B\ ) = P(A) + P(B)$$

Note: The probability of the intersection of A and B, $P(A \cap B)$, is zero.

➢ **EXAMPLE 4.4.8:**

a. What is the probability of selecting an 8 or a face card from a standard 52-card deck?
b. What is the probability of rolling a number less than 3 or a number greater than 4 on a standard six-sided die?

SOLUTION:

a. The probability of selecting one of the four 8s in a standard 52-card deck is $\frac{4}{52}$. The probability of selecting a face card is $\frac{12}{52}$. Since there are no face cards that are 8s, the probability of selecting an 8 or a face card is $\frac{4}{25} + \frac{12}{52} = \frac{16}{52} = \frac{4}{13}$.

b. The probability of rolling a number less than 3 (1 or 2) on a standard six-sided die is $\frac{2}{6}$. The probability of rolling a number greater than 4 (5 or 6) is $\frac{2}{6}$. Since there are no numbers that are both less than 2 and greater than 4, the probability of rolling a number less than 2 or a number greater than 4 is $\frac{2}{6} + \frac{2}{6} = \frac{4}{6} = \frac{2}{3}$.

❖ **YOU TRY IT 4.4.D:**

a. What is the probability of selecting a red 9 or a king on standard 52-card deck?
b. What is the probability of rolling an even number or an odd number on a standard six-sided die?

➢ **EXAMPLE 4.4.9:** The following table lists the number of factory workers by age and employment status (part-time or full-time) in a small town. If one person is randomly selected from the group,

a. find the probability of selecting a person between the ages of 21 – 25 or 36 – 40 years old.

b. find the probability of selecting a full-time person between 41 – 45 or a part-time person between 46 – 50 years old.

Age	Part-time	Full-time	Total
16 – 20	27	24	51
21 – 25	47	43	90
26 – 30	33	49	82
31 – 35	31	53	84
36 – 40	36	45	81
41 – 45	25	47	72
46 – 50	19	46	65
51 – 55	15	40	55
56 – 60	10	35	45
61 – 65	10	30	40
Above 65	27	22	49
Total	280	434	714

SOLUTION:

a. The probability of selecting a person between the ages of 21 – 25 is $\frac{90}{714}$. The probability of selecting a person between the ages of 36 – 40 is $\frac{81}{714}$. Notice that there are no people who are between the ages of 21 – 25 and 36 – 40 years old simultaneously. Thus, they are mutually exclusive events. So, the probability of selecting a person between the ages of 21 – 25 or 36 – 40 years old is $\frac{90}{714} + \frac{81}{714} = \frac{171}{714} = \frac{57}{238}$.

Age	Part-time	Full-time	Total
16 – 20	27	24	51
21 – 25	47	43	90
26 – 30	33	49	82
31 – 35	31	53	84
36 – 40	36	45	81
41 – 45	25	47	72
46 – 50	19	46	65
51 – 55	15	40	55
56 – 60	10	35	45
61 – 65	10	30	40
Above 65	27	22	49
Total	280	434	714

b. The probability of selecting a full-time person between 41 – 45 years old is $\frac{47}{714}$. The probability of selecting a part-time person between 46 – 50 years old is $\frac{19}{714}$. Since a person from this table is not in both of these categories simultaneously, the events are mutually exclusive. Therefore, the probability is $\frac{47}{714} + \frac{19}{714} = \frac{66}{714} = \frac{11}{119}$.

Age	Part-time	Full-time	Total
16 – 20	27	24	51
21 – 25	47	43	90
26 – 30	33	49	82
31 – 35	31	53	84
36 – 40	36	45	81
41 – 45	25	47	72
46 – 50	19	46	65
51 – 55	15	40	55
56 – 60	10	35	45
61 – 65	10	30	40
Above 65	27	22	49
Total	280	434	714

❖ **YOU TRY IT 4.4.E:** Using the table from Example 4.4.9, if one person is randomly selected from the group,

a. what is the probability of selecting a full-time person between 16 – 20 years old or a part-time person between 21 – 25 years old?
b. what is the probability of selecting a person 61 – 65 years old or above 65?

Probability of an Event Happening At Least Once

Suppose Tara and Danny want five children. What is the probability that they will have at least one girl? To answer this question, consider all the possibilities. For example, Tara and Danny can have all boys, all girls, two girls and three boys, two boys and three girls, and so on. Since the question asks for the probability of ***at least*** one girl, think about what it means to NOT have at least one girl. To NOT have at least one girl, Tara and Danny must have all boys. For each of the five children there are two options: boy or girl. Using the Fundamental Counting Principle from Section 4.1, the sample space for this situation includes $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 2^5 = 32$ total outcomes. There is only one event out of the 32 possible outcomes in which Tara and Danny have all boys. So, the probability of them having at least one girl is $1 - \frac{1}{32} = \frac{31}{32}$. This means that if Tara and Danny have five children, the probability of having at least one girl is $\frac{31}{32}$. This is an example of calculating the probability using the complement. In this case, having at least one girl and having have no girls are complements.

The Probability of an Event Happening At Least Once

The probability of an event happening at least once is 1 minus the probability that the event does not happen at all.

$$P(\text{event happening at least once}) = 1 - P(\text{event does not happen})$$

➢ **EXAMPLE 4.4.10:** Suppose Wayne rolls two six-sided dice. Using the following figure, find the probability that Wayne rolls at least one number greater than 2.

		First Roll					
		1	**2**	**3**	**4**	**5**	**6**
Second Roll	**1**	(1, 1)	(2, 1)	(3, 1)	(4, 1)	(5, 1)	(6, 1)
	2	(1, 2)	(2, 2)	(3, 2)	(4, 2)	(5, 2)	(6, 2)
	3	(1, 3)	(2, 3)	(3, 3)	(4, 3)	(5, 3)	(6, 3)
	4	(1, 4)	(2, 4)	(3, 4)	(4, 4)	(5, 4)	(6, 4)
	5	(1, 5)	(2, 5)	(3, 5)	(4, 5)	(5, 5)	(6, 5)
	6	(1, 6)	(2, 6)	(3, 6)	(4, 6)	(5, 6)	(6, 6)

SOLUTION:

There are six possible outcomes for the first die rolled. Similarly, there are six possible outcomes when the second die is rolled. By the Fundamental Counting Principle there are $6 \cdot 6 = 36$ possible outcomes when two dice are rolled. Here it would be easier to find the probability where no number greater than 2 is rolled. Using the above figure, this would occur with the following four outcomes: (1,1), (1,2), (2,1), and (2,2). The probability of not rolling at least one number greater than 2 is $\frac{4}{36}$. Thus, the probability that at least one number greater than 2 is rolled is $1 - P(\text{no number greater than 2}) = 1 - \frac{4}{36} = \frac{32}{36} = \frac{8}{9}$.

❖ **YOU TRY IT 4.4.F:** What is the probability that Rita gets at least one tail when flipping six coins.

Quick Review

- Given event A, the **complement** of event A consists of the outcomes from the sample space that are not part of event A. The **probability of the complement** of A, notated as $P(\bar{A})$, can be calculated in the following ways: $P(\bar{A}) = \frac{n(\bar{A})}{n(S)}$, $P(\bar{A}) = 1 - \frac{n(A)}{n(S)}$ where S is the sample space, or $P(\bar{A}) = 1 - P(A)$.

- The **union** of two events A and B consists of the outcomes of event A, event B, or both events A and B. As a reminder, this is the inclusive definition of *or*. Given events A and B, the **probability of the union** of A and B, notated as $A \cup B$, is given by $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

- **Mutually exclusive events** are events that have no common outcomes. Given mutually exclusive events A and B, the probability of $A \cup B$ is given by $P(A \cup B) = P(A) + P(B)$. For mutually exclusive events, the probability of the intersection of events A and B, $P(A \cap B)$, is zero.

- The **probability of an event happening at least once** is given by $P(\text{event happening at least once}) = 1 - P(\text{event does not happen})$

YOU TRY IT 4.4.A SOLUTION:

a. $\frac{9}{13}$

b. $\frac{5}{6}$

c. $\frac{7}{8}$

d. 0.000000315

YOU TRY IT 4.4.B SOLUTION:

a. $\frac{5}{6}$

b. $\frac{2}{13}$

YOU TRY IT 4.4.C SOLUTION:

a. $\frac{461}{714}$

b. $\frac{160}{357}$

YOU TRY IT 4.4.D SOLUTION:

a. $\frac{3}{26}$

b. 1

YOU TRY IT 4.4.E SOLUTION:

a. $\frac{71}{714}$

b. $\frac{89}{714}$

YOU TRY IT 4.4.F SOLUTION: $1 - \frac{1}{2^6} = 1 - \frac{1}{64} = \frac{63}{64}$

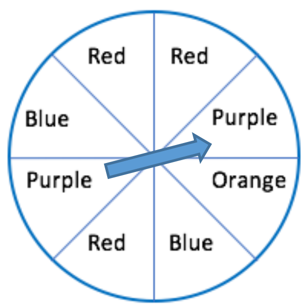
Section 4.4 | Exercises

In Exercises 1 – 4, using a standard 52-card playing deck, find the probability of NOT choosing

1. a black jack.
2. a red 4, 5, or 6.
3. a face card.
4. an odd numbered card (remember that an ace is not considered an odd number).

In Exercises 5 – 8, given the spinner below, find the probability of NOT landing on

5. red.
6. blue.
7. purple or orange.
8. a color on the American flag.



In Exercises 9 – 12, given in the following table are the income levels of students at a local college. If a person is randomly selected from this group, find the probability of

Income	Number
Under $10,000	97
$10,000 - $11,999	57
$12,000 - $13,999	45
$14,000 - $15,999	37
$16,000 - $17,999	27
$18,000 - $19,999	17
Above $20,000	5
Total	285

9. not selecting a student with an income under $10,000.
10. not selecting a student with an income above $20,000.
11. selecting a student who does not have an income between $16,000 and $17,999.
12. selecting a student who does not have an income $16,000 or higher.

In Exercises 13 – 16, given a standard six-sided die, find the probability of rolling

13. a prime number or 5.
14. an even prime number or 2.
15. an odd number or a number less than 4.
16. an even number or a number less than 6.

In Exercises 17 – 24, given in the following table is the number of social media users in a small town based on age and employment status (part-time or full-time). If a person is randomly selected from this group, find the probability of selecting

Age	Part-time	Full-time	Total
Under 10	12	27	39
11 – 20	79	127	206
21 – 30	59	151	210
31 – 40	48	110	158
41 – 50	48	75	123
51 – 60	53	61	114
61 – 70	34	45	79
Above 70	10	19	29
Total	343	615	958

17. a person who is between the ages of 21 – 30 or part-time.
18. a person who is full-time or above 70.
19. a person who is full-time or between the ages of 61 – 70.
20. a person who is between the ages of 51 – 70 or part-time.
21. a person who is under the age of 31 or is full-time.
22. a person who is part-time or not between the ages of 31 and 60.
23. a person who is between the ages 21 – 40 or full-time.
24. a person who is under the age of 41 or over the age of 70.

In Exercises 25 – 28, if a card from a standard 52-card deck is dealt, find the probability of

25. being dealt a red ace or a black queen.
26. being dealt a red 2, 3, or 4 or a black 8, 9, or 10.
27. being dealt a face card or an even-numbered card less than 8.
28. being dealt a black ace or even-numbered heart card.

In Exercises 29 – 32, given in the table below are drink preferences of patrons of a local coffee shop. If one patron from this group is randomly selected, find the probability of selecting

Drink	Hot	Cold/Iced	Total
Black Coffee	151	24	175
Flavored Coffee	123	78	201
Tea	47	88	135
Chocolate	23	49	72
Juices	5	60	65
Other Drink Options	63	65	128
Total	412	364	776

29. a patron who ordered a hot black coffee or an iced tea.
30. a patron who ordered hot chocolate or an iced flavored coffee.
31. a patron who didn't order a hot chocolate or who didn't order an iced flavored coffee.
32. a patron who ordered a tea or cold juice.

In Exercises 33 – 40, find the probability of each of the following.

33. A couple wants four children. What is the probability of them having at least one boy?
34. If two six-sided die are rolled, what is the probability of rolling a three at least once?
35. Mimi flips eight coins. What is the probability that she gets at least one head?
36. If two dice are rolled, what is the probability of not rolling at least one prime number?
37. If two six-sided die are rolled, what is the probability of rolling an odd number at least once?
38. If two six-sided die are rolled, what is the probability of not rolling a six at least once.
39. Given a true/false quiz with 10 questions, what is the probability of at least one false answer?
40. A female spider will lay about 100 eggs in her egg sac. Find the probability that at least one will become a female spider. (Leave your answer in exponent form).

In Exercises 41 – 50, using a standard 52-card deck, find each of the probabilities if one card is chosen.

41. What is the probability of drawing a face card or an ace?
42. What is the probability of not drawing a heart?
43. What is the probability of not drawing a red jack?
44. What is the probability of drawing one king?
45. What is the probability of drawing an odd-numbered card between 2 and 10 or a black queen?

46. What is the probability of drawing a face card or a red card?
47. What is the probability of drawing a red card or a spade?
48. What is the probability of drawing a 5 or a black card?
49. What is the probability of not drawing a red 8 or a black 9?
50. What is the probability of drawing a queen or an odd numbered red card. (Remember that an ace is not considered an odd numbered card.)

In Exercises 51 – 60, the table below lists the claims typically made to a given insurance company any given year. If one claim is randomly selected, answer the following.

Insurance Claim	Home	Business	Total
Fire	13	37	50
Flood	157	23	180
Lightening	3	5	8
Tornado	4	4	8
Accident	20	57	77
No Claim	303	174	477
Total	500	300	800

51. Find the probability that a claim is not for a business.
52. Find the probability that the claim is not for lightening.
53. Find the probability that the claim is for a business or for a flood.
54. Find the probability that the claim is for a home accident or a business flood.
55. Find the probability of an insurance claim that is for a home or no claim.
56. Find the probability that an insurance claim is for a tornado at home or an accident at home.
57. Find the probability that there is a home insurance claim.
58. Find the probability that the claim is not for a home fire.
59. Find the probability that the claim is for a tornado or for a business.
60. Find the probability that the claim is for an accident or home fire.

Section 4.4 Exercise Solutions

1. $\frac{50}{52} = \frac{25}{26}$
2. $\frac{46}{52} = \frac{23}{26}$
3. $\frac{40}{52} = \frac{10}{13}$
4. $\frac{36}{52} = \frac{9}{13}$
5. $\frac{5}{8}$
6. $\frac{6}{8} = \frac{3}{4}$
7. $\frac{5}{8}$
8. $\frac{3}{8}$
9. $\frac{188}{285}$
10. $\frac{280}{285} = \frac{56}{57}$
11. $\frac{258}{285} = \frac{86}{95}$
12. $\frac{236}{285}$
13. $\frac{3}{6} = \frac{1}{2}$
14. $\frac{1}{6}$
15. $\frac{4}{6} = \frac{2}{3}$
16. 1
17. $\frac{494}{958} = \frac{247}{479}$
18. $\frac{625}{958}$
19. $\frac{649}{958}$
20. $\frac{449}{958}$
21. $\frac{765}{958}$
22. $\frac{712}{958} = \frac{356}{479}$
23. $\frac{722}{958} = \frac{361}{479}$
24. $\frac{642}{958} = \frac{321}{479}$
25. $\frac{4}{52} = \frac{1}{13}$
26. $\frac{12}{52} = \frac{3}{13}$
27. $\frac{24}{52} = \frac{6}{13}$
28. $\frac{7}{52}$
29. $\frac{239}{776}$
30. $\frac{101}{776}$
31. $\frac{675}{776}$
32. $\frac{195}{776}$
33. $\frac{15}{16}$
34. $\frac{11}{36}$
35. $\frac{255}{256}$
36. $\frac{9}{36} = \frac{1}{4}$
37. $\frac{27}{36} = \frac{3}{4}$
38. $\frac{25}{36}$
39. $\frac{1023}{1024}$
40. $1 - \frac{1}{2^{100}}$
41. $\frac{16}{52} = \frac{4}{13}$
42. $\frac{39}{52} = \frac{3}{4}$
43. $\frac{25}{26}$
44. $\frac{4}{52} = \frac{1}{13}$
45. $\frac{18}{52} = \frac{9}{26}$
46. $\frac{32}{52} = \frac{8}{13}$
47. $\frac{39}{52} = \frac{3}{4}$
48. $\frac{28}{52} = \frac{7}{13}$
49. $\frac{48}{52} = \frac{12}{13}$
50. $\frac{12}{52} = \frac{3}{13}$
51. $\frac{500}{800} = \frac{5}{8}$
52. $\frac{792}{800} = \frac{99}{100}$
53. $\frac{457}{800}$
54. $\frac{43}{800}$
55. $\frac{674}{800} = \frac{337}{400}$
56. $\frac{24}{800} = \frac{3}{100}$
57. $\frac{500}{800} = \frac{5}{8}$
58. $\frac{787}{800}$
59. $\frac{304}{800} = \frac{19}{50}$
60. $\frac{90}{800} = \frac{9}{80}$

Section 4.5 Probability with Permutations & Combinations

Objectives

- Solve probability problems involving permutations
- Solve probability problems involving combinations

Permutations and combinations are counting techniques presented in Section 4.2. How can these counting methods be incorporated into probability problems? For example, a student might wonder what her chances are of being selected first if a teacher randomly selects the order in which students will deliver a speech to the class. A person playing a card game might be interested in the probability of being dealt all hearts. Or a person might wonder how their chances of winning a state lottery change if they purchase 100 tickets instead of one. In this section, permutations and combinations will be used to calculate probabilities like these.

Probabilities with Permutations

Suppose that Ellie is one of eight students who must give a speech during a particular class period. If her teacher randomly selects the order in which the students will present, what is the probability that Ellie is selected first? This problem involves permutations since students are arranged in order and no student will occupy more than one spot of the lineup. The probability will be calculated as follows:

$$\text{P(Ellie is first to deliver a speech)} = \frac{\text{number of permutations with Ellie first}}{\text{total number of permutations}}$$

To find the number of permutations with Ellie first, consider the options for each spot of the lineup. If Ellie is to deliver her speech first, then there is only one option for the first spot. This leaves seven other students who could fill the second spot of the lineup. Once the second spot is filled, then six other students remain who could fill the third spot of the lineup. This process continues until all eight spots are filled. Using the Fundamental Counting Principle, the number of permutations with Ellie first is:

$$1 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5{,}040$$

Notice that this is the same as calculating $1 \cdot P(7,7) = 1 \cdot 5{,}040 = 5{,}040$.

To find the total number of permutations, use the formula for $P(8,8)$ since eight students are randomly arranged to deliver speeches to the class.

$$P(8,8) = 40{,}320$$

Now, the probability that Ellie is randomly chosen to deliver her speech first can be calculated using these permutations.

$$\text{P(Ellie is first)} = \frac{\text{number of permutations with Ellie first}}{\text{total number of permutations}}$$

$$= \frac{1 \cdot P(7,7)}{P(8,8)} = \frac{5{,}040}{40{,}320} = \frac{1}{8} = 0.125$$

Thus, Ellie has a $\frac{1}{8}$, or 12.5%, chance of randomly being selected to deliver her speech first to the class. This makes sense, because if there are eight students who must deliver speeches then each student will have a $\frac{1}{8}$ chance of randomly being selected to go first.

- **EXAMPLE 4.5.1:** Matt and Kai are two of 15 artists who have submitted their work for an exhibit at a gallery. The exhibit will be held in a long room in which artwork will be arranged in a line. If eight pieces of art are selected and arranged randomly, what is the probability that Matt's piece appears first and Kai's piece appears last?

  SOLUTION: First, consider the number of possible arrangements in which Matt's piece appears first and Kai's piece appears last. In this arrangement, there is only one option for the first spot and one option for the last spot of the arrangement. This leaves 13 pieces of art that can fill the second spot. Once this spot is filled, there are 12 remaining pieces of art that can fill the third spot. This continues until all six spots between Matt and Kai are filled. Using the Fundamental Counting Principle, the number of ways in which eight pieces of art can be arranged with Matt's piece first and Kai's piece last is as follows:

  $$1 \cdot 13 \cdot 12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 1 = 1{,}235{,}520$$

  Notice this is the same as calculating $1 \cdot P(13,6) \cdot 1 = 1{,}235{,}520$.

To find the total number of possible arrangements of eight pieces of art from a collection of 15 pieces, use the formula for $P(15, 8)$.

$$P(15, 8) = 259{,}459{,}200$$

Now, the probability that Matt's piece appears first and Kai's piece appears last can be calculated using these probabilities.

$$\text{P(Matt's work is first and Kai's work is last)}$$

$$= \frac{\text{number of permutations with Matt's work first and Kai's work last}}{\text{total number of permutations}}$$

$$= \frac{1 \cdot P(13, 6) \cdot 1}{P(15, 8)} = \frac{1{,}235{,}520}{259{,}459{,}200} = \frac{1}{210} \approx 0.0048$$

Thus, the probability that Matt's artwork is randomly selected to appear first and Kai's artwork is randomly selected to appear last is $\frac{1}{210}$ or approximately 0.48%.

➢ **EXAMPLE 4.5.2:** A digital keypad for a garage door has a code that consists of six digits from 0 – 9 where digits cannot be repeated. If a code is randomly created, what is the probability that it alternates between even and odd digits, starting with an even digit?

SOLUTION: First, consider the number of six-digit arrangements that alternate between even and odd digits. Since there are five even digits possible (0, 2, 4, 6, or 8) and five odd digits possible (1, 3, 5, 7, or 9), the number of possibilities of even-odd alternating arrangements is as follows:

$$5 \cdot 5 \cdot 4 \cdot 4 \cdot 3 \cdot 3 = 3{,}600$$

Notice this is the same as calculating $P(5, 3) \cdot P(5, 3) = 3{,}600$.

To find the total number of possible codes consisting of six digits from a group of ten digits, use the formula for $P(10, 6)$.

$$P(10, 6) = 151{,}200$$

Now, the probability that an even-odd alternating code is randomly created can be calculated using these probabilities.

$$\text{P(code alternates between even and odd digits)}$$

$$= \frac{\text{number of codes that alternate between even and odd digits}}{\text{total number of permutations}}$$

$$= \frac{P(5,3) \cdot P(5,3)}{P(10,6)} = \frac{3{,}600}{151{,}200} = \frac{1}{42} \approx 0.0238$$

Thus, the probability that a randomly created code alternates between even and odd digits, starting with an even digit, is $\frac{1}{42}$ or approximately 2.38%.

❖ **YOU TRY IT 4.5.A:** Jamar and Sam are in a group of eight students who are competing in the 400-meter race at a track meet. Assuming there are no ties, what is the probability that Jamar finishes first and Sam finishes second?

Probabilities with Combinations

Suppose a person is playing a state lottery game in which six numbers must be chosen from 1 – 50 and the order of selection doesn't matter. What is the probability of winning this lotto if one ticket is purchased?

This is a combination problem because numbers are chosen from a group without regard to order. The winning ticket will contain all six winning numbers. Thus, there is only one way in which the winning numbers can be selected. This can be confirmed using the formula for $C(6,6)$.

$$C(6,6) = 1$$

Since six numbers are chosen from a group of 50, the total number of possible combinations can be found by calculating $C(50,6)$. This represents the number of possible unique lottery tickets that could exist for the lottery.

$$C(50,6) = 15{,}890{,}700$$

Thus, the probability of winning this lottery with one ticket is calculated as follows:

$$\text{P(win with one ticket)} = \frac{C(6,6)}{C(50,6)} = \frac{1}{15{,}890{,}700} \approx 0.000000063$$

The probability of winning this with one ticket is $\frac{1}{15{,}890{,}700}$ or approximately 0.0000063%.

➢ **EXAMPLE 4.5.3:** To play the Mega Millions® in 2023, a person must select five unique numbers from 1 – 70 without regard to order (the white lottery balls) and one additional number from 1 – 25 (the gold Mega Ball®). What is the probability of winning with one ticket? What is the probability of winning with 100 different tickets?

SOLUTION: The winning ticket will contain all six winning numbers. Thus, there is only one way in which the five winning white balls and one winning gold Mega Ball can be selected. This can be confirmed using the formulas for $C(5,5)$ and $C(1,1)$, along with the Fundamental Counting Principle.

$$C(5,5) \cdot C(1,1) = 1 \cdot 1 = 1$$

To find the total number of ways in which the numbers can be selected for the Mega Millions, consider the number of ways in which five white lottery ball numbers can be taken from a group of 70 and the number of ways in which one gold Mega Ball number can be taken from a group of 25. The formulas for $C(70,5)$ and $C(25,1)$, along with the Fundamental Counting Principle, can be used.

$$C(70,5) \cdot C(25,1) = 12{,}103{,}014 \cdot 25 = 302{,}575{,}350$$

Thus, the probability of winning with one ticket is as follows:

$$\text{P(win with one ticket)} = \frac{C(5,5) \cdot C(1,1)}{C(70,5) \cdot C(25,1)} = \frac{1}{302{,}575{,}350} \approx 0.0000000033$$

Purchasing 100 different tickets increases the chances of winning by a factor of 100. Thus, the probability of winning with 100 different tickets is as follows:

$$\text{P(win with 100 tickets)} = 100 \cdot \frac{1}{302{,}575{,}350} = \frac{2}{6{,}051{,}507} \approx 0.00000033$$

Since this solution is so small, a calculator may display the answer using scientific notation. For example, a calculator may display the answer in scientific notation as 3.304961888 E -7. This is the same as $3.304961888 \cdot 10^{-7}$ or approximately 0.00000033.

❖ **YOU TRY IT 4.5.B:** To play a lottery, a person must select five unique numbers from 1 – 60 (without regard to order) and one additional number from 1 – 35. What is the probability of winning with 10 different tickets?

➢ **EXAMPLE 4.5.4:** Suppose a person is playing a card game in which five cards are dealt to each player at random from a standard deck of 52 cards. What is the probability that the hand of cards consists of all hearts?

SOLUTION: This problem involves combinations since the order in which cards are dealt does not matter. The probability is calculated by dividing the number of combinations involving all hearts by the total number of possible five-card hands.

$$\text{P(dealt five hearts)} = \frac{\text{number of combinations with five hearts}}{\text{total number of combinations of five cards}}$$

Since there are 13 possible hearts, the number of five-card hands consisting of all hearts can be found by calculating $C(13, 5) = 1{,}287$.

Since there are 52 cards in the deck, the number of possible five-card hands can be found by calculating $C(52, 5) = 2{,}598{,}960$.

Thus, the probability of being dealt five hearts from a standard deck of 52 cards is as follows:

$$\text{P(dealt five hearts)} = \frac{C(13,5)}{C(52,5)} = \frac{1{,}287}{2{,}598{,}960} = \frac{33}{66{,}640} \approx 0.0005$$

Therefore, the probability of being dealt five hearts is $\frac{33}{66{,}640}$ or approximately 0.05%.

❖ **YOU TRY IT 4.5.C:** Suppose a person is playing a card game in which six cards are dealt to each player at random from a standard deck of 52 cards. What is the probability that the hand consists of only face cards (jack, queen, or king)?

➢ **EXAMPLE 4.5.5:** The U.S. Senate consisted of 49 Republicans, 48 Democrats, and 3 Independents in 2023. Suppose a six-person committee is to be formed by random selection. What is the probability that the committee consists of three Republicans and three Democrats?

SOLUTION: This problem involves combinations since the order in which members are selected to form a committee does not matter. The probability is calculated by dividing the number of combinations involving three Republicans and three Democrats by the total number of possible combinations of six senators.

$$\text{P(3 Reps and 3 Dems)} = \frac{\text{number of combinations with 3 Reps and 3 Dems}}{\text{total number of possible combinations of 6 senators}}$$

First, find the number of combinations of three Republicans taken from a group of 49 by calculating $C(49, 3) = 18{,}424$. Then, consider the number of combinations of three Democrats taken from a group of 48 by calculating $C(48, 3) = 17{,}296$. Then, the Fundamental Counting Principle can be used to find the number of possible groups containing three Republicans and three Democrats.

$$C(49, 3) \cdot C(48, 3) = 18{,}424 \cdot 17{,}296 = 318{,}661{,}504$$

Since there are 100 members of the U.S. Senate, the number of possible six-person committees can be found by calculating $C(100, 6) = 1{,}192{,}052{,}400$.

Thus, the probability that a randomly selected group of six U.S. Senators consists of three Republicans and three Democrats is as follows:

$$\text{P(3 Reps and 3 Dems)} = \frac{C(49, 3) \cdot C(48, 3)}{\text{C}(100, 6)} = \frac{318{,}661{,}504}{1{,}192{,}052{,}400} \approx 0.2673$$

Therefore, the probability that the randomly selected group consists of three Republicans and three Democrats is approximately 26.73%.

❖ **YOU TRY IT 4.5.D:** A high school robotics club consists of eight freshman, 12 sophomores, 12 juniors, and 10 seniors. A group of six will be selected at random to represent the club at a competition. What is the probability that the group will consist of two freshman and four seniors?

YOU TRY IT 4.5.A SOLUTION: $\frac{1}{56} \approx 0.0179$

YOU TRY IT 4.5.B SOLUTION: $\frac{1}{19{,}115{,}292} \approx 0.000000052$

YOU TRY IT 4.5.C SOLUTION: $\frac{33}{727{,}090} \approx 0.000045$

YOU TRY IT 4.5.D SOLUTION: $\frac{420}{374{,}699} \approx 0.0011$

Section 4.5 | Exercises

In Exercises 1 – 18, solve.

1. Katarina is one of seven students chosen to attempt a half-court shot during the half time break at a basketball game. The first student to make the shot wins. If the seven students are randomly lined up for this opportunity, what is the probability that Katarina is last in the line up?
2. Paula and Paul would like to be president and vice president of their student body at their college. There are six people running for the two positions. What is the probability that Paula is president and Paul is vice president?
3. Hugo and Linus are in a group of nine people at a Harry Potter exhibit where they can choose their own wand. If the order in which people are arranged to select their wand is determined at random, what is the probability that Hugo is first and Linus is second in the line up?
4. The math department awards four scholarships: one each for $1,000, $750, $500, and $250. If 7 engineering majors and 10 math majors apply for the scholarships, what is the probability that all four scholarships are awarded to students who major in math?
5. A five-digit code is used to open a garage door. Assuming that no digit can be repeated in the code, what is the probability that the code alternates between even and odd digits, starting with an even digit. (Note: Zero is an even number.)
6. Esperanza needs to choose a four-digit pin for her debit card. Assuming digits can be repeated, what is the probability that she chooses a four-digit pin where all four digits are different?
7. The combination to a lock consists of four digits. Assuming that no digit can be repeated in the combination, what is the probability that the combination is an even number greater than 9,000?
8. The combination to a lock consists of four digits. Assuming that no digit can be repeated in the combination, what is the probability that the combination contains only even digits and is a multiple of five?
9. The resource center at a college has a box of 20 calculators to loan out, but five of the calculators are missing batteries. If three are selected at random, what is the probability that **none** are missing batteries?
10. The resource center at a college has a box of 20 calculators to loan out, but five of the calculators are missing batteries. If three are selected at random, what is the probability that **all** are missing batteries?

11. A pizzeria offers the following meat toppings: sausage, pepperoni, ham, bacon, ground beef, and chicken. The pizzeria also offers the following non-meat toppings: mushroom, green pepper, black olives, green olives, onion, pineapple, tomato, roasted red pepper, and spinach. A person orders a three-topping pizza and asks the pizzeria to surprise them with the toppings. Assuming no topping appears more than once, what is the probability that the pizza ends up being a vegetarian dish (without meat)?
12. A box of identically shaped chocolates contains 18 pieces of chocolate. In the box, 1/3 of the chocolates are filled with caramel, 1/3 of the chocolates are filled with vanilla cream, and 1/3 of the chocolates are filled with raspberry cream. If a person selects five pieces at random, what is the probability that none of the pieces are filled with caramel?
13. An animal shelter has 10 dogs and 16 cats available for adoption. Eight of these animals are selected at random to be brought to an adoption event at a pet store. What is the probability that the group of animals being brought to the adoption event consists of only cats?
14. A pet store has 15 dogs, 20 cats, three snakes, 10 rabbits, and two guinea pigs. If a family chooses three pets to take home, what is the probability the pets are not snakes and are not guinea pigs?
15. A parent-teacher organization is made up of 15 parents and seven teachers. A 10-person subcommittee must be formed to lead a fundraiser. If members of this subcommittee are selected at random, what is the probability that the subcommittee doesn't contain a teacher?
16. A jewelry store has 30 pairs of earrings. Ten of the earrings are diamonds, seven are rubies, three are jades, and 10 are pearls. If a buyer purchases three pairs of earrings, what is the probability that none of them are diamonds?
17. Kristi has 17 books on her reading wish list. Nine books are historical fiction, five books are fiction, two books are non-fiction, and one book is a mystery. If she randomly chooses four books to read in order, find the probability that the first book she reads is a mystery book and the last book is historical fiction.
18. A local scout group is selling cookies. They have 10 boxes of caramel coconut cookies, five boxes of peanut butter thumbprint cookies, eight boxes of shortbread cookies, and six boxes of chocolate mint cookies. Sarah is allergic to peanuts. If she buys six randomly selected boxes of cookies, what is the probability that none contain peanut butter?

In Exercises 19 – 24, solve.

19. A student club consists of five freshman, 10 sophomores, 13 juniors, and 16 seniors. Ten of these students are randomly selected to present at a conference. What is the probability that the group consists of two freshman and eight seniors?

20. A bag contains 27 marbles where five are blue, five are green, seven are purple, three are red, six are orange, and one is brown. If four marbles are removed at once from the bag, what is the probability that one brown and three red marbles were removed?
21. To play a state lottery game, a person must select four numbers from 1 – 45 and one additional number from 1 – 15. If a person purchases 15 tickets containing different combinations of numbers, what is the probability of winning the grand prize?
22. In a raffle, a person must select five numbers from 1 – 60 and two letters from A – Z. If a person purchases three tickets containing different combinations of numbers and letters, what is the probability of correctly selecting all five numbers and two letters?
23. In 2023, the U.S. Senate consisted of 49 Republicans, 48 Democrats, and three Independents. A committee consisting of nine senators is randomly selected. What is the probability that the committee consists of an equal number of Republicans, Democrats, and Independents?
24. A college architectural advisory board consists of eight professional architects, four professors of architecture, and two college administrators. A six-person committee was created to write a grant proposal. What is the probability that the committee consists of an equal number of architects, professors, and administrators?

In Exercises 25 – 32, a standard deck of 52 cards is used.

25. If three cards are dealt from a standard deck of 52 cards, what is the probability that all three cards are a 7?
26. If four cards are dealt from a standard deck of 52 cards, what is the probability that all four cards are jacks?
27. If five cards are dealt from a standard deck of 52 cards, what is the probability that none are face cards (jack, queen, or king)?
28. If five cards are dealt from a standard deck of 52 cards, what is the probability that none are clubs?
29. If four cards are dealt from a standard deck of 52 cards, what is the probability of being dealt two aces and two kings?
30. If four cards are dealt from a standard deck of 52 cards, what is the probability of being dealt three hearts and one diamond?
31. If five cards are dealt from a standard deck of 52 cards, what is the probability that two cards are face cards and the other three cards are not face cards?
32. If five cards are dealt from a standard deck of 52 cards, what is the probability of being dealt three queens and two kings?

In Exercises 33 – 36, probabilities will be considered for winning the Mega Millions® lottery in 2023. To play Mega Millions, one must select five numbers from 1 – 70 (the white balls) and one additional number from 1 – 25 (the gold Mega Ball®).

33. A person will win $1,000,000 if they match all five numbers for the white lottery balls but don't match the number for the gold Mega Ball. What is the probability of winning this prize? (Hint: For the numerator, consider the number of ways to select all five winning numbers for the white lottery balls and the number of ways to select a non-winning number for the gold Mega Ball.)
34. A person will win $500 if they match four of the five numbers for the white lottery balls and don't match the number for the gold Mega Ball. What is the probability of winning this prize? (Hint: For the numerator, consider the number of ways to select four winning numbers and one non-winning number for the white lottery balls and the number of ways to select a non-winning number for the gold Mega Ball.)
35. A person will win $10,000 if they match four of the five numbers for the white lottery balls and match the number for the gold Mega Ball. What is the probability of winning this prize? (Hint: For the numerator, consider the number of ways to select four winning numbers and one non-winning number for the white lottery balls and the number of ways to select the winning number for the gold Mega Ball.)
36. A person will win $200 if they match three of the five numbers for the white lottery balls and match the number for the gold Mega Ball. What is the probability of winning this prize? (Hint: For the numerator, consider the number of ways to select three winning numbers and two non-winning numbers for the white lottery balls and the number of ways to select the winning number for the gold Mega Ball.)

Section 4.5 Exercise Solutions

1. $\frac{1}{7}$
2. $\frac{1}{30}$
3. $\frac{1}{72}$
4. $\frac{3}{34}$
5. $\frac{5}{126}$
6. $\frac{63}{125}$
7. $\frac{1}{18}$
8. $\frac{1}{210}$
9. $\frac{91}{228}$
10. $\frac{1}{114}$
11. $\frac{12}{65}$
12. $\frac{11}{119}$
13. $\frac{18}{2{,}185}$
14. $\frac{1{,}419}{1{,}960}$
15. $\frac{3}{646}$
16. $\frac{57}{203}$
17. $\frac{9}{272}$
18. $\frac{33{,}649}{118{,}755}$
19. $\frac{450}{8{,}675{,}723}$
20. $\frac{1}{17{,}550}$
21. $\frac{1}{148{,}995}$
22. $\frac{1}{591{,}663{,}800}$
23. $\frac{2{,}632}{15{,}711{,}575}$
24. $\frac{8}{143}$
25. $\frac{1}{5{,}525}$
26. $\frac{1}{270{,}725}$
27. $\frac{2{,}109}{8{,}330}$
28. $\frac{2{,}109}{9{,}520}$
29. $\frac{36}{270{,}725}$
30. $\frac{286}{20{,}825}$
31. $\frac{209}{833}$
32. $\frac{1}{108{,}290}$
33. $\frac{4}{50{,}429{,}225}$
34. $\frac{52}{2{,}017{,}169}$
35. $\frac{13}{12{,}103{,}014}$
36. $\frac{416}{6{,}051{,}507}$

Section 4.6 | Conditional Probability & Probability of Intersections

Objectives

- Identify independent events
- Identify dependent events
- Compute conditional probabilities
- Compute probabilities of intersections of events

Sometimes it is desirable to calculate the probability of events in succession. For example, cards might be repeatedly dealt from a deck or a die might be rolled multiple times. What is the probability of being dealt two aces in a row or rolling a 1 two times in a row? These situations may seem similar, but there is an important difference when calculating the probabilities. Sometimes information is already known about a situation that would change the sample space of an event. For example, suppose a card is dealt and it is known that the card is a face card. How does the probability of getting an ace change considering this new information? In this section, techniques for calculating probabilities such as these are examined.

Conditional Probability

Suppose a person is dealt a card from a standard 52-card deck. Before looking at the card, the person is told by the dealer that it is a face card (jack, queen, or king). How does this knowledge change the probability that a queen was dealt? Before extra information about the card is provided, the probability of being dealt a queen is $\frac{4}{52} = \frac{1}{13}$. However, since it is revealed that a face card was dealt, the sample space changes from 52 cards to 12 cards. Of these 12 face cards, four are queens. Thus, it becomes more likely than originally thought that the card dealt is a queen. This is an example of conditional probability.

DEFINITION: The probability of event B given that event A has occurred is the **conditional probability** of B given A. This is represented symbolically as $P(B|A)$ and is read “The probability of B given A.”

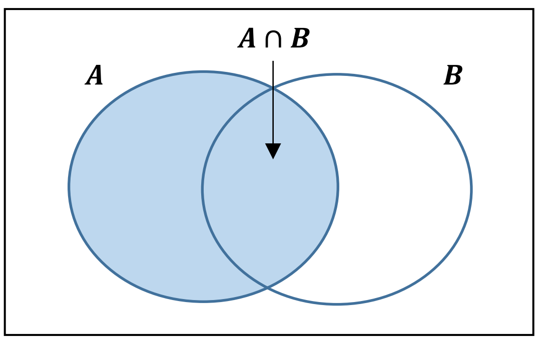
Knowing in advance that a face card is dealt reduces the sample space to 12 cards, four of which are queens. Thus, the probability that the card is a queen given a face card was dealt is:

$$P(\text{queen}|\text{face card}) = \frac{4}{12} = \frac{1}{3}$$

Notice that the probability of being dealt a queen given a face card was dealt is larger than the probability of being dealt a queen when the sample space contains all 52 cards, which is $\frac{1}{13}$.

To develop a formula for calculating conditional probability, a Venn diagram may be used. The Venn diagram in Figure 4.6.1 shows overlapping sets A and B that represent events in an equally likely sample space.

FIGURE 4.6.1



When finding the probability of B given that A has occurred, the sample space will only contain the outcomes of event A (since it is known that A has occurred). This region is highlighted in blue in Figure 4.6.1. If A and B are events in an equally likely sample space, the probability of event B occurring given that event A has occurred can be found using the number of outcomes in the intersection of A and B and the number of outcomes in A.

Calculating Conditional Probability from an Equally Likely Sample Space

If events A and B are from an equally likely sample space, the probability of B given that A has occurred is:

$$P(B|A) = \frac{n(A \cap B)}{n(A)}$$

This is the **conditional probability** of B given A.

➢ **EXAMPLE 4.6.1:** If a six-sided die is rolled twice, what is the probability of rolling a sum of eight given that both numbers are even?

SOLUTION: If A is the event of rolling two even numbers, the set of possible outcomes is given by A $= \{(2,2),(2,4),(2,6),(4,2),(4,4),(4,6),(6,2),(6,4),(6,6)\}$. If B is the event of rolling a sum of eight, the intersection of A and B is $A \cap B = \{(2,6),(4,4),(6,2)\}$.

There are nine outcomes for event A, and there are three outcomes for A ∩ B. Thus, the probability of rolling a sum of eight given that both numbers are even can be calculated as follows:

$$P(\text{sum of eight}|\text{both numbers are even}) = P(B|A) = \frac{n(A \cap B)}{n(A)} = \frac{3}{9} = \frac{1}{3}$$

❖ **YOU TRY IT 4.6.A:** A six-sided die is rolled twice. What is the probability of rolling an odd number each time given that the sum of numbers rolled is less than nine?

➢ **EXAMPLE 4.6.2:** An illness is prevalent in adults within a community. Some with the illness show symptoms, whereas others may have the illness without showing symptoms. The number of people who have the illness in a particular week are listed in the table below.

Age	With symptoms	Without symptoms	Total
18 – 24	64	32	96
25 – 34	45	51	96
35 – 44	35	47	82
45 – 54	57	36	93
55 – 64	86	67	153
65 and above	72	58	130
Total	359	291	650

a. If one person is selected at random, what is the probability that they have symptoms given that they are in the 25 – 34 age category?

b. If one person is selected at random, what is the probability that they are aged 18 – 24 or 65 and above given that they do not have symptoms?

SOLUTION:

a. The probability that someone has symptoms *given* that they are in the 25 – 34 age category is to be calculated. It is important to take note of the *given* statement because it is known the event in this statement has occurred. Thus, the sample space here is the set of those aged 25 – 34 (highlighted in yellow below). The probability of selecting a person with symptoms is found from this sample space.

Age	With symptoms	Without symptoms	Total
18 – 24	64	32	96
25 – 34	45	51	96
35 – 44	35	47	82
45 – 54	57	36	93
55 – 64	86	67	153
65 and above	72	58	130
Total	359	291	650

Number of people aged 25 – 34 with symptoms

Total number of people aged 25 – 34

If A is the event of selecting someone aged 25 – 34 and B is the event of selecting someone with symptoms, the probability of selecting a person with symptoms given that they are aged 25 – 34 is calculated is as follows:

$$P(\text{with symptoms}|\text{aged } 25-34) = P(B|A) = \frac{n(A \cap B)}{n(A)} = \frac{45}{96} = \frac{15}{32}$$

b. The probability that someone aged 18 – 24 or 65+ is selected *given* that they don't have symptoms is to be found. The sample space is restricted to the *given* statement which is people without symptoms (in yellow below). The probability of selecting someone aged 18 – 24 or 65+ is calculated from this sample space.

Age	With symptoms	Without symptoms	Total
18 – 24	64	32	96
25 – 34	45	51	96
35 – 44	35	47	82
45 – 54	57	36	93
55 – 64	86	67	153
65 and above	72	58	130
Total	359	291	650

Number of people 18 – 24 or 65+ without symptoms

Total number of people without symptoms

If A is the event of selecting someone without symptoms and B is the event of selecting someone aged 18 – 24 or 65 and above, the probability of selecting someone aged 18 – 24 or 65 and above given that they don't have symptoms is as follows:

$$P(18-24 \text{ or } 65 \text{ and above}|\text{without symptoms}) = P(B|A) = \frac{n(A \cap B)}{n(A)}$$
$$= \frac{32+58}{291} = \frac{90}{291} = \frac{30}{97}$$

❖ **YOU TRY IT 4.6.B:** Use the data presented in Example 4.6.2 to solve. If one person is randomly selected from this group, what is the probability that they are not aged 35 – 44 given that they have symptoms of the illness?

If the outcomes of a sample space are not equally likely or if it is not possible to count the number of outcomes of an event, then a different formula for computing conditional probability can be used. In this general formula, probabilities are used instead of the number of outcomes of an event. This is summarized in the box below.

Calculating Conditional Probability

If A and B are events, the probability of B given that A has occurred is:

$$P(B|A) = \frac{P(A \cap B)}{P(A)}$$

This is the **conditional probability** of B given A.

➢ **EXAMPLE 4.6.3:** At a school, 60% of students are involved in student clubs, 10% take honors classes, 30% are involved in student clubs and play sports, and 5% are involved in student clubs, take honors classes, and play sports. What is the probability that a student who is involved in student clubs also plays sports?

SOLUTION: Finding the probability that a student involved in clubs also plays sports is the same as finding the probability that a student plays sports *given* that they are involved in clubs. Here, the probability that a randomly selected student is involved in clubs is 60%, or 0.6. The probability that a randomly selected student is involved in clubs and plays sports (the intersection of those sets) is 30%, or 0.3.

Let A represent the set of students involved in clubs and B represent the set of students who play sports. Then, the probability that a student involved in clubs also plays sports is:

$$P(\text{plays sports}|\text{involved in clubs}) = P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{0.3}{0.6} = \frac{1}{2}$$

Note that the other probabilities provided were not relevant to the question at hand and, thus, were not used in the calculation.

❖ **YOU TRY IT 4.6.C:** At a firm in a large city, 78% of workers take public transportation to work, 42% bring their own lunch, 58% take public transportation and live more than 25 miles away from the workplace, and 35% take public transportation, live more than 25 miles away, and bring their own lunch. What is the probability that a worker who takes public transportation to work also lives more than 25 miles away?

Probabilities of Intersections

The formula for calculating conditional probability can be used to calculate the probability of intersections. Recall that for events A and B, the probability of B given A is:

$$\frac{P(A \cap B)}{P(A)} = P(B|A)$$

Multiplying both sides of this equation by $P(A)$ gives the probability of the intersection of A and B.

$$P(A) \cdot \frac{P(A \cap B)}{P(A)} = P(A) \cdot P(B|A)$$

$$P(A \cap B) = P(A) \cdot P(B|A)$$

Calculating the Probability of an Intersection

If A and B are events, the probability of the intersection of A and B is:

$$P(A \cap B) = P(A) \cdot P(B|A)$$

Dependent Events

A game is being played in which two cards are drawn from a standard 52-card deck. It is said that these cards are drawn *without replacement* since the first card is not placed back into the deck before the second card is drawn. If the first card drawn is an ace, then the probability of drawing an ace again becomes smaller. If the first card drawn is not an ace, then the probability of drawing an ace next becomes larger.

Suppose the first card drawn is an ace. Then, there will be only three aces remaining out of 51 cards. Thus, the probability that the second card drawn is an ace is
$\text{P(ace second|ace first)} = \frac{3}{51}$.

On the other hand, suppose the first card drawn is not an ace. Then, there will still be four aces remaining out of 51 cards. Thus, the probability that the second card drawn is an ace is
$\text{P(ace second|not ace first)} = \frac{4}{51}$.

If the cards are instead drawn *with replacement*, then the probability of drawing an ace is $\frac{4}{52}$ each time a card is drawn. Note that this is not equal to either probability given above when cards are drawn without replacement. In the case where cards are drawn without replacement, the first card drawn will influence the possible outcomes for the second card drawn. This is an example of dependent events.

DEFINITION: If A and B are events and the outcome of one has an effect on the outcome of the other, then events A and B are **dependent**. If $P(B|A) \neq P(B)$, then events A and B are dependent.

The formula for finding the probability of the intersection of events is used to find the probability of dependent events. This is summarized in the following box.

Calculating the Probability of Dependent Events

If A and B are dependent events, then

$$P(A \cap B) = P(A) \cdot P(B|A).$$

NOTE: The process of using multiplication to calculate the probability of dependent events can be used with two or more events.

If two cards are drawn without replacement, what is the probability of getting an ace both times? This can be considered the intersection of two events, A and B, where A is drawing an ace first and B is drawing an ace second. The formula for finding the probability of the intersection of events can be used as follows:

$$P(A \cap B) = P(A) \cdot P(B|A)$$

$$P(\text{ace first and then ace second}) = P(\text{ace first}) \cdot P(\text{ace second}|\text{ace first})$$

$$= \frac{4}{52} \cdot \frac{3}{51} = \frac{1}{13} \cdot \frac{1}{17} = \frac{1}{221}$$

This can be confirmed using combinations from Section 4.5. Since order doesn't matter when drawing these two cards, the probability is calculated by dividing the number of combinations of two aces from a group of four by the number of combinations of two cards from 52.

$$P(\text{two aces}) = \frac{C(4,2)}{C(52,2)} = \frac{6}{1{,}326} = \frac{1}{221}$$

Thus, the probability of drawing two aces without replacement is $\frac{1}{221}$.

NOTE: If the cards had been drawn *with replacement*, meaning the first card is replaced in the deck before the second card is drawn, then the events would not be dependent. In this case, the probability of drawing an ace would remain $\frac{4}{52}$ each time a card is drawn.

- **EXAMPLE 4.6.4:** A box contains 12 pieces of identically shaped chocolate. In the box, $\frac{1}{4}$ of the chocolates are filled with caramel, $\frac{1}{4}$ are filled with vanilla cream, $\frac{1}{4}$ are filled with raspberry cream, and $\frac{1}{4}$ are filled with coconut. If three pieces of candy are selected at random without replacement, what is the probability that the first two are filled with coconut and the third is filled with caramel?

  SOLUTION:

  The probability of first selecting a coconut piece is $\frac{3}{12}$, because three of the 12 pieces are filled with coconut. Assuming a coconut piece is selected first, then there would only be two coconut pieces left out of 11 remaining pieces of candy. Assuming coconut pieces are selected first and second, there would still be three pieces filled with caramel out of 10 remaining pieces of candy. Since this is an example of dependent events, this probability can be calculated using multiplication as follows:

$$P(\text{coconut 1st and then coconut 2nd and then caramel 3rd})$$
$$= P(\text{coconut 1st}) \cdot P(\text{coconut 2nd}|\text{coconut 1st}) \cdot P(\text{caramel 3rd}|\text{coconut 1st and 2nd})$$
$$= \frac{3}{12} \cdot \frac{2}{11} \cdot \frac{3}{10} = \frac{3}{220}$$

Thus, the probability of selecting two coconut-filled pieces of candy followed by a caramel piece is $\frac{3}{220}$.

❖ **YOU TRY IT 4.6.D:** A bag contains 50 marbles, all identically shaped. Nine of the marbles are blue, 12 are yellow, 14 are red, and 15 are green. If four marbles are selected at random without replacement, what is the probability of getting two green marbles followed by two yellow marbles?

➢ **EXAMPLE 4.6.6:** The birthday problem is a classic problem that involves finding the probability that at least two people in a group share the same birthday. Suppose there is a group of five students.

a. What is the probability that no members in this group share a birthday?
b. What is the probability that some (at least two) in this group share a birthday?

SOLUTION:

a. The first person's birthday can fall on any of the 365 days of the year. The second person's birthday is limited to 364 possible days out of the 365 days of the year. (The birthday can fall on any day but the first person's birthday.) The third person's birthday is limited to 363 possible days out of the year. (It can fall on any day except the first person's and second person's birthday.) This process continues for all members of the group. Thus, the probability that none in a group of five people share a birthday is calculated as follows:

$$P(\text{five different birthdays}) = \frac{365}{365} \cdot \frac{364}{365} \cdot \frac{363}{365} \cdot \frac{362}{365} \cdot \frac{361}{365} \approx 0.973$$

There is about a 97.3% chance that none in a group of five will share a birthday.

b. In a group of people, either none will share a birthday or at least two will share a birthday. These events are complements of one another, as presented in Section 4.4. Thus, the following is true:

$$P(\text{at least two share a birthday}) = 1 - P(\text{none share a birthday})$$

From part a, the probability that none in a group of five people share a birthday is approximately 0.973. Thus, the probability that at least two share a birthday is as follows:

$$\begin{aligned} P(\text{at least two share a birthday}) &= 1 - P(\text{none share a birthday}) \\ &\approx 1 - 0.973 = 0.027 \end{aligned}$$

There is an approximately 2.7% chance that at least two people in a group of five will share a birthday.

❖ **YOU TRY IT 4.6.E:** In a group of eight people, what is the probability that at least two share a birthday?

Examples with dependent events involve situations in which the outcome of one event influences the outcome of another. This is not always the case when considering the probability of the intersection of events. Sometimes, the outcome of one event does not affect the outcome of another.

Independent Events

Consider the act of repeatedly rolling a six-sided die. What the die lands on the first time it is rolled will have no effect on what it lands on the second time it is rolled. If a 1 is rolled, for instance, then rolling a 1 will not be any more or less likely during subsequent rolls. Each time the die is rolled, the probability of rolling a 1 is $\frac{1}{6}$. Rolling a die multiple times is an example of independent events.

DEFINITION: If A and B are events in which the outcome of one has no effect on the outcome of the other, then events A and B are **independent**. If $P(B|A) = P(B)$, then events A and B are independent.

If a six-sided die is rolled twice, what is the probability of rolling a 1 both times? In other words, what is the probability of the intersection of two events, A and B, where A is rolling a 1 first and B is rolling a 1 second given that a 1 is rolled the first time? Just like with dependent events, the formula for finding the probability of the intersection of events can be used as follows:

$$P(A \cap B) = P(A) \cdot P(B|A)$$
$$P(1 \text{ on first roll and } 1 \text{ on second roll}) =$$
$$P(1 \text{ on first roll}) \cdot P(1 \text{ on second roll}|1 \text{ on first roll}) = \frac{1}{6} \cdot \frac{1}{6} = \frac{1}{36}$$

Notice that the probability of rolling a 1 on the second toss given that a 1 is rolled on the first toss is still $\frac{1}{6}$. The outcome of the first roll has no effect on the outcome of the second. This result can be verified by considering the sample space. In the case of rolling a die two times, the sample space consists of 36 possible outcomes as shown in Figure 4.6.2.

FIGURE 4.6.2

		First Roll					
		1	**2**	**3**	**4**	**5**	**6**
Second Roll	**1**	(1, 1)	(2, 1)	(3, 1)	(4, 1)	(5, 1)	(6, 1)
	2	(1, 2)	(2, 2)	(3, 2)	(4, 2)	(5, 2)	(6, 2)
	3	(1, 3)	(2, 3)	(3, 3)	(4, 3)	(5, 3)	(6, 3)
	4	(1, 4)	(2, 4)	(3, 4)	(4, 4)	(5, 4)	(6, 4)
	5	(1, 5)	(2, 5)	(3, 5)	(4, 5)	(5, 5)	(6, 5)
	6	(1, 6)	(2, 6)	(3, 6)	(4, 6)	(5, 6)	(6, 6)

The probability of rolling a 1 on both the first and second roll is $\frac{1}{36}$. This is because out of the 36 outcomes shown in Figure 4.6.2, only one outcome involves rolling a 1 both times.

Even though it is not necessary to do so, a variation of the formula for finding the probability of the intersection of events can be written for independent events. It is reflected in this formula that for independent events A and B, $P(B|A) = P(B)$.

Calculating the Probability of Independent Events

If A and B are independent events, then

$$P(A \cap B) = P(A) \cdot P(B),$$

where $P(B|A) = P(B)$.

NOTE: Similar to dependent events, the process of using multiplication to calculate the probability of independent events can be used with two or more events.

- **EXAMPLE 4.6.7:** If a coin is flipped three times in a row, what is the probability that it lands on heads all three times?

  SOLUTION: Since flipping a coin repeatedly is an example of independent events, this probability can be calculated using multiplication as follows:

$$P(\text{heads three times}) = P(\text{heads}) \cdot P(\text{heads}) \cdot P(\text{heads}) = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8}$$

  This can be verified by considering the sample space for flipping a coin three times. If a coin landing on heads is represented by H and tails is represented by T, then this sample space is as follows:

$$\{\text{HHH}, \text{HHT}, \text{HTH}, \text{THH}, \text{TTH}, \text{THT}, \text{HTT}, \text{TTT}\}$$

  The first outcome listed represents a coin landing on heads three times in a row. This outcome is one out of the eight outcomes of the sample space. Thus, the probability of landing on heads three times in a row is $\frac{1}{8}$.

- **YOU TRY IT 4.6.F:** A standard six-sided die is rolled five times in a row. What is the probability of landing on an even number all five times?

Recall from Section 4.4, an event not happening and an event happening at least once are complements. So, for event A, the probability that event A happens at least once is equal to one minus the probability that event A does not happen.

- **EXAMPLE 4.6.8:** Carlito has a math final at 8 am. He is worried he may oversleep, so he sets three alarms. There is a 0.4% chance that a single alarm will malfunction. Assume all three alarms have the same probability of malfunctioning. What is the probability that at least one of his alarms will ring?

  SOLUTION: This is an example of independent events since the functionality of one alarm has no impact on the functionality of another. The probability that one single alarm will not ring is 0.4%, or 0.004. So, the probability that all three alarms don't ring is given by $0.004 \cdot 0.004 \cdot 0.004 = 0.004^3$. Then, the probability that at least one of his alarms will ring is given by $1 - 0.004^3 = 0.999999936$.

❖ **YOU TRY IT 4.6.G:** Suppose the probability that a tornado hits a region in February in any single year is $\frac{2}{99}$. What is the probability that the region is hit with a tornado in February at least once in the next four years?

Quick Review

- The probability of event B given that event A has occurred is the **conditional probability** of B given A. Symbolically this is represented as $P(B|A)$.

  The conditional probability of event B given event A is found by:

$$P(B|A) = \frac{P(A \cap B)}{P(A)}$$

- If A and B are events and the outcome of one has an effect on the outcome of the other, then events A and B are **dependent**. If $P(B|A) \neq P(B)$, then A and B are dependent.

  If A and B are dependent events, then $P(A \cap B) = P(A) \cdot P(B|A)$.

- If A and B are events and the outcome of one does not have an effect on the outcome of the other, then events A and B are **independent**. If $P(B|A) = P(B)$, then A and B are independent.

  If A and B are independent events, then $P(A \cap B) = P(A) \cdot P(B)$.

YOU TRY IT 4.6.A SOLUTION: $\frac{4}{13}$

YOU TRY IT 4.6.B SOLUTION: $\frac{324}{359}$

YOU TRY IT 4.6.C SOLUTION: $\frac{29}{39}$

YOU TRY IT 4.6.D SOLUTION: $\frac{33}{6{,}580}$

YOU TRY IT 4.6.E SOLUTION: Approximately 0.074

YOU TRY IT 4.6.F SOLUTION: $\frac{1}{32}$

YOU TRY IT 4.6.G SOLUTION: Approximately 0.07839216

Section 4.6 Exercises

In Exercises 1 – 4, two six-sided dice are rolled.

1. Find the probability that a sum of 8 is rolled, given that both numbers rolled are even.
2. Find the probability that one die displays and even number and the other displays an odd number, given that a sum less than 9 is rolled.
3. Find the probability that an absolute difference of two is rolled given that both numbers are odd.
4. Find the probability that at least one die displays a prime number given that a sum of seven is rolled.

In Exercises 5 – 8, one card is dealt from a standard deck of 52 cards. Find each probability.

5. $\mathrm{P}(\text{face card} \mid \text{black card})$
6. $\mathrm{P}(\text{card numbered 2 through 10} \mid \text{heart})$
7. $\mathrm{P}(\text{ace} \mid \text{not a club})$
8. $\mathrm{P}(\text{not face card} \mid \text{diamond})$

In Exercises 9 – 12, a group of students are getting ready to play flag football. A bag contains 20 jerseys for the students to wear. Ten of the jerseys are orange and have even numbers printed on them from 2 – 20. Ten of the jerseys are blue and have odd numbers printed on them from 1 – 19. A student selects a jersey at random.

9. What is the probability of getting a jersey with a number greater than 7 printed on it given that the jersey is orange?
10. What is the probability of getting a jersey with a double-digit number printed on it given that the jersey is blue?
11. What is the probability of getting an orange jersey given that the jersey has an odd number printed on it?
12. What is the probability of getting a blue jersey given that the jersey has a single-digit number printed on it?

In Exercises 13 – 18, use the data provided to find each probability. A housing commission analyzed housing situations for those aged 18 and older in a community. The results are summarized in the table below.

Age	Own	Rent	Other	Total
18 – 24	153	579	456	1,188
25 – 34	675	543	324	1,542
35 – 44	756	409	125	1,290
45 – 54	897	419	96	1,412
55 – 64	835	356	123	1,314
65 and above	643	234	259	1,136
Total	3,959	2,540	1,383	7,882

13. If one person is selected at random, what is the probability that they rent given that they are in the 35 – 44 age group?
14. If one person is selected at random, what is the probability that they are 65 and above given that they are in the other category?
15. If one person is selected at random, what is the probability that they are in the 18 – 24 or 25 – 34 age group given that they own their residence?
16. If one person is selected at random, what is the probability that they rent given that they are in the 35 – 44 or 55 – 64 age group?
17. If one person is selected at random, what is the probability that they own their own residence given they are not in the 18 – 24 age group?
18. Which is more likely?
    a. A person rents their residence given they are in the 65 and above age group.
    b. A person is in the 65 and above age group given they rent their residence.

In Exercises 19 – 22, find each probability.

19. In a group of high school seniors, 65% were planning to enroll in college, 46% were planning to get a part-time job, and 39% were planning to enroll in college and get a part-time job. What is the probability that a student from this group who plans to enroll in college also plans to get a part-time job?
20. At a local coffee shop, 63% will purchase a hot drink, 22% will purchase a cold drink, and 27% will purchase a hot drink and a food item. What is the probability that a customer who purchases a hot drink will also purchase a food item?
21. In a group of teachers, 32% have completed a doctoral program and 58% are enrolled in an advanced degree program. It is also known that 18% have completed their doctoral program and continue to enroll in an advanced degree program. What is the probability that a teacher selected at random is enrolled in an advanced degree program given they have completed a doctoral program?

22. A running magazine found that 83% of subscribers have completed a 5K race, 63% have completed a half-marathon, and 31% have completed a marathon. The magazine also found that 25% of subscribers have completed both a half-marathon and a marathon, while 52% have completed both a 5K and a half-marathon. If one magazine subscriber is selected at random, what is the probability that the person has completed a 5K given they have completed a half-marathon?

In Exercises 23 – 28, a six-sided die is rolled multiple times in succession.

23. If the die is rolled three times, what is the probability of landing on a 6 each time?
24. If the die is rolled three times, what is the probability of landing on an odd number greater that 1 each time?
25. If the die is rolled four times, what is the probability of landing on numbers that alternate between even and odd, starting with an even number?
26. If a die is rolled five times, what is the probability of landing on a different number each time?
27. If the die is rolled three times, what is the probability of landing on a number less than 3 the first time, a number greater than 3 the second time, and a 3 the third time?
28. If the die is rolled four times, what is the probability of landing on an even number first, an odd number second, a prime number third, and 6 last?

In Exercises 29 – 32, a game involves a spinner containing eight regions of equal size. Two of the regions are blue, two are pink, two are green, and two are purple.

29. If the pointer is spun twice in a row, what is the probability that it lands on blue and then pink?
30. If the pointer is spun twice in a row, what is the probability that it lands on purple both times?
31. If the pointer is spun three times in a row, what is the probability that it lands on a color other than green each time?
32. If the pointer is spun four times in a row, what is the probability that it lands on a color other than purple each time?

In Exercises 33 – 36, a coin is flipped and a six-sided die is rolled.

33. What is the probability of landing on tails and rolling a 5
34. What is the probability of landing on heads and rolling an even number?
35. What is the probability of landing on heads and rolling a prime number?
36. What is the probability of landing on heads and rolling a number greater than four?

In Exercises 37 – 42, a standard deck of 52 cards is used. Assume that cards are dealt without replacement.

37. If four cards are dealt, what is the probability that they are all face cards?
38. If four cards are dealt, what is the probability that they are all diamonds?
39. If five cards are dealt, what is the probability that none are aces?
40. If five cards are dealt, what is the probability that none are face cards?
41. If six cards are dealt, what is the probability that they are all 5s or red 9s?
42. If five cards are dealt, what is the probability that each one has a different value?

In Exercises 43 – 46, determine if the events are dependent or independent. Then, calculate the probability.

43. An envelope contains five $50 bills, eight $20 bills, 15 $5 bills, and 27 $1 bills. If three bills are randomly selected without replacement, what is the probability of getting a $50 bill, followed by another $50 bill, and then a $1 bill?
44. A game requires players to use a spinner containing ten regions of the same size, numbered 1 – 10. If the spinner is used by four players in a row, what is the probability that it lands on a number less than three all four times?
45. A game is played in which four cards are drawn from a standard 52-card deck with replacement. What is the probability that the cards that are drawn alternate between face cards and non-face cards? (Hint: There are two ways this can happen, either starting with a face card or starting with a non-face card.)
46. A table at a buffet contains 12 cupcakes, eight slices of cake, and seven cookies. If three people select a dessert one at a time, what is the probability they all choose cupcakes?

In Exercises 47 – 52, find each probability rounded to five decimal places.

47. a. The probability that one light bulb doesn't work is 0.168. What is the probability that none of the light bulbs work in a box of four?
    b. What is the probability that at least one light bulb works?
48. a. The probability that a battery doesn't work is 0.35. What is the probability that none of the batteries work in a box of 10?
    b. What is the probability that at least one of the batteries works?
49. a. The probability that a certain region will be hit by a hurricane in any given year is 33%. What is the probability that the region is not hit by a hurricane in the next ten years?
    b. What is the probability that the region is hit by a hurricane at least once in the next ten years?

50. a. The probability that a river floods in any given year is 57%. What is the probability that the river doesn't flood in the next six years?
    b. What is the probability that the river floods at least once in the next six years?
51. a. If three cards are dealt at random from a 52-card deck without replacement, what is the probability that no kings are dealt?
    b. If three cards are dealt at random from a 52-card deck without replacement, what is the probability that at least one king is dealt?
52. a. If four cards are dealt at random from a 52-card deck, what is the probability that no 8s or 10s are dealt?
    b. If four cards are dealt at random from a 52-card deck, what is the probability that at least one 8 or 10 is dealt?

In Exercises 53 – 56, refer to Example 4.6.6 for a description of the birthday problem. In Exercises 53 and 54, round solutions to the nearest thousandth.

53. a. In a group of 10 students, what is the probability that no members share a birthday?
    b. What is the probability that some (at least two) in this group share a birthday?
54. a. In a group of 15 students, what is the probability that no members share a birthday?
    b. What is the probability that same (at least two) in this group share a birthday?
55. How many people would need to be in a group so that the probability that some (at least two) share a birthday is at least 40%?
56. How many people would need to be in a group so that the probability that some (at least two) share a birthday is at least 50%?

Review

In Exercises 57 – 64, use the data provided to find each probability. In the table, the number of active duty and reserve U.S. military force personnel in 2021, by service branch and reserve component, is listed[22].

Military Branch	Active Duty	Reserves	Total
Air Force	328,888	70,570	399,458
Army	482,416	184,358	666,774
Marine Corps	179,378	35,240	214,618
Navy	343,223	57,632	400,855
Total	1,333,905	347,800	1,681,705

[22] Statista Research Department. (2023, January 31). U.S. military force numbers 2021, by service branch and reserve component. https://www.statista.com/statistics/232330/us-military-force-numbers-by-service-branch-and-reserve-component/

57. If a person is randomly selected from this group, what is the probability that they are not a reserve member of any branch?
58. If a person is randomly selected from this group, what is the probability that they are not in the Marine Corps?
59. If a person is randomly selected from this group, what is the probability that they are an active-duty member of the Air Force or Marines?
60. If a person is randomly selected from this group, what is the probability that they are a reserve member of the Army or Navy?
61. If a person is randomly selected from this group, what is the probability that they are an active-duty member of any branch or in the Marine Corps?
62. If a person is randomly selected from this group, what is the probability that they are in the reserves of any branch or in the Army?
63. If a person is randomly selected from this group what is the probability that they are in the Navy given that they are in the reserves?
64. If a person is randomly selected from this group, what is the probability that they are an active-duty military personnel given that they are in the Marine Corps?

Section 4.6 Exercise Solutions

1. $\frac{1}{3}$
2. $\frac{6}{13}$
3. $\frac{4}{9}$
4. $\frac{2}{3}$
5. $\frac{3}{13}$
6. $\frac{9}{13}$
7. $\frac{1}{13}$
8. $\frac{10}{13}$
9. $\frac{7}{10}$
10. $\frac{1}{2}$
11. 0
12. $\frac{5}{9}$
13. $\frac{409}{1,290}$
14. $\frac{259}{1,383}$
15. $\frac{828}{3,959}$
16. $\frac{255}{868}$
17. $\frac{1,903}{3,347}$
18. a. $\frac{117}{568}$ b. $\frac{117}{1,270}$
    Option a is more likely.
19. $\frac{3}{5}$
20. $\frac{3}{7}$
21. $\frac{9}{16}$
22. $\frac{52}{63}$
23. $\frac{1}{216}$
24. $\frac{1}{27}$
25. $\frac{1}{16}$
26. $\frac{5}{54}$
27. $\frac{1}{36}$
28. $\frac{1}{48}$
29. $\frac{1}{16}$
30. $\frac{1}{16}$
31. $\frac{27}{64}$
32. $\frac{81}{256}$
33. $\frac{1}{12}$
34. $\frac{1}{4}$
35. $\frac{1}{4}$
36. $\frac{1}{6}$
37. $\frac{99}{54,145}$
38. $\frac{11}{4,165}$
39. $\frac{35,673}{54,145}$
40. $\frac{2,109}{8,330}$
41. $\frac{1}{20,358,520}$
42. $\frac{52}{52}\cdot\frac{48}{51}\cdot\frac{44}{50}\cdot\frac{40}{49}\cdot\frac{36}{48}=$
    $\frac{C(13,5)\cdot 4^5}{C(52,5)}=\frac{2,112}{4,165}$
43. Dependent; $\frac{2}{583}$
44. Independent; $\frac{1}{625}$
45. Independent; $\frac{1,800}{28,561}$
46. Dependent; $\frac{44}{585}$
47. a. 0.00080
    b. 0.99920
48. a. 0.000028
    b. 0.999972
49. a. 0.01823
    b. 0.98177
50. a. 0.00632
    b. 0.99368
51. a. 0.78262
    b. 0.21738
52. a. 0.50144
    b. 0.49856
53. a. 0.883
    b. 0.117
54. a. 0.747
    b. 0.253
55. 20 people
56. 23 people
57. $\frac{266,781}{336,341}$
58. $\frac{1,467,087}{1,681,705}$
59. $\frac{508,266}{1,681,705}$
60. $\frac{48,398}{336,341}$
61. $\frac{273,829}{336,341}$
62. $\frac{830,216}{1,681,705}$
63. $\frac{7,204}{43,475}$
64. $\frac{89,689}{107,309}$

Section 4.7 | Odds & Expected Value

Objectives

- Calculate odds
- Convert between odds and probability
- Calculate expected value
- Solve problems involving expected value

When discussing the likelihood of an event, some people will use the term "odds." For example, if the odds of winning the Mega Millions® lottery was advertised as 1 in 302 million[23], how should this be interpreted? First, it can be said that winning the large lottery prize is not a common occurrence. More specifically, with the odds of winning stated as 1 in 302 million, only one lottery ticket will win the large prize with 302 million other tickets not winning.

Odds and Probability

Odds can be expressed in favor of an event or against an event. What are the odds of winning the NCAA Frozen Four hockey tournament? The odds of a team winning are tied to many factors such as which teams are competing and how the players are performing at that point in their season. What are the odds against winning tickets to Lollapalooza on a radio program? The odds of winning concert tickets, just as the odds against winning concert tickets, would be dependent on the number of people who entered the contest.

The odds in favor of an event and the odds against an event are related to the probability or likelihood of that event.

[23] Jay, P. (2023, January 10). *The Math of Mega Millions: How to tilt the odds in your favor*. Davidson. Retrieved April 24, 2023, from https://www.davidson.edu/news/2023/01/10/math-mega-millions-how-tilt-odds-your-favor

DEFINITION: The **odds in favor** of an event E are the ratio of the probability of event E happening compared to the probability of event E not happening. The **odds in favor** of event E can also be expressed as the number of times event E occurred compared to the number of times event E did not occur.

DEFINITION: The **odds against** event E are the ratio of the probability of event E not happening compared to the probability of event E happening. The **odds against** event E can also be expressed as the number of times event E did not occur compared to the number of times event E did occur.

Odds can be expressed in several ways. Common ways odds are expressed include, but are not limited to, separating the two values using a fraction bar, a colon, the word "in", or the word "to". It is important to remember that when representing odds two values are always being compared. For example, the odds of 3 to 7 could be written as follows:

$\frac{3}{7}$ 3:7 3 in 7 3 to 7

Calculating Odds

$$\text{Given event E, the odds in favor of event E} = \frac{\mathrm{P(E)}}{\mathrm{P(not\ E)}} = \frac{\mathrm{n(E)}}{\mathrm{n(not\ E)}}.$$

$$\text{Given event E, the odds against event E} = \frac{\mathrm{P(not\ E)}}{\mathrm{P(E)}} = \frac{\mathrm{n(not\ E)}}{\mathrm{n(E)}}.$$

- **EXAMPLE 4.7.1:** In 2022, the Chicago Sky women's basketball team had 26 wins and 10 losses during conference play. Find the odds in favor of the Chicago Sky winning a game. Also, state the odds against the Chicago Sky winning a game.

  SOLUTION: The odds in favor of the Chicago Sky winning are the ratio of the probability of a win compared to the probability of the team losing (not winning). To calculate the probability, the total number of games is needed. The Chicago Sky had 26 wins and 10 losses, so there was a total of 36 games played. It is helpful to organize this information in a table separating the information into two distinct categories: wins and losses.

Summary of 2022 Chicago Sky conference games	
Number of wins	26
Number of losses	10
Total	**36**

26 wins + 10 losses
= 36 games total

The probability of the Chicago Sky winning is the number of wins divided by the number of games or $\frac{26}{36} = \frac{13}{18}$. The probability of the Chicago Sky losing is the number of losses divided by the number of games or $\frac{10}{36} = \frac{5}{18}$.

The odds in favor of the Chicago Sky winning are calculated as the ratio below.

$$\text{Odds in favor of a win} = \frac{\text{probability of winning}}{\text{probability of losing}} =$$

$$\frac{\frac{26}{36}}{\frac{10}{36}} = \frac{26}{36} \div \frac{10}{36} = \frac{26}{36} \cdot \frac{36}{10} = \frac{26}{10} = \frac{13}{5} \text{ or } 13:5$$

This odds in favor of the Chicago Sky winning could also be calculated as the number of wins compared to the number of losses.

$$\text{Odds in favor of a win} = \frac{\text{number of win}}{\text{number of losses}} = \frac{26}{10} = \frac{13}{5} \text{ or } 13:5$$

The odds against the Chicago Sky winning are calculated as the ratio below.

$$\text{Odds against a win} = \frac{\text{probability of losing}}{\text{probability of winning}} =$$

$$\frac{\frac{10}{36}}{\frac{26}{36}} = \frac{10}{36} \div \frac{26}{36} = \frac{10}{36} \cdot \frac{36}{26} = \frac{10}{26} = \frac{5}{13} \text{ or } 5:13$$

The odds against the Chicago Sky winning could also be calculated as the number of losses compared to the number wins.

$$\text{Odds against a win} = \frac{\text{number of losses}}{\text{number of wins}} = \frac{10}{26} = \frac{5}{13} \text{ or } 5:13$$

The work for converting between odds and probability can be streamlined as shown in the next example.

➢ **EXAMPLE 4.7.2:** The high school scholar bowl team won 17 of their last 25 matches.

a. Find the odds in favor of the high school scholar bowl team winning their match.
b. Find the odds against the high school scholar bowl team winning their match.
c. Find the probability of the high school scholar bowl team winning their match.
d. Find the probability of the high school scholar bowl team losing their match.

SOLUTION: It is helpful to organize the information provided in a table separating the information into two distinct categories: wins and losses.

Summary of scholar bowl team matches	
Number of wins	17
Number of losses	8
Total	**25**

25 total games – 17 wins = 8 losses

a. To find the odds in favor of the high school scholar bowl team winning their match, create the ratio of the number of wins compared to the number of losses.

$$\text{Odds in favor of a win} = \frac{\text{number of wins}}{\text{number of losses}} = \frac{17}{8} \text{ or } 17:8$$

b. To find the odds against the high school scholar bowl team winning their match, create the ratio of the number of losses compared to the number of wins.

$$\text{Odds against a win} = \frac{\text{number of losses}}{\text{number of wins}} = \frac{8}{17} \text{ or } 8:17$$

c. To find the probability of the high school scholar bowl team winning their match, create the ratio of the number of wins compared to the total number of matches.

$$\text{P(win)} = \frac{\text{number of wins}}{\text{total number of matches}} = \frac{17}{25}$$

d. To find the probability of the high school scholar bowl team losing their match, create the ratio of the number of losses compared to the total number of matches.

$$\text{P(loss)} = \frac{\text{number of losses}}{\text{total number of matches}} = \frac{8}{25}$$

➢ **EXAMPLE 4.7.3:** It is known that 19% of Boston Marathon races have had at least 0.01 inches of rainfall.

a. Find the odds in favor of rain during the Boston Marathon.
b. Find the odds against rain during the Boston Marathon.
c. Find the probability of rain during the Boston Marathon.
d. Find the probability against rain during the Boston Marathon.

SOLUTION: It is helpful to organize the information provided in a table separating the information into two distinct categories: rain and not rain. In this case, 19% could be interpreted as 19 out of 100.

Summary of information provided	
Number of times rain occurred	19
Number of times rain did not occur	81
Total	**100**

100 total – 19 rain = 81 not rain

a. To find the odds in favor of rain during the Boston Marathon, create the ratio of the number of times rain occurred compared to the number of times rain did not occur.

$$\text{Odds in favor of a rain} = \frac{\text{number of times rain occurred}}{\text{number of times rain did not occur}} = \frac{19}{81} \text{ or } 19{:}81$$

b. To find the odds against rain during the Boston Marathon, create the ratio of the number of times rain did not occur compared to the number of times rain did occur.

$$\text{Odds against rain} = \frac{\text{number of times rain did not occur}}{\text{number of times rain occurred}} = \frac{81}{19} \text{ or } 81{:}19$$

c. To find the probability of rain during the Boston Marathon, create the ratio of the number of times rain occurred compared to the total number of observations.

$$\text{P(rain)} = \frac{\text{number of times rain occurred}}{\text{total number of observations}} = \frac{19}{100} \text{ or 19\% (which was given)}$$

d. To find the probability that rain will not occur during the Boston Marathon, create the ratio of the number of times rain did not occur compared to the total number of observations.

$$\text{P(not rain)} = \frac{\text{number of times rain did not occur}}{\text{total number of observations}} = \frac{81}{100} \text{ or 81\%}$$

❖ **YOU TRY IT 4.7.A:** Several baseball players competed during a homerun hitting competition where 840 balls were pitched to the players. If 292 homeruns were hit during the competition,

a. Find the odds in favor of a homerun ball being hit during this competition.
b. Find the odds against a homerun ball being hit during this competition.
c. Find the probability of a homerun ball being hit during this competition.
d. Find the probability against a homerun ball being hit during this competition.

➢ **EXAMPLE 4.7.4:** The odds of being struck by lightning are 1 in 15,300.

a. Find the odds in favor of being struck by lightning.
b. Find the odds against being struck by lightning.
c. Find the probability of being struck by lightning.
d. Find the probability of not being struck by lightning.

SOLUTION: It is helpful to organize the information provided in a table separating the information into two distinct categories: being hit by lightning and not being hit by lightning. In this case, the odds of being struck by lightning are provided. This indicates that the 1 represents the number of times a person was struck by lightning while the 15,300 represents the number of times a person was not struck by lightning.

Summary of information provided	
Number of times a person was struck by lightning	1
Number of times a person was not struck by lightning	15,300
Total	**15,301**

a. The odds in favor of being struck by lightning are provided.

$$\text{Odds in favor of being struck by lightning} = \frac{\text{number of times struck by lightning}}{\text{number of times not struck by lightning}}$$

$$= \frac{1}{15{,}300} \text{ or } 1:15{,}300 \text{ (which was given)}$$

b. To find the odds against being struck by lightning, create the ratio of the number of times a person was not struck by lightning compared to the number of times a person was struck by lightning.

$$\text{Odds against being struck by lightning} = \frac{\text{number of times not struck by lightning}}{\text{number of times struck by lightning}}$$

$$= \frac{15{,}300}{1} \text{ or } 15{,}300:1$$

c. To find the probability of being struck by lightning, create the ratio of the number of times a person was struck by lightning compared to the total number of observations.

$$\text{P(struck by lightning)} = \frac{\text{number of times struck by lightning}}{\text{total number of observations}} = \frac{1}{15{,}301} \text{ or } 0.0065\%$$

d. To find the probability of not being struck by lightning, create the ratio of the number of times a person was not struck by lightning compared to the total number of observations.

$$\text{P(not struck by lightning)} = \frac{\text{number of times not struck by lightning}}{\text{total number of observations}}$$

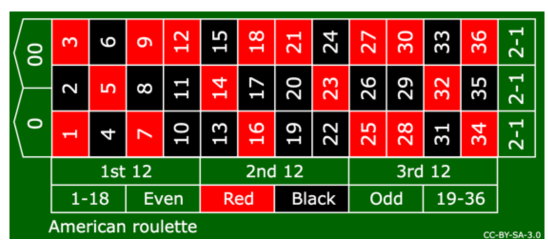
$$= \frac{15{,}300}{15{,}301} \text{ or } 99.99\%$$

NOTE: In part b of Example 4.7.4, the odds of $\frac{15{,}300}{1}$ or 15,300: 1 does not simplify to 15,300 because odds are expressed as a ratio comparing two values. In this case, the 15,300 in the numerator refers to the number of times a person was not struck by lightning and the 1 in the denominator refers to the number of times a person was struck by lightning.

Probability and Expected Value

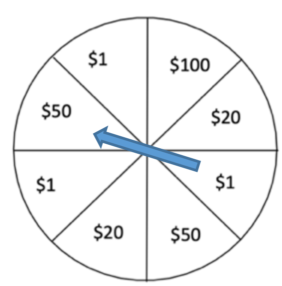
Probability calculations can be used to express the outcome expected from an experiment. In many cases, the phrase "in the long run" or "average" is used to describe expected value. The expected value can be thought of as the average outcome when an experiment is repeated over many trials.

For example, suppose Eric placed a $100 bet on a red number on a roulette table (image on the right). What are his expected winnings? Would his expected winnings be any different if Eric bet on the red number 18?



"American roulette layout" by Betzaar is licensed under CC BY-SA 3.0. Roulette wheel image deleted.

As another example, consider the spinner shown on the right. If the pointer is spun many times in a row and the results are recorded, what is the average outcome?



These questions can be answered using probability to calculate the expected value.

DEFINITION: The **expected value** is a number that represents the anticipated long-term average of the outcomes if an experiment is repeated over many trials.

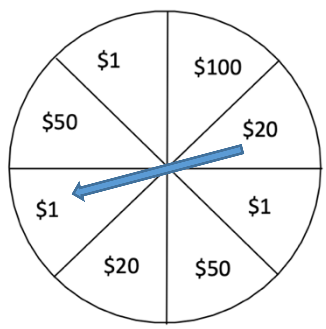
To calculate expected value, multiply the probability of each outcome by the value of that outcome and then sum the results.

Calculating Expected Value

Suppose an event had outcomes $r_1, r_2, r_3, \ldots, r_n$ with respective values of $v_1, v_2, v_3, \ldots, v_n$, then the expected value of the event can be calculated as

Expected Value = $P(r_1) \cdot v_1 + P(r_2) \cdot v_2 + P(r_3) \cdot v_3 + \ldots + P(r_n) \cdot v_n$.

- **EXAMPLE 4.7.5:** At the local fair, Jazmyn was asked to spin the spinner shown below. She would win the amount shown when the pointer stopped. What is the expected value for this game?



SOLUTION: First, it is important to know the probability of each possible outcome. Since Jazmyn could win \$100, \$50, \$20 or \$1, the probability of spinning the \$100 prize is $\frac{1}{8}$, the probability of spinning the \$50 prize is $\frac{2}{8} = \frac{1}{4}$, the probability of spinning the \$20 prize is also $\frac{2}{8} = \frac{1}{4}$, and the probability of spinning the \$1 prize is $\frac{3}{8}$. Notice that the probability of $\frac{2}{8}$ could be simplified; however, it is sometimes useful to leave fractions with their original denominators to quickly see that the sum of all probabilities equals one, such as $\frac{1}{8} + \frac{2}{8} + \frac{2}{8} + \frac{3}{8} = \frac{8}{8} = 1$.

To find the expected value, multiply the probability of an outcome by the value of that outcome. Then find the sum of these products.

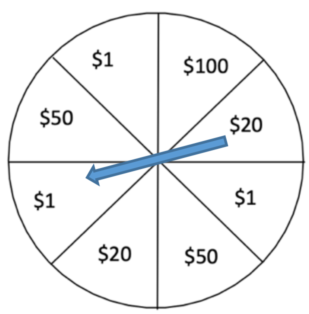
$$\mathrm{P(spin\ \$100)} \cdot \$100 + \mathrm{P(spin\ \$50)} \cdot \$50 + \mathrm{P(spin\ \$20)} \cdot \$20 + \mathrm{P(spin\ \$1)} \cdot \$1$$
$$= \frac{1}{8} \cdot \$100 + \frac{2}{8} \cdot \$50 + \frac{2}{8} \cdot \$20 + \frac{3}{8} \cdot \$1$$

$$= \frac{\$100}{8} + \frac{\$100}{8} + \frac{\$40}{8} + \frac{\$3}{8}$$
$$= \frac{\$243}{8}$$
$$= \$30.375$$
$$\approx \$30.38$$

Therefore, in the long run, the expected value is $30.38.

NOTE: The expected value does not represent the expected payout from one spin but rather what payout is expected on average after repeatedly playing the spinner game over a long period of time.

➢ **EXAMPLE 4.7.6:** Suppose Jazmyn needed to pay $5 to play the spinner game in Example 4.7.5. How would that affect the expected value?



SOLUTION: The probabilities for each outcome have not changed. However, the amount Jazmyn wins is $5 less than the amount the spinner lands on since she pays $5 to play the game. For example, if the spinner lands on $100, then Jazmyn will take home $95. If the spinner lands on $1, then Jazmyn will lose $4, which is represented using a negative number.

To find the expected value, multiply the probability of an outcome by the value of that outcome. Then find the sum of these products.

$$\mathrm{P(spin\ \$100)} \cdot \$95 + \mathrm{P(spin\ \$50)} \cdot \$45 + \mathrm{P(spin\ \$20)} \cdot \$15 + \mathrm{P(spin\ \$1)} \cdot -\$4$$
$$= \frac{1}{8} \cdot \$95 \ + \ \frac{2}{8} \cdot \$45 \ + \ \frac{2}{8} \cdot \$15 \ + \ \frac{3}{8} \cdot -\$4$$
$$= \frac{\$95}{8} + \frac{\$90}{8} + \frac{\$30}{8} + \frac{-\$12}{8}$$

$$= \frac{\$203}{8}$$
$$= \$25.375$$
$$\approx \$25.38$$

Therefore, in the long run, the expected value is $25.38.

This is the same as taking the answer from Example 4.7.5 and subtracting the $5 that Jazmyn paid to play the game.

- **EXAMPLE 4.7.7:** Ondro challenged his friend Kiara to a dice game. If Kiara rolled two dice and the sum was greater than nine, she would win $5. However, if the sum was less than or equal to nine, Kiara would have to pay $5. What is the expected value of this game? Who does the game favor? It might be helpful to use the table of outcomes for rolling two dice found in Figure 4.3.3.

SOLUTION: First, it is important to know the probability of each possible outcome. In this case, it is possible to put the outcomes into two categories: rolling a sum greater than nine or rolling a sum less than or equal to nine. The probability of rolling a sum greater than nine is $\frac{6}{36}$. The probability of rolling a sum less than or equal to nine is $\frac{30}{36}$.

To find the expected value, multiply the probability of an outcome by the value of that outcome. Then find the sum of these products.

$$\mathrm{P(roll\ sum\ greater\ than\ nine)} * \$5 + \mathrm{P(roll\ sum\ less\ than\ or\ equal\ to\ nine)} * -\$5$$
$$= \frac{6}{36} \cdot \$5 + \frac{30}{36} \cdot -\$5$$
$$= \frac{\$30}{36} + \frac{-\$150}{36}$$
$$-\frac{\$120}{36}$$
$$\approx -\$3.33$$

Therefore, in the long run, the expected value is -$3.33. Kiara would expect to lose $3.33 in the long run if repeatedly playing this game, so the game favors Ondro.

❖ **YOU TRY IT 4.7.B:** Emma decided to play an envelope game where she is allowed to choose one envelope and keep the prize inside the envelope. In one envelope, the prize listed is a television worth \$350. In another envelope, she will win ear buds valued at \$200. A third envelope contains a \$50 gift card. Two envelopes contain \$20 cash while six envelopes are empty. Emma must pay \$25 to play the game. What is the expected value of the game? Is the game in Emma's favor?

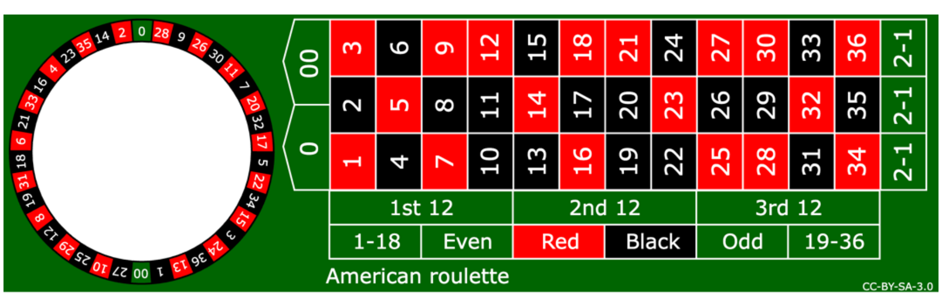
Sports betting or betting on games played in casinos are also times when expected value can be helpful. For the purposes of this textbook, the specific odds or amount that can be won on a game will be provided.

For the following scenario, consider that it costs \$1 to play a game. When a bet pays 1 to 1, it indicates that a winner is paid \$1 to replace the amount for the initial bet and an additional \$1 for a total of \$2. At the same time, a losing bet would receive \$0. However, it is important to remember that in each scenario, the player paid \$1 to play the game. So, the take home value (profit) of a winning bet is $\$2 - \$1 = \$1$ and the take home value for a losing bet is $\$0 - \$1 = -\$1$.

In summary, after paying to play the game, a bet that pays 1 to 1 indicates that a winning bet will take home 1 times the amount of the initial bet while a losing bet will cost the player 1 times the amount of the initial bet. If a player pays \$9 to play a game with a 1 to 1 payout, a winning bet (after paying for the game) will take home $(1) \cdot \$9 = \$9$ while a losing bet will cost the player \$9 which is calculated as $(-1) \cdot \$9 = -\$9$.

As another example, consider a \$5 bet that pays 17 to 1. This indicates that a win would take home $(17) \cdot \$5 = \$85$ while a losing bet will cost the player \$5 which is calculated as $(-1) \cdot \$5 = -\$5$.

➢ **EXAMPLE 4.7.8:** Eric went to Las Vegas and played roulette. He placed a \$150 bet on the number 18. If betting on a single number pays 35 to 1, what is the expected value of the game? Is the game in Eric's favor? Note: On a U.S. roulette wheel, there are 18 red numbers, 18 black numbers and 2 green numbers. The player places their bet on the table; however, the winning number is found by spinning a ball on a spinner wheel.



"American roulette layout" by Betzaar is licensed under CC BY-SA 3.0.

SOLUTION: It is important to know the probability of winning the bet. The probability of landing on the number 18 is $\frac{1}{38}$. In this case, the remaining numbers can be thought of as losing numbers with $\mathrm{P(loss)} = \frac{37}{38}$. If Eric pays $150 to play the game with a 35 to 1 payout, a winning bet (after paying for the game) will take home $(35) \cdot \$150 = \$5{,}250$ while a losing bet will cost $150 which is calculated as $(-1) \cdot \$150 = -\$150$.

To find the expected value, multiply the probability of an outcome by the value of that outcome. Then find the sum of these products.

$$\begin{aligned}
&\mathrm{P(spin\ the\ number\ 18)} * \$5{,}250 + \mathrm{P(not\ spin\ the\ number\ 18)} * -\$150 \\
&= \frac{1}{38} \cdot \$5{,}250 + \frac{37}{38} \cdot -\$150 \\
&= \frac{\$5{,}250}{38} + \frac{-\$5{,}550}{38} \\
&- \frac{\$300}{38} \\
&\approx -\$7.89
\end{aligned}$$

Therefore, in the long run, the expected value is -$7.89. If Eric were to make this same bet many times, he would expect to average a loss of $7.89 on each bet. The roulette game is in favor of the casino not the player.

❖ **YOU TRY IT 4.7.C:** At a casino, Becky decided to place a $100 bet that the ball on the roulette wheel would land on a red number. If betting on a red color pays 1 to 1, what is the expected value for her bet?

Expected value can also be used in non-betting situations. It can be used to help make decisions about taking a class when a teacher uses a grading curve or whether to risk getting a parking ticket. Expected value can also be used to determine whether a business decision is profitable.

➢ **EXAMPLE 4.7.9:** Yukiko typically earns B grades in history class. She heard that Professor Plum uses a grading curve, so she wants to take his class. In Professor Plum's class 5% of his students receive an A, 15% receive a B, 50% receive a C, 25% receive a D and 5% receive an F. If Yukiko's school uses a 4.0 grading scale, determine her expected grade from the class. Also, decide whether it would be to her advantage to take Professor Plum's history class. Note: If the school uses a 4.0 grading scale, then an A is worth 4.0 points, a B is worth 3.0 points, a C is worth 2.0 points, a D is worth 1.0 points, and an F receives 0 points.

SOLUTION: It is important to know the probability of each possible outcome. In this case, $\text{P(A)} = 5\% = 0.05$, $\text{P(B)} = 15\% = 0.15$, $\text{P(C)} = 50\% = 0.50$, $\text{P(D)} = 25\% = 0.25$, and $\text{P(F)} = 5\% = 0.05$. The value of each outcome is the value assigned to the letter grade. Since the school uses a 4.0 grading scale, an A is worth 4.0 points, a B is worth 3.0 points, a C is worth 2.0 points, a D is worth 1.0 points, and an F receives 0 points.

To find the expected value, multiply the probability of an outcome by the value of that outcome. Then find the sum of these products.

$$\begin{gathered}\text{P(A)} \cdot 4.0 + \text{P(B)} \cdot 3.0 + \text{P(C)} \cdot 2.0 + \text{P(D)} \cdot 1.0 + \text{P(F)} \cdot 0 \\ = (0.05 \cdot 4.0) + (0.15 \cdot 3.0) + (0.50 \cdot 2.0) + (0.25 \cdot 1.0) + (0.05 \cdot 0) \\ = 0.20 + 0.45 + 1.0 + 0.25 + 0 \\ = 1.90\end{gathered}$$

Therefore, the expected grade value is 1.90 which represents a grade of D. Since Yukiko is typically a B level student, she may not want to take Professor Plum's course.

- **EXAMPLE 4.7.10:** Varsha works for an advertising agency and hopes to create a proposal for a large client. Her advertising agency will spend \$75,000 on creating a proposal to present to the client. However, there are four different advertising agencies expected to make proposals, so earning the large client account isn't guaranteed. If Varsha's company can earn the business of the large client, it will pay her advertising agency \$620,000. Find the expected value for this business decision.

SOLUTION: First, it is important to know the probability of each possible outcome. In this case, there are four advertising companies making presentations, so the probability that Varsha's company earns the client's business is $\frac{1}{4}$. Using complements, the probability that Varsha's company will not earn the client's business is $\frac{3}{4}$. The take home value of earning the client is $\$620{,}000 - \$75{,}000 = \$545{,}000$ after paying their expenses. If they don't earn the client's business, the advertising agency would still have the \$75,000 expense of creating the proposal.

To find the expected value, multiply the probability of an outcome by the value of that outcome. Then find the sum of these products.

$$\begin{gathered}\text{P(win client)} \cdot \$545{,}000 + \text{P(not win client)} \cdot -\$75{,}000 \\ \frac{1}{4} \cdot \$545{,}000 + \frac{3}{4} \cdot -\$75{,}000\end{gathered}$$

$$= \$136{,}250 + \ -\$56{,}250$$
$$= \$80{,}000$$

Therefore, if the company made multiple proposals with the same stipulations, then in the long run the expected value of making the proposal is $80,000. The advertising company can now use this information to determine whether they want to pursue making the proposal. Since the company is expected to gain an average of $80,000 for proposals of this type, it is in their best interest to make the proposal.

❖ **YOU TRY IT 4.7.D:** Koa is visiting a friend and must park his car on the street. He heard that parking on this street is only monitored four days a week, so he is considering the risk of parking on the street without paying for the parking space. A parking fine would cost $75. What is the expected value for this risk?

Quick Review

- The **odds in favor** of event E are the ratio of the probability of event E happening compared to the probability of event E not happening. It can also be expressed as the number of times event E occurs compared to the number of times event E did not occur.
- $\text{Odds in favor of event E} = \frac{\text{P(E)}}{\text{P(not E)}} = \frac{\text{n(E)}}{\text{n(not E)}}$
- The **odds against** event E are the ratio of the probability of event E not happening compared to the probability of event E happening. It can also be expressed as the number of times event E did not occur compared to the number of times event E did occur.
- $\text{Odds against event E} = \frac{\text{P(not E)}}{\text{P(E)}} = \frac{\text{n(not E)}}{\text{n(E)}}$
- The **expected value** is a number that represents the anticipated long-term average of the outcomes if an experiment is repeated over many trials.
- To calculate expected value, multiply the probability of each outcome by the value of that outcome and then sum the results.

YOU TRY IT 4.7.A SOLUTION:

- **a.** $73:137$
- **b.** 137:73
- **c.** $\frac{73}{210}$
- **d.** $\frac{137}{210}$

YOU TRY IT 4.7.B SOLUTION: Expected value = \$33.18. The game is in Emma's favor.

YOU TRY IT 4.7.C SOLUTION: Expected value = - \$5.26. The game is in the casino's favor.

YOU TRY IT 4.7.D SOLUTION: Expected value = - \$42.86.

Section 4.7 | Exercises

In Exercises 1 – 4, given the odds in favor of an event, state the odds against the event.

1. The odds in favor of snow are 4 to 7.
2. The odds in favor of Sam winning the race are 11 to 5.
3. The odds in favor of winning a lottery are 1 to 177,099.
4. The odds in favor of becoming president of a city are 23 to 517,967.

In Exercises 5 – 8, given the odds against an event, state the odds in favor of the event.

5. The odds against rain are 5 to 12.
6. The odds against winning the lottery are 2,352,187 to 2.
7. The odds against being bit by a shark are 4,300,000 to 1.
8. The odds against vacationing in the Maldives for free are 33,759 to 19.

In Exercises 9 – 18, find the odds in favor of and against each event.

9. A single die is rolled.
    a. Find the odds in favor of rolling a six.
    b. Find the odds against rolling a six.

10. A single die is rolled.
    a. Find the odds in favor of rolling a number less than three.
    b. Find the odds against rolling a number less than three.

11. A card is chosen from a standard 52-card deck.
    a. Find the odds in favor of selecting a diamond.
    b. Find the odds against selecting a diamond.

12. A card is chosen from a standard 52-card deck.
    a. Find the odds in favor of selecting a black 6, 7, 8, or 9.
    b. Find the odds against selecting a black 6, 7, 8, or 9.

13. Two dice are rolled.
    a. Find the odds in favor of rolling at least one odd number.
    b. Find the odds against rolling at least one odd number.

14. Two dice are rolled.
    a. Find the odds in favor of rolling a sum of 10.
    b. Find the odds against rolling a sum of 10.

15. One card is chosen from a standard 52-card deck.
    a. Find the odds in favor of selecting a face card.
    b. Find the odds against selecting a face card.

16. One card is chosen from a standard 52-card deck.
    a. Find the odds in favor of selecting a black 7 or a red 9.
    b. Find the odds against selecting a black 7 or a red 9.

17. One card is chosen from a standard 52-card deck.
    a. Find the odds in favor of selecting a black card or an ace.
    b. Find the odds against selecting a black card or an ace.

18. One card is chosen from a standard 52-card deck.
    a. Find the odds in favor of selecting a face card or a heart.
    b. Find the odds against selecting a face card or a heart.

In Exercises 19 – 26, given the following probability, (a) find the odds in favor of the event and (b) the odds against the event.

19. The probability of being awarded a car loan with bad credit is $\frac{4}{17}$.
    a. Find the odds in favor of being awarded a car loan with bad credit.
    b. Find the odds against being awarded a car loan with bad credit.

20. The probability of winning a scratch off ticket lottery game is $\frac{1}{1,000,000}$.
    a. Find the odds in favor of winning the scratch off ticket lottery game.
    b. Find the odds against winning the scratch off ticket lottery game.

21. The probability that it rains on any given day in Illinois in May is $\frac{4}{10}$.
    a. Find the odds in favor of rain on any given day in Illinois in May.
    b. Find the odds against rain on any given day in Illinois in May.

22. The probability of a golfer getting a hole-in-one is $\frac{1}{12,501}$.
    a. Find the odds in favor of a golfer getting a hole-in-one.
    b. Find the odds against a golfer getting a hole-in-one.

23. The probability of a television show NOT being cancelled after five seasons is $\frac{92}{114}$.
    a. Find the odds in favor of a television show being cancelled after five seasons.
    b. Find the odds against a television show being cancelled after five seasons.

24. In soccer, the probability of getting a goal from a penalty kick is $\frac{19}{25}$.
    a. Find the odds in favor of getting a goal from a penalty kick.
    b. Find the odds against getting a goal from a penalty kick.

25. In Monopoly, the probability of landing on Boardwalk is $\frac{1}{40}$.
    a. Find the odds in favor of landing on Boardwalk.
    b. Find the odds against landing on Boardwalk.

26. The probability of NOT having rain on a wedding day is $\frac{13}{157}$.
    a. Find the odds in favor of having rain on a wedding day.
    b. Find the odds against having rain on a wedding day.

In Exercises 27 – 31, given the odds, find the following probabilities.

27. The odds of being in a car accident on a rainy day are 13 to 47.
    a. Find the probability of being in a car accident on a rainy day.
    b. Find the probability of not being in a car accident on a rainy day.

28. A new medicine claims the odds of reducing a patient's blood pressure are 2 to 9.
    a. Find the probability that the new medicine will reduce a patient's blood pressure.
    b. Find the probability that the new medicine will not reduce a patient's blood pressure.

29. The odds against being dealt a royal flush (a ten, jack, queen, king, and ace of the same suit) are 649,739 to 1.
    a. Find the probability of being dealt a royal flush.
    b. Find probability of not being dealt a royal flush.

30. If a person randomly guessed every game outcome, the odds against having a perfect March Madness college basketball bracket are 223,372,036,854,775,808 to 1.
    a. Find the probability of having a perfect March Madness college basketball bracket.
    b. Find the probability of not having a perfect March Madness college basketball bracket.

31. The odds of finishing a 739-page book in two hours is 16 to 232,980,121.
    a. Find the probability of finishing the 739-page book in two hours.
    b. Find the probability of not finishing the 739-page book in two hours.

In Exercises 32 – 39, given the scenario, find the odds or probability.

32. Olivia is playing Monopoly and landed in jail. To get out of jail, she can roll doubles (both dice have the same number).
    a. Find the probability that Olivia will roll doubles.
    b. Find the probability that Olivia will not roll doubles.
    c. Find the odds in favor of rolling doubles.
    d. Find the odds against rolling doubles.

33. Josiah is playing Monopoly with his sister Olivia. If he can roll a sum of four using two dice, then he will land on and purchase the last remaining railroad space.
    a. Find the probability that Josiah will roll a sum of four.
    b. Find the probability that Josiah will not roll a sum of four.
    c. Find the odds in favor of rolling a sum of four.
    d. Find the odds against rolling a sum of four.

34. Elisabeth is taking a 10-question true/false quiz. She ends up guessing the answer to each question and doesn't leave any answers blank.
    a. Find the probability that Elisabeth answers all questions correctly.
    b. Find the probability that Elisabeth answers at least one question incorrectly.
    c. Find the odds in favor of Elisabeth answering all questions correctly.
    d. Find the odds against Elisabeth answering all questions correctly.

35. Paris applied to a job at a big law firm. She believes she has a 27% chance of being offered the job.
    a. Find the probability that Paris is offered the job.
    b. Find the probability that Paris is not offered the job.
    c. Find the odds in favor of Paris being offered the job.
    d. Find the odds against Paris being offered the job.

36. Madeline is anxiously waiting to hear if she was accepted into her first-choice college. Based on her college entrance application, she thinks she has a 62% chance of being accepted.
    a. Find the probability of Madeline being accepted into her first-choice college.
    b. Find the probability of Madeline not being accepted into her first-choice college.
    c. Find the odds in favor of Madeline being accepted into her first-choice college.
    d. Find the odds against Madeline being accepted into her first-choice college.

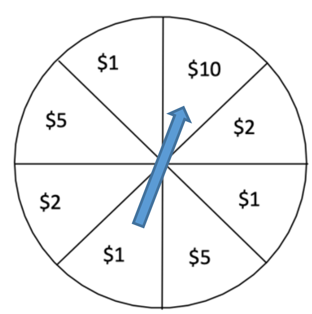
37. Ten turtles are in a race, and Matt places a bet on the first three turtles to finish. To win the bet, Matt must correctly predict the first, second and third place winner.
    a. Find the probability that Matt wins his bet.
    b. Find the probability that Matt loses his bet.
    c. Find the odds in favor of Matt winning his bet.
    d. Find the odds against Matt winning his bet.

38. Conrad is choosing three cards from a standard 52-card deck. Choosing one pair means choosing two cards with the same denomination such as two kings or two eights. The third card can be any card but not the same denomination as the pair.
    a. Find the probability that Conrad chooses exactly one pair.
    b. Find the probability that Conrad does not choose exactly one pair.
    c. Find the odds in favor of Conrad choosing exactly one pair.
    d. Find the odds against Conrad choosing exactly one pair.

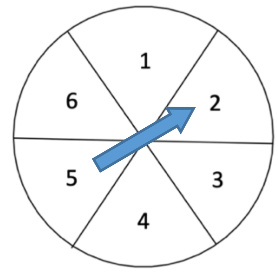
39. Anna and Andrew are playing a game of war with a standard 52-card deck. A war occurs when two or more players have a card of the same value, such as both fives.
    a. Find the probability of having a war (two cards of the same denomination selected).
       Hint: the first card can be any card. Then, the second card must match the first.
    b. Find the probability of not having a war (not having two cards of the same denomination selected).
    c. Find the odds in favor of having a war.
    d. Find the odds against having a war.

In Exercises 40 – 62, find the expected value.

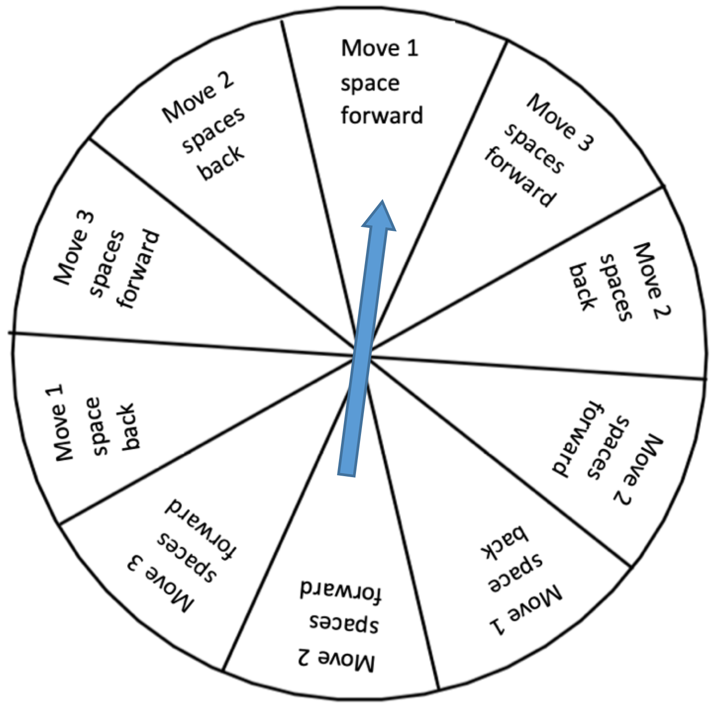
40. Laura is playing the spinner game (shown on the right) at the school fair. She spins the pointer and wins the prize shown when the pointer stops. What is the expected value for this game?



41. To play a board game, a player spins a spinner to determine the number of spaces to move (shown at right). Find the expected value for the number of spaces to move.



42. To play a board game, a player spins a spinner to determine the number of spaces to move (shown at right). Find the expected value for the number of spaces to move.



43. At the scholarship fundraiser, several prizes will be given away. The prizes are one television (worth \$350), one pair of ear pods (worth \$199), two \$100 restaurant gift cards, and four \$25 gift cards for the gas station. If 360 people are expected to attend the fundraiser, find the expected value of prizes.

44. A game is played in which three cards are selected at random from a 52-card deck. The player will win \$50 for every face card drawn. It costs \$20 to play the game. What is the expected value for this game? (Hint: the possible outcomes are no face cards, one face card, two face cards, and three face cards.)

45. A game is played using one die. It costs the player \$15 to play. If the player rolls 1, the player wins \$5. If the player rolls 2, the player wins \$10. Rolling a 3, the player wins \$15. A roll of 4, the players wins \$20. Rolling a 5, the player wins \$25. Lastly, rolling 6, the player wins \$30. What is the expected value of this game? Is this a fair game for the player?

46. On Fridays, the student union offers a “Choose the Envelope” game. It costs \$2 to play and the player wins what is in the envelope. There is one envelope that contains \$50, three envelopes that contain \$10, ten envelopes that contain \$5, and twenty envelopes that do not contain any money. Find the expected value (to the nearest cent) for the envelope game.

47. For homecoming, the school is offering a “Choose the Envelope” game. It costs \$5 to play and the player wins what is in the envelope. There is one envelope that contains \$250, two envelopes that contain \$100, four envelopes that contain \$50, ten envelopes that contain \$10, and 30 envelopes that do not contain any money but do contain a slip of paper with a typed inspirational quote.
    a. Find the expected value for the envelope game.
    b. How does the expected value change if the \$250 envelope was not part of the choose the envelope game?

48. It costs \$20 to play a dice game. For this game, two dice are rolled. If doubles are rolled, the player receives \$150. If the player rolls a sum of five, then they receive \$100. If the player rolls exactly one five, then they receive \$20. For all other rolls, the player does not receive any prize. Find the expected value of the dice game. In the long run, does the game favor the player?

49. It costs \$10 to play a dice game. For this game, two dice are rolled. If a sum greater than 10 is rolled, the player receives \$20. If a sum less than six is rolled, the player receives \$10. If a player rolls two odd numbers, then they receive \$5. A player can only receive one prize. Therefore, if a roll meets the description of more than one prize, the player only receives the higher prize value (not both). Find the expected value of the dice game. In the long run, does the game favor the player?

50. The parent teacher association (PTA) is holding a back-to-school raffle. Tickets cost \$3 each and they will give away five prizes. The first prize is a backpack of school supplies valued at \$40. The second prize is a new lunchbox valued at \$25. They will also give away three prize packages containing markers and pencils valued at \$10 each.
    a. If 280 people purchase raffle tickets, what is the expected value of the raffle?
    b. Will the raffle earn money or lose money for the parent-teacher association? How much money will be gained or lost for the parent-teacher association?

51. A multiple-choice exam contains questions that each have five answer choices. One point is awarded for each correct answer, and 1/4 point is deducted for each incorrect answer. If an answer is left blank, then points are not awarded or deducted.
    a. What is the expected value if a student guesses on a question?
    b. Is it advantageous to guess if an answer is unknown?
    c. How does the expected value change if 1/2 point is deducted for each incorrect answer?

52. A game is played using one die. It costs the player $10 to play. If the player rolls 1, the player wins $1. If the player rolls 2, the player wins $2. Rolling a 3, the player wins $3. A roll of 4, the player wins $4. Rolling a 5, the player wins $20. Lastly, rolling 6, the player wins $25.
    a. What is the expected value of this game?
    b. Is this a fair game for the player?
    c. What would the cost of the game be for this to be a fair game? Note: To be a fair game, the expected value would be zero.

53. Mikayla typically earns C grades in science class. She heard that Professor White has a grading curve. In Professor White's class 15% of her students receive an A, 15% receive a B grade, 40% receive a C grade, 20% receive a D grade and 10% receive an F grade. If the school uses a 4.0 grading scale, determine her expected grade from the class. Also decide whether it would be to her advantage to take Professor White's science class. Note: If the school uses a 4.0 grading scale, then an A is worth 4.0 points, a B is worth 3.0 points, a C is worth 2.0 points, a D is worth 1.0 points and an F receives 0 points.

54. An electronic store offers a warranty costing $25 for a $200 pair of headphones. If the headphones become defective within a year, the store will replace the headphones for a customer who has purchased the warranty. There is an 8% chance that the headphones become defective within a year. What is the customer's expected value if the warranty is purchased? Is it in the customer's best interest to pay for the warranty?

55. A college has a raffle in which 300 tickets are sold. There is one prize of a $1,500 computer, two $1000 prizes of flat screen televisions, and three $500 gift cards. If each raffle ticket costs $25, what is the expected value of the raffle? What should a raffle ticket cost so that a person buying the raffle ticket has an expected value of $3?

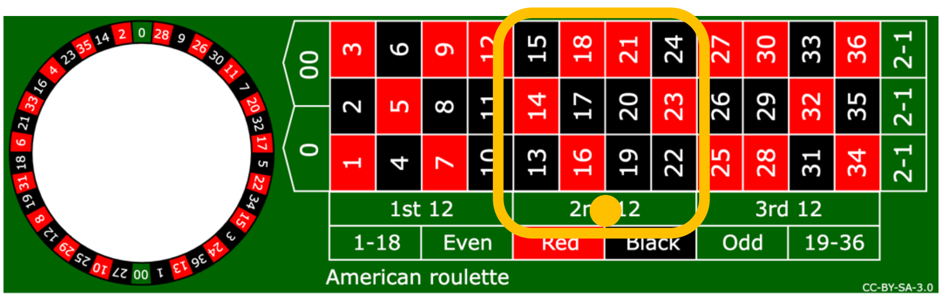
56. A travel company held a sweepstakes promotion with four winners. One grand prize winner will receive a seven-night all-inclusive international vacation for two people valued at $7,500. The first-place prize winner will receive a five-night all-inclusive international vacation valued at $5,000. One second-place prize winner will receive a four-night all-inclusive vacation to California valued at $2,800. And one third-place prize winner will receive a two-night all-inclusive vacation to Florida valued at $1,500. If the company expects 40,000 entries, find the expected value for the sweepstakes. There is no cost to enter the sweepstakes.

57. An architectural firm would like to place a bid to draft plans for a new building in a city. It costs the firm $30,000 to draft the plans, but they will make $250,000 if their bid is accepted. Nine **<u>other</u>** firms have also placed bids for the same job.
    a. Assuming each firm has an equal chance of being selected, what is the expected value for this business decision?
    b. Is it in this firm's best interest to place a bid on this job?
    c. The firm would like to increase their client base and will place a bid for the job if they can at least break even ($0 expected value). How many other firms could place bids for the same job for it to be in the firm's interest to bid on the job?

58. An insurance company will issue a life insurance policy for $10,000 for a person 18 – 27 years old. The charge for the life insurance policy is $100 per year. If there is a 99.97% chance that someone who is 18 – 27 years old will live for the year, what is the expected value of the policy. If the insurance company would like to profit $200 per person, what would they charge per insurer?

In Exercises 57 – 62, a U.S. Roulette table is used (image below). On a U.S. roulette wheel, there are 18 red numbers, 18 black numbers and 2 green numbers. The player places their bet on the table; however, the winning number is found by spinning a ball on a spinner wheel.

59. In roulette, playing a group of 12 means that a player is betting on 12 connected numbers that form a group. As an example, the yellow marker on the roulette table below is betting on the second 12 group highlighted in yellow. Playing a group of 12 numbers pays 2 to 1. If a player bets $100 on a group of 12 connected numbers, what is the expected value of the play?



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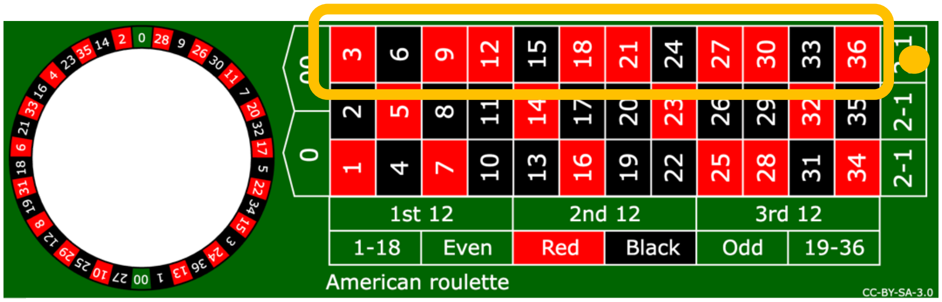
60. In roulette, playing a column means that a player is betting on a line of 12 numbers. As an example, the yellow marker on the roulette table below is betting one column of 12 numbers highlighted in yellow. Playing a column of 12 numbers pays 2 to 1. If a player bet $70 on a column of numbers, what is the expected value of the play?



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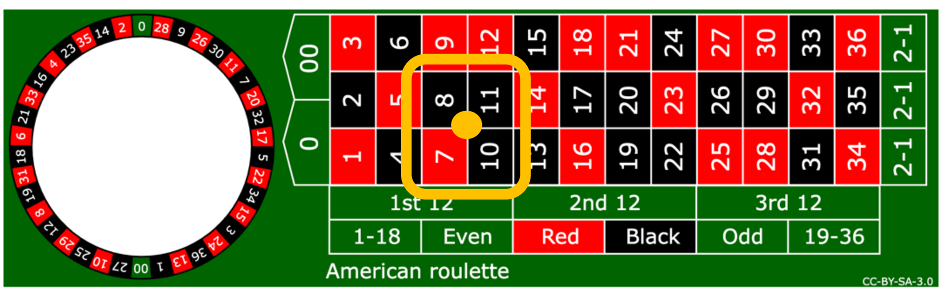
61. In roulette, playing a square means that a player is betting on four connected numbers that form a square. As an example, the yellow marker on the roulette table below is betting on the four connected numbers 7, 8, 10 and 11. Playing a square pays 8 to 1. If a player bet $20 on a square of four connected numbers, what is the expected value of the play?



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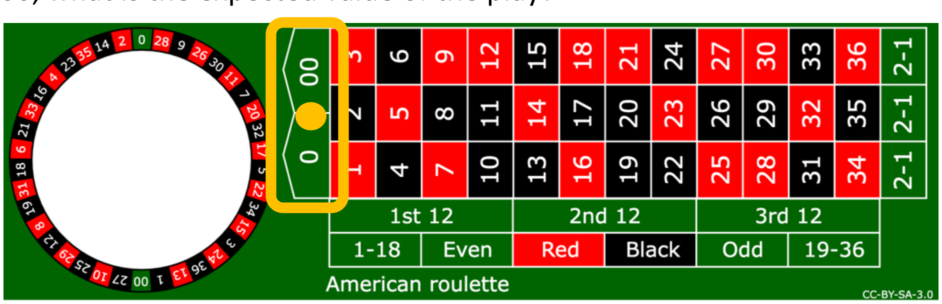
62. In roulette, playing the green spaces means that a player is betting on both the green 0 and green 00 spaces. As an example, the yellow marker on the roulette table below is betting on the two green spaces. The bet pays 17 to 1. If a player bet $50 on numbers 0 and 00, what is the expected value of the play?



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Section 4.7 Exercise Solutions

1. 7 : 4
2. 5 : 11
3. 177,099 : 1
4. 517,967 : 23
5. 12 : 5
6. 2 : 2,352,187
7. 1 : 4,300,000
8. 19 : 33,759
9. a. 1 : 5 b. 5 : 1
10. a. 1 : 2 b. 2 : 1
11. a. 1 : 3 b. 3 : 1
12. a. 2: 11 b. 11 : 2
13. a. 3 : 1 b. 1 : 3
14. a. 1 : 11 b. 11 : 1
15. a. 3 : 10 b. 10 : 3
16. a. 1 : 12 b. 12 : 1
17. a. 7 : 6 b. 6 : 7
18. a. 11 : 15 b. 15 : 11
19. a. 4 : 13 b. 13 : 4
20. a. 1 : 999,999 b. 999,999 : 1
21. a. 2 : 3 b. 3 : 2
22. a. 1 : 12,500 b. 12,500 : 1
23. a. 11 : 46 b. 46 : 11
24. a. 19 : 6 b. 6 : 19
25. a. 1 : 39 b. 39 : 1
26. a. 144 : 13 b. 13 : 144
27. a. $\frac{13}{60}$ b. $\frac{47}{60}$
28. a. $\frac{2}{11}$ b. $\frac{9}{11}$
29. a. $\frac{1}{649{,}740}$ b. $\frac{649{,}739}{649{,}740}$
30. a. $\frac{1}{223{,}372{,}036{,}854{,}775{,}809}$
    b. $\frac{223{,}372{,}036{,}854{,}775{,}808}{223{,}372{,}036{,}854{,}775{,}809}$
31. a. $\frac{16}{232{,}980{,}137}$ b. $\frac{232{,}980{,}121}{232{,}980{,}137}$
32. a. $\frac{1}{6}$ b. $\frac{5}{6}$
    c. 1 : 5 d. 5 : 1
33. a. $\frac{1}{12}$ b. $\frac{11}{12}$
    c. 1 : 11 d. 11 : 1
34. a. $\frac{1}{1{,}024}$ b. $\frac{1{,}023}{1{,}024}$
    c. 1 : 1,023 d. 1,023 : 1
35. a. $\frac{27}{100}$ b. $\frac{73}{100}$
    c. 27 : 73 d. 73 : 27
36. a. $\frac{31}{50}$ b. $\frac{19}{50}$
    c. 31 : 19 d. 19 : 31
37. a. $\frac{1}{720}$ b. $\frac{719}{720}$
    c. 1 : 719 d. 719 : 1
38. a. $\frac{72}{425}$ b. $\frac{353}{425}$
    c. 72 : 353 d. 353 : 72
39. a. $\frac{1}{17}$ b. $\frac{16}{17}$
    c. 1 : 16 d. 16 : 1
40. $3\frac{3}{8}$ or \$3.38
41. 3.5

42. 0.8
43. $2.36
44. $14.62
45. $2.50, yes
46. $1.82
47. a. $10.96
    b. $5.87
48. $21.67. Favors the player.
49. - $4.72. Does not favor the player.

50. a. -$2.66
    b. PTA earns $745
51. a. 0
    b. Guessing does not give an advantage or a disadvantage.
    c. The expected value changes to - 0.2, which means that guessing would not be advantageous.
52. a. - $.83
    b. no
    c. $9.16
53. 2.05, C grade;
    yes, advantageous to take the course
54. -$9. It is not in the customer's best interest.
55. -$8.33; $13.67
56. $0.42
57. a. -$5,000.
    b. It is not in the firm's best interest.
    c. 7 other firms (total of 8 firms)
58. - $97, $203
59. - $5.26
60. - $3.68
61. - $1.05
62. - $2.63

Section 4.8 | Probability Trees & Binomial Experiment

Objectives

- Use probability trees to calculate probability of conjunctions
- Use probability trees to calculate probability of disjunctions
- Use probability trees to calculate conditional probabilities
- Use binomial experiment to calculate probabilities

In Section 4.1 and 4.3, tree diagrams were used as a visual way to construct the sample space of an experiment or to know the total number of outcomes from an experiment. This section will provide further discussion on how the tree diagrams can be used to calculate many of the probabilities found in the previous sections.

Recall from Section 4.6, the probability of events in succession or events occurring simultaneously can be found using the probability of intersections. The probability of intersections is calculated by multiplying the probability of the first event with the conditional probability of the second event, given the outcome of the first event. For example, when rolling a single coin, the probability of rolling a one is $\frac{1}{6}$. Therefore, the probability of rolling a one and then rolling a one again on a second roll is $\frac{1}{6} \cdot \frac{1}{6} = \frac{1}{36}$.

Probability Trees

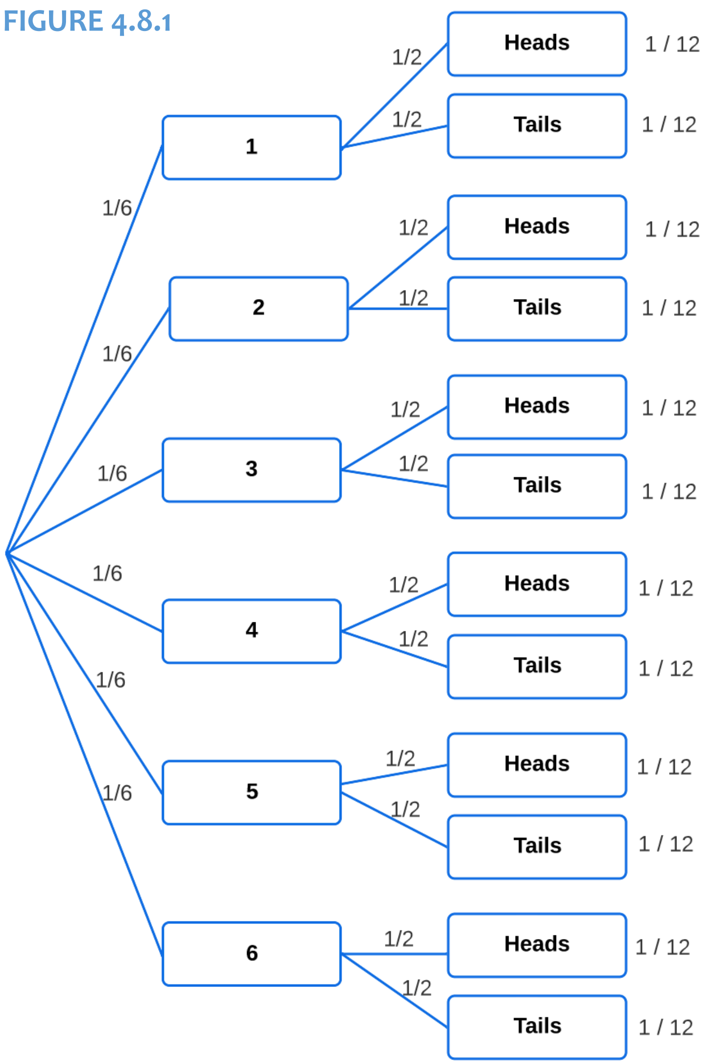
Suppose the experiment is rolling a single die and then flipping one coin. To make the tree diagram a probability tree, the probability of each outcome is placed on the branch in front of that outcome. Since a single six-sided die is rolled first, the sample space is $\{1, 2, 3, 4, 5, 6\}$. These are the six branches drawn to the right of the starting point with the outcome listed at the end of the branch. The probability of each outcome is $\frac{1}{6}$. Notice that the sum of all probabilities on this set of branches is $\frac{6}{6}$ or 1.

NOTE: The probability trees in this section have been created horizontally, left to right. However, probability trees can also be created vertically.

The experiment continues with flipping a coin. When flipping a coin, the sample space is $\{\text{Heads}, \text{Tails}\}$ or abbreviated as $\{\text{H}, \text{T}\}$. Two branches are drawn to the right of the outcomes from rolling a die, one for heads and one for tails. Since $\text{P(H)} = \frac{1}{2}$ and $\text{P(T)} = \frac{1}{2}$, these probabilities are shown on each branch leading to outcomes for heads and tails. Again, notice that each set of heads and tails branches total $\frac{2}{2}$ or 1.

Finally, in Section 4.6, the formula used to calculate the probability of the intersection of events is $P(A \cap B) = P(A) \cdot P(B|A)$. This formula indicates that the probabilities of successive branches are multiplied together to find the overall probability of the complete branch representing one specific intersection of events. As shown in the branch from left to right at the top of the probability tree, the probability of rolling a one and then flipping heads is $\frac{1}{6} \cdot \frac{1}{2} = \frac{1}{12}$. Notice that outcomes for rolling a die and flipping a coin are equally likely, making the 12 possible outcomes in the probability tree of these successive events also equally likely.

FIGURE 4.8.1

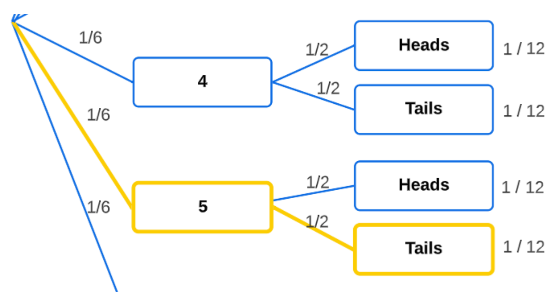


➢ **EXAMPLE 4.8.1:** Use the probability tree in Figure 4.8.1 to find the following probabilities.

a. $P(\text{five}, \text{T})$
b. P(an even number and heads)
c. P(a five or tails)
d. P(heads | three was rolled first)
e. P(four | heads was flipped)

SOLUTION:

a. When results are listed in sequential order separated by commas, the order is being specified as five first and then tails second. There is one possible result when rolling a five and flipping tails. Follow the path leading to the outcome of rolling a five and then to the outcome of flipping tails in the section of the tree diagram highlighted below. Multiply the individual probabilities of each branch to calculate $P(\text{five}, \text{T}) = \frac{1}{6} \cdot \frac{1}{2} = \frac{1}{12}$.



b. When more than one complete branch meets the description, find the probability of each complete branch that meets the description, and then find the sum of these probabilities. In the probability tree shown in Diagram 4.8.2B, complete branches where an even number was rolled and heads was flipped are highlighted in yellow. Each of the three complete branches have a probability of $\frac{1}{12}$; therefore,
$\text{P(an even number and heads)} = \frac{1}{12} + \frac{1}{12} + \frac{1}{12} = 3 \cdot \frac{1}{12} = \frac{3}{12} = \frac{1}{4}$.

DIAGRAM 4.8.2B

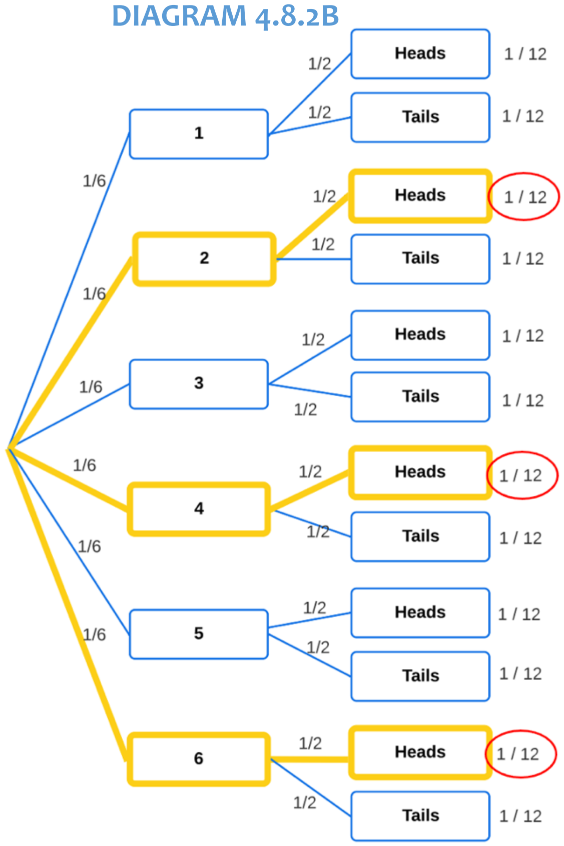
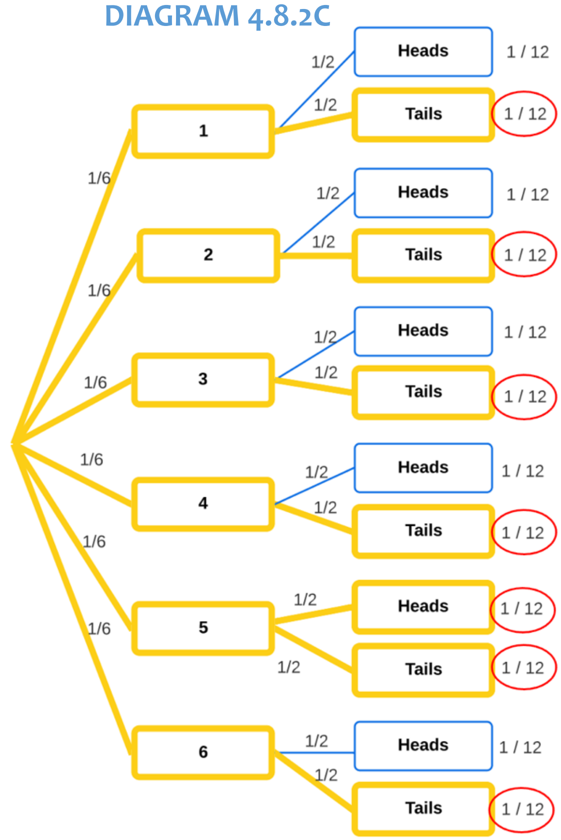


DIAGRAM 4.8.2C

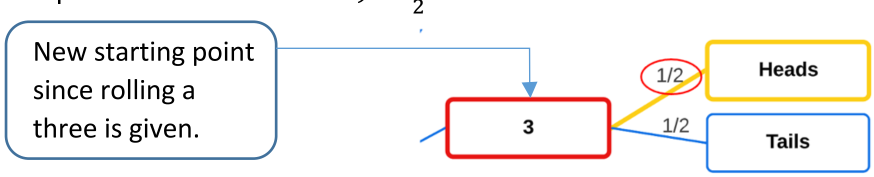


c. When more than one complete branch meets the description, find the probability of each complete branch that meets the description, and then find the sum of these probabilities. In Diagram 4.8.2C above, complete branches where a five was rolled or where tails was flipped are highlighted in yellow. Each of the seven complete branches have a probability of $\frac{1}{12}$; therefore,

$$\text{P(rolling a five or flipping tails)} = 7 \cdot \frac{1}{12} = \frac{7}{12}.$$

d. Finding conditional probability changes the sample space since some information is known. In this case, it is known that a three was rolled first. This means only branches that lead to rolling a three should be considered while the rest of the probability tree should be ignored. Since rolling a single die was the first event, it is possible to consider adjusting the starting point to where the three was rolled, highlighted in red in the section of the probability tree shown below. The probability of getting heads after it is known that a three was rolled is shown on the branch leading to the heads outcome. Therefore,

$$\text{P(heads | three was rolled first)} = \frac{1}{2}.$$

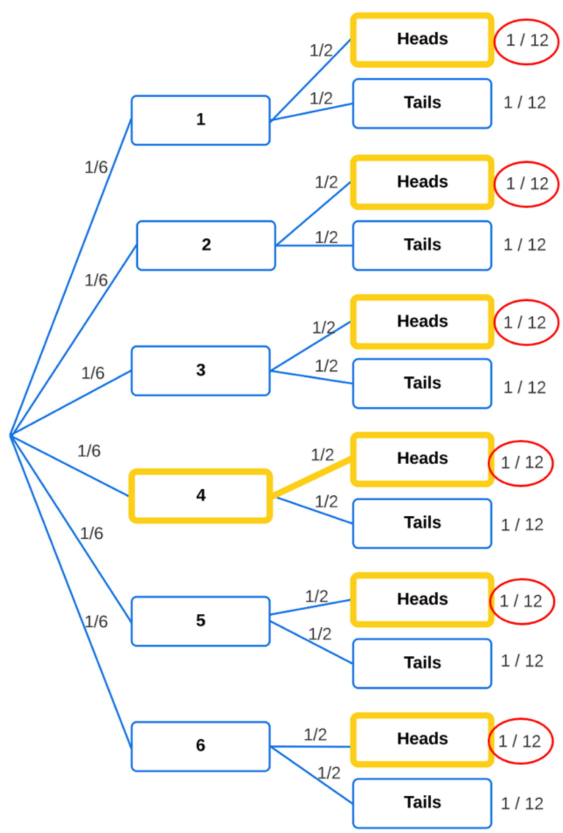


It is also possible to obtain this same answer using the definition of conditional probability

$$P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{\text{P(three and heads)}}{\text{P(three)}} = \frac{\frac{1}{12}}{\frac{1}{6}} = \frac{1}{12} \div \frac{1}{6} = \frac{1}{12} \cdot \frac{6}{1} = \frac{6}{12} = \frac{1}{2}$$

e. Finding conditional probability changes the sample space since some information is known. In this case, it is known that the coin landed on heads. Notice this is not the first set of branches after the starting point, so it is important to use the the definition of conditional probability:

$$P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{\text{P(four and heads)}}{\text{P(heads)}} = \frac{\frac{1}{12}}{\frac{6}{12}} = \frac{1}{12} \div \frac{6}{12} = \frac{1}{12} \cdot \frac{12}{6} = \frac{1}{6}$$



NOTE: When an event becomes more specific through the intersection of events, the overall probability typically decreases with each successive detail. For example, the probability of rolling a three first and then rolling a four second is more specific and less likely to occur than just rolling a three. In part a of Example 4.8.1, where the result must include a both a five on the die **and** tails on the coin, multiplying their individual probabilities makes the probability for the complete branch smaller than the individual probabilities in their product. Specifically, $P(5) = \frac{1}{6}$ and $P(\mathrm{T}) = \frac{1}{2}$; however, the probability for the complete branch is $P(5, \mathrm{T}) = \frac{1}{6} \cdot \frac{1}{2} = \frac{1}{12}$ which is smaller.

NOTE: When an event has more than one way to meet the description, the probability increases. With more options to meet the event description, it is more likely to occur. In part c of Example 4.8.1, where the result can include a five on the die **or** a tails on the coin **or** both, adding the probabilities from the complete branches that meet the description makes the overall probability larger.

- **YOU TRY IT 4.8.A:** Use the probability tree in Figure 4.8.1 to find the following probabilities:

  a. $\mathrm{P(three, T)}$
  b. $\mathrm{P(a\ three\ or\ heads)}$
  c. $\mathrm{P(an\ even\ number\ |\ tails\ was\ flipped)}$

Probability Trees with Replacement

Suppose an experiment requires that two cards be chosen from a standard 52-card deck. To determine the probability of choosing two hearts, it is important to know whether the first card is recorded and replaced back into the deck or recorded and set aside. The next two examples show how replacing an item or not replacing an item affects the probability tree and probability calculations.

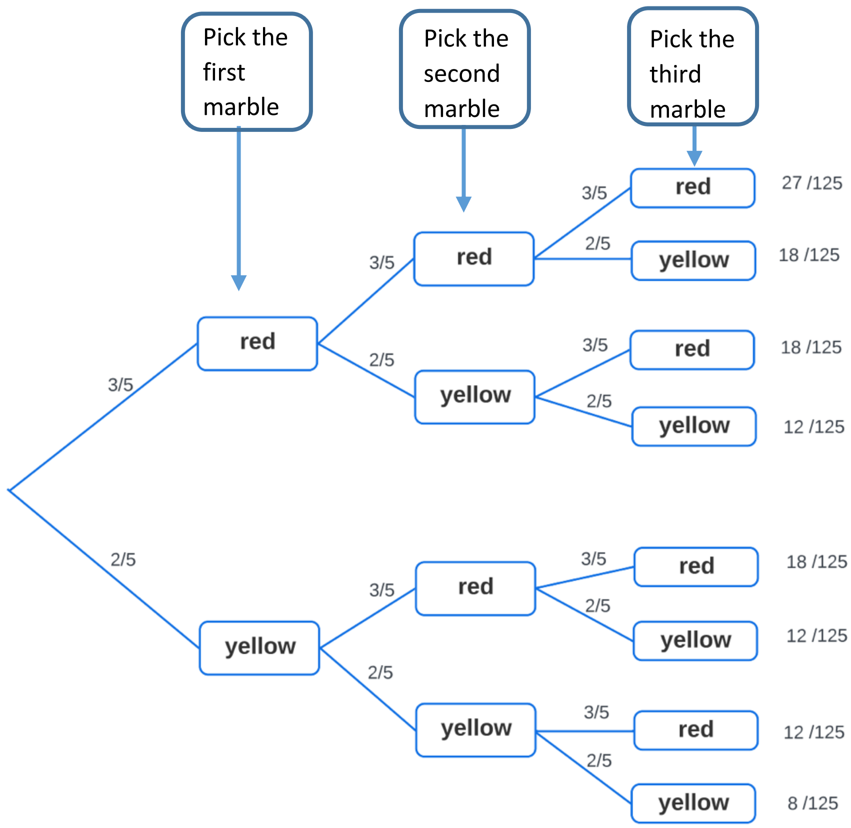
- **EXAMPLE 4.8.2:** Create a probability tree for the following experiment and then find the following probabilities.

  A bag of marbles contains three red and two yellow marbles. Create a probability tree for the experiment of choosing three marbles (with replacement).

  a. $\mathrm{P(yellow, yellow, red)}$
  b. $\mathrm{P(two\ red\ marbles)}$
  c. $\mathrm{P(at\ least\ one\ yellow\ marble)}$
  d. $\mathrm{P(at\ least\ one\ yellow\ |\ red\ first)}$

SOLUTION:

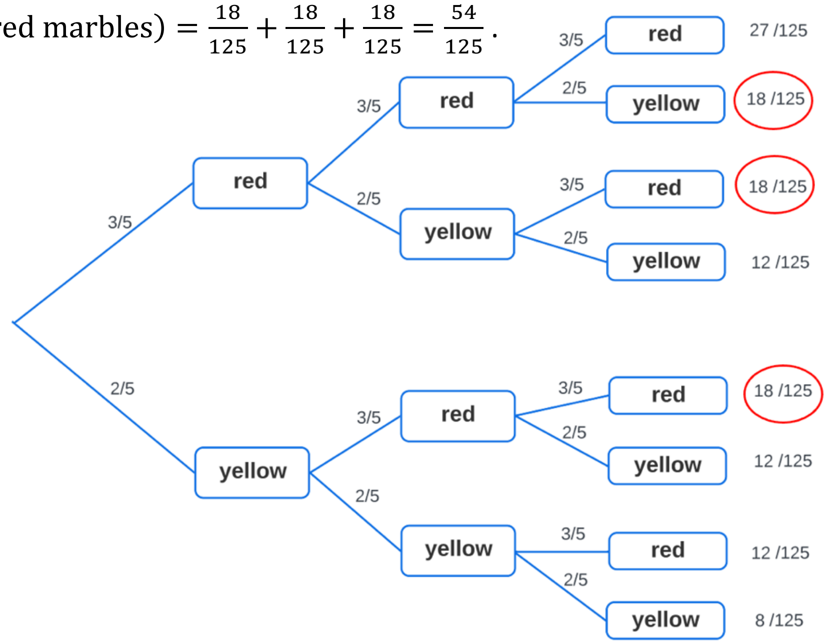
First, create the probability tree. The first set of branches will represent the outcomes possible when selecting the first marble. Therefore, two branches should extend from the starting point, one for red and one for yellow. Next, write the probability of selecting a red marble on the branch leading to the red outcome and write the probability of selecting a yellow marble on the branch leading to the yellow outcome. It is known that the color of the marble is noted and then replaced back into the marble bag. Therefore, the probability of choosing a red marble remains $\frac{3}{5}$ and the probability of choosing a yellow marble remains $\frac{2}{5}$ each time a marble is chosen. Next, the probabilities of successive branches are multiplied together to find the overall probability of the complete branch.



a. The results are listed in order yellow first, yellow second, and red third. There is one possible result when choosing yellow, yellow, and then red. Follow the branches left to right leading to yellow and then to yellow and finally to red. Multiply the successive probabilities to get $\mathrm{P(yellow, yellow, red)} = \frac{2}{5} \cdot \frac{2}{5} \cdot \frac{3}{5} = \frac{12}{125}$.

b. When more than one complete branch meets the description, find the probability of each complete branch that meets the description and then find the sum of these probabilities. In the probability tree on the next page, complete branches where two red marbles were chosen (in any order) are highlighted with a red oval. Find the sum of the complete branch probabilities where two red marbles were chosen.

$\text{P(two red marbles)} = \frac{18}{125} + \frac{18}{125} + \frac{18}{125} = \frac{54}{125}$.

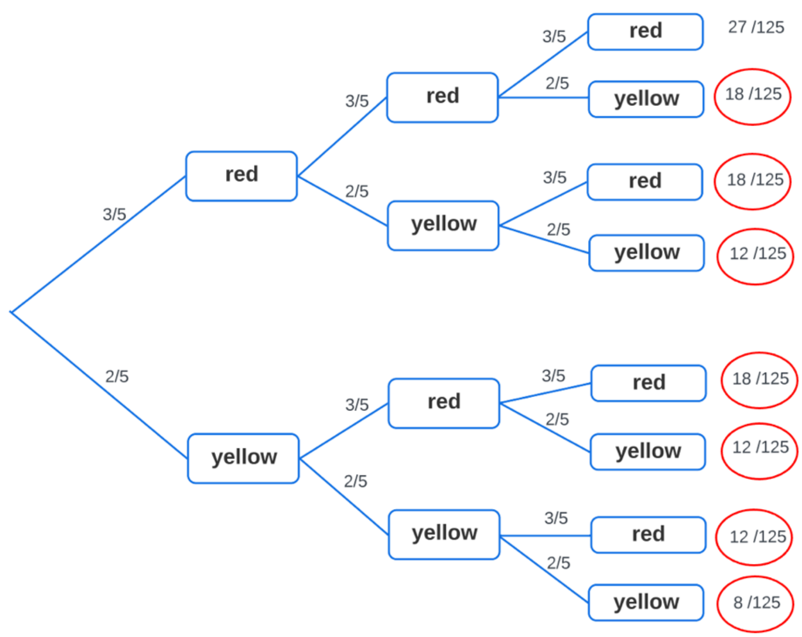


c. When more than one complete branch meets the description, find the probability of each complete branch that meets the description and then and find the sum of these probabilities. In the probability tree below, complete branches where at least one yellow marble was chosen are highlighted with a red oval. Find the sum of the complete branch probabilities where at least one yellow marble was chosen.

$\text{P(at least one yellow)} = \frac{18}{125} + \frac{18}{125} + \frac{12}{125} + \frac{18}{125} + \frac{12}{125} + \frac{12}{125} + \frac{8}{125} = \frac{3}{12} = \frac{98}{125}$.

It is also possible to obtain this same answer using complements.

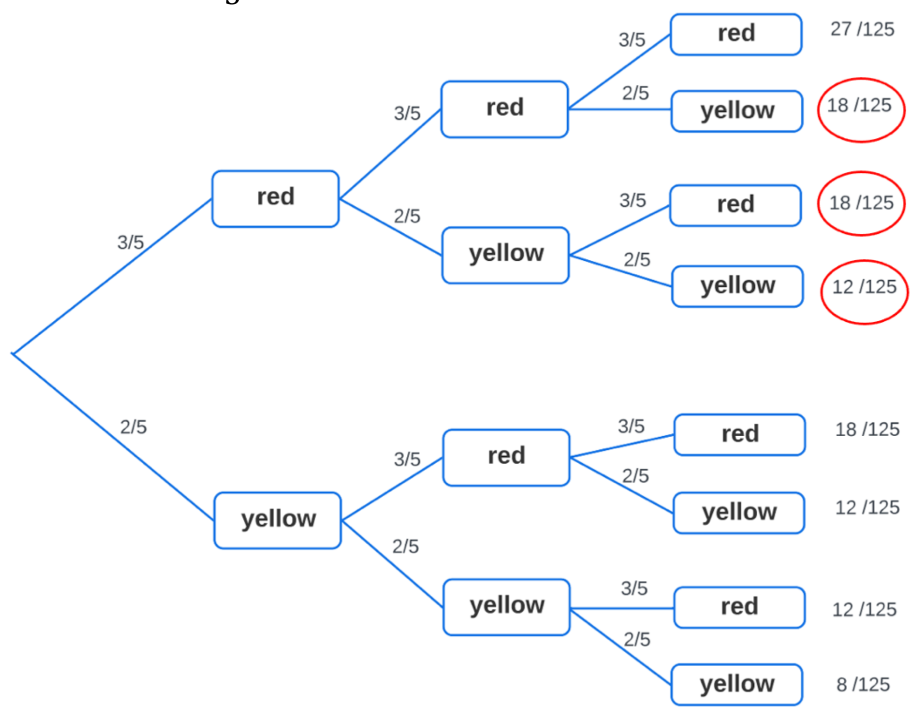
$$\text{P(at least one yellow marble)} = 1 - \text{P(no yellow marble)} = 1 - \frac{27}{125} = \frac{98}{125}$$



d. Finding conditional probability changes the sample space since some information is known. In this case, it is known that first marble chosen was red. Using the definition of conditional probability:

$$P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{\text{P(at least one yellow and red first)}}{\text{P(red first)}} = \frac{\frac{18}{125} + \frac{18}{125} + \frac{12}{125}}{\frac{3}{5}}$$

$$= \frac{\frac{48}{125}}{\frac{3}{5}} = \frac{48}{125} \div \frac{3}{5} = \frac{48}{125} \cdot \frac{5}{3} = \frac{16}{25}$$

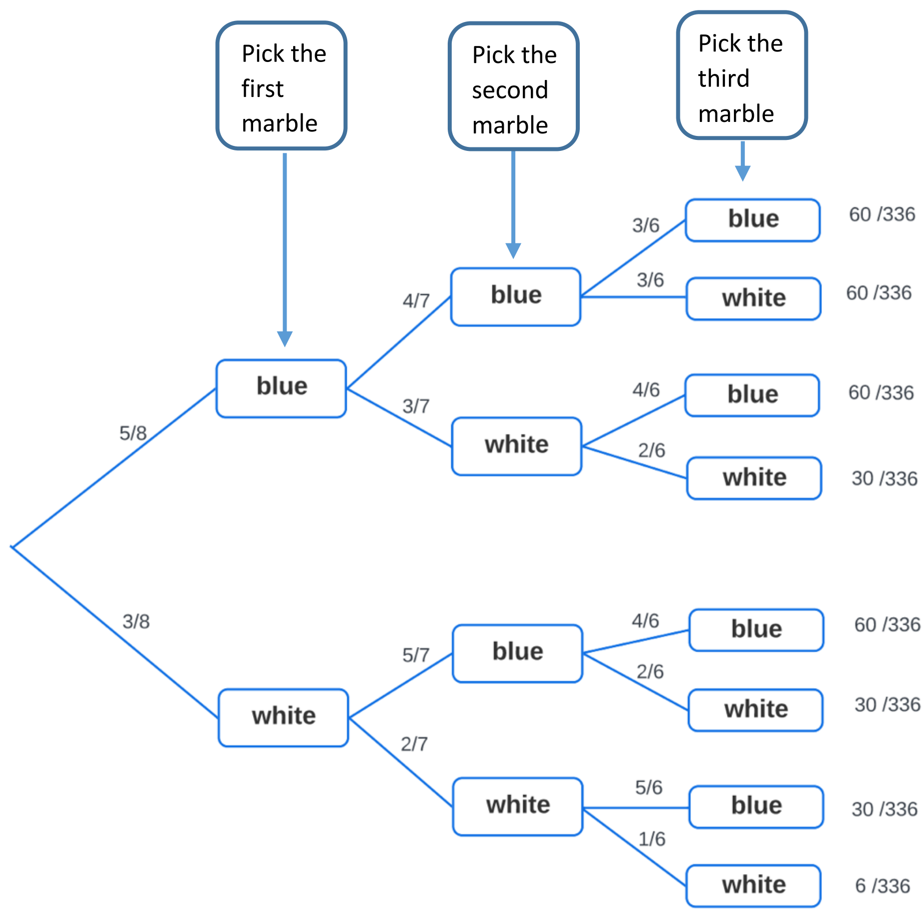


Probability Trees without Replacement

Consider the experiment where three marbles are chosen without replacement from a bag containing five blue and three white marbles. When a marble is chosen, the color is noted and the marble is left out of the bag. Therefore, when a second marble is chosen, the sample space has changed making the probabilities for choosing the second marble dependent on the color of marble chosen first.

As with creating any probability tree, the first set of branches will represent the outcomes possible when selecting the first marble. Therefore, two branches should extend from the starting point, with $\text{P(blue)} = \frac{5}{8}$ and $\text{P(white)} = \frac{3}{8}$.

However, once the first marble color is noted, it is set aside and left out of the marble bag leaving only seven marbles when choosing the second marble. Follow each first branch and consider the marbles remaining to find the probability for blue and the probability for white in each scenario. If a blue marble was chosen first, the probability of choosing blue on the second pick (given blue was chosen first) is $\frac{4}{7}$ and the probability of choosing white on the second pick is $\frac{3}{7}$. Going back to the starting point and following the branch leading to a white marble selected first leaves all five blue marbles and two white marbles remaining, so the probability of choosing blue on the second pick (given white was chosen first) is $\frac{5}{7}$ and the probability of choosing white on the second pick is $\frac{2}{7}$. Finally, follow the same careful reasoning for each possible scenario when determining the probabilities for choosing the third marble. Again, the probabilities are purposely not reduced to make it easier to see that the sum of the probabilities of a set of branches equals 1.



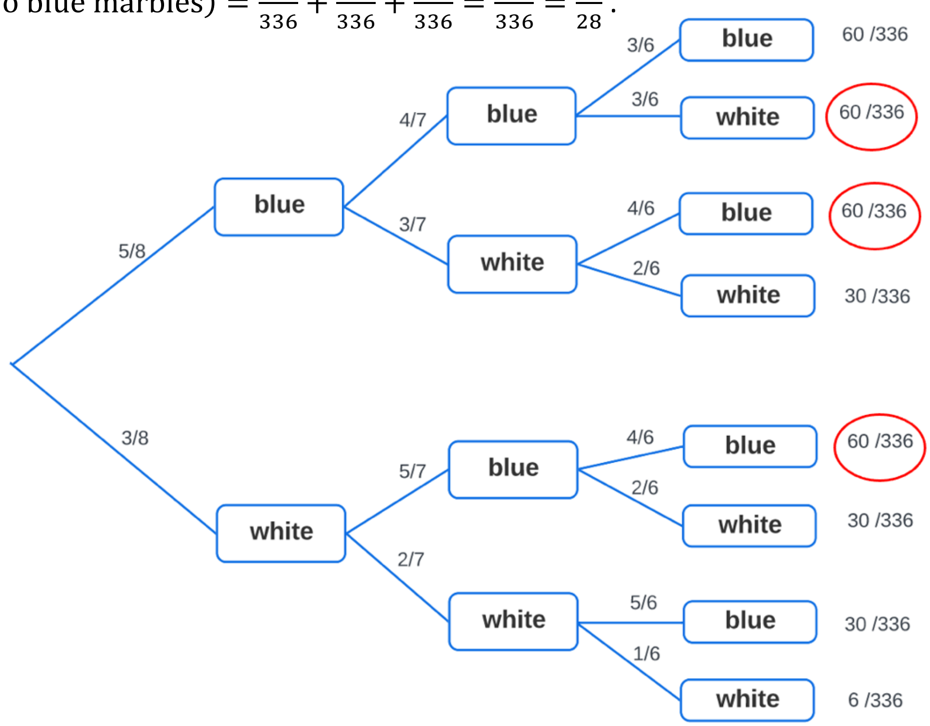
➢ **EXAMPLE 4.8.3:** Use the probability tree created for the experiment choosing three marbles from a bag containing five blue and three white marbles (without replacement), then find the following probabilities.

a. $\mathrm{P(white, blue, blue)}$.
b. $\mathrm{P(exactly\ two\ blue\ marbles)}$
c. $\mathrm{P(at\ least\ one\ white\ marble)}$
d. $\mathrm{P(at\ least\ one\ blue\ marble\ |\ blue\ first)}$

SOLUTION:

a. The results are listed in order with white first, blue second, and blue third. There is one possible result when choosing white, blue, and then blue. Follow the branches from left to right leading to white, and then to blue, and finally to blue. Multiply the successive probabilities to get $\mathrm{P(white, blue, blue)} = \frac{3}{8} \cdot \frac{5}{7} \cdot \frac{4}{6} = \frac{60}{336} = \frac{5}{28}$.

b. In the probability tree below, complete branches where two blue marbles were chosen (in any order) are highlighted with a red oval. Then find the sum to get $\mathrm{P(two\ blue\ marbles)} = \frac{60}{336} + \frac{60}{336} + \frac{60}{336} = \frac{180}{336} = \frac{15}{28}$.

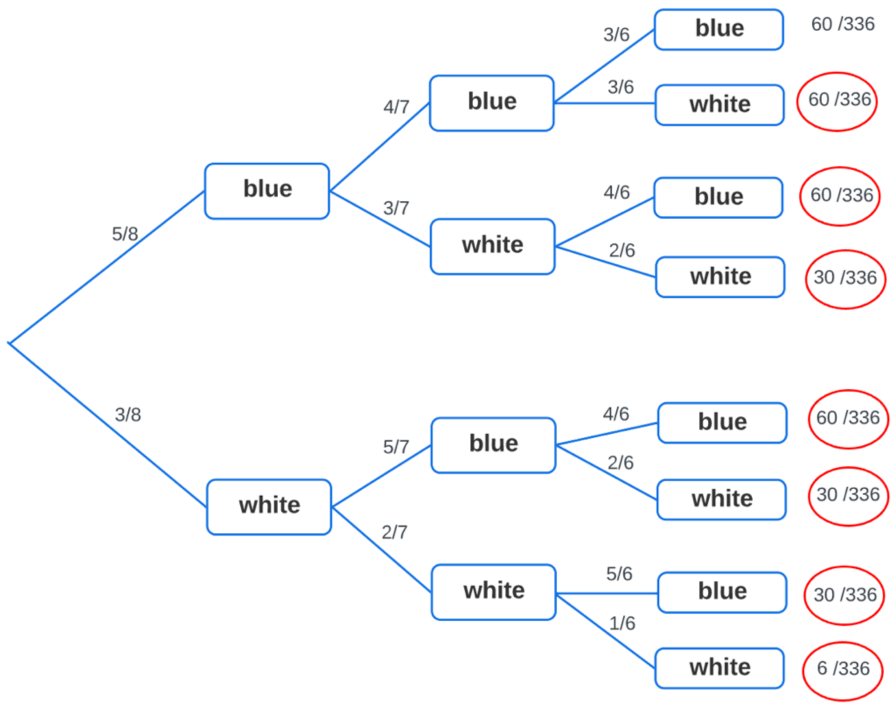


c. In the probability tree on the next page, complete branches where at least one white marble was chosen are highlighted with a red oval. Then find the sum of the complete branch probabilities where at least one white marble was chosen.

$$P(\text{at least one white marble}) = \frac{60}{336} + \frac{60}{336} + \frac{30}{336} + \frac{60}{336} + \frac{30}{336} + \frac{30}{336} + \frac{6}{336} = \frac{276}{336} = \frac{23}{28}.$$

It is also possible to calculate this same answer using complements.

$$P(\text{at least one white marble}) = 1 - P(\text{no white marbles}) = 1 - \frac{60}{336} = \frac{276}{336} = \frac{23}{28}$$



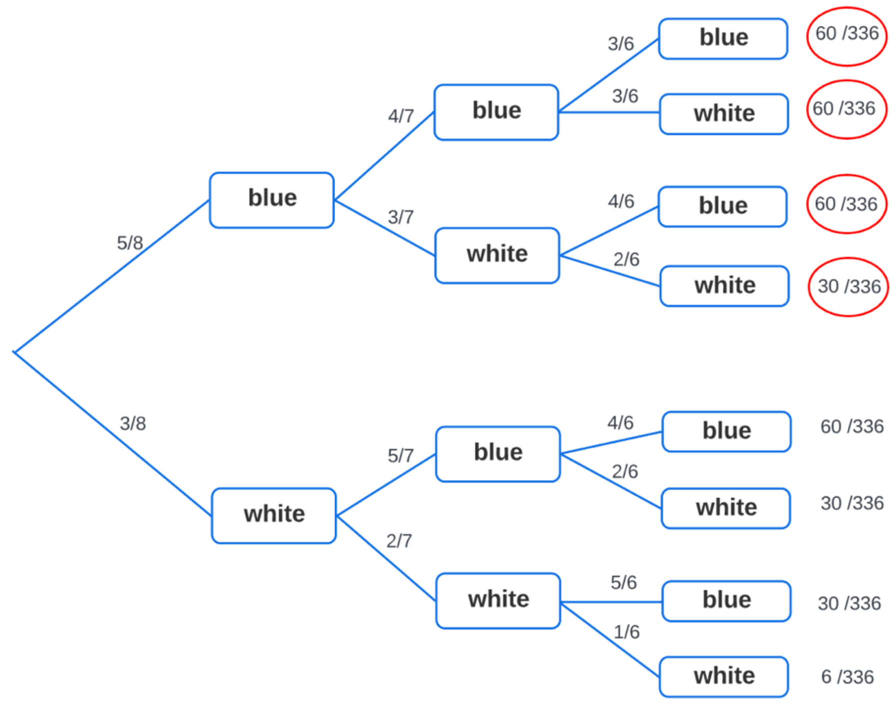
d. Finding conditional probability changes the sample space since some information is known. In this case, it is known that the first marble chosen was blue. Using the definition of conditional probability and the probability tree on the next page:

P(at least one blue marble | blue first)=

$$P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{\text{P(at least one blue and blue first)}}{\text{P(blue first)}}$$

$$= \frac{\frac{60}{336} + \frac{60}{336} + \frac{60}{336} + \frac{30}{336}}{\frac{5}{8}} = \frac{\frac{210}{336}}{\frac{5}{8}} = \frac{210}{336} \div \frac{5}{8} = \frac{210}{336} \cdot \frac{8}{5} = \frac{1{,}680}{1{,}680} = 1$$

NOTE: If it is known that a blue marble was chosen first, then all complete branches where a blue marble was chosen first would have at least one blue marble. Thus, having at least one blue marble is certain to occur giving a probability of 1.



➢ **YOU TRY IT 4.8.B:** Create a probability tree for the following experiment and then find the following probabilities.

A bag of marbles contains five blue, three green, and two white marbles. Create a probability tree for the experiment of choosing two marbles without replacement.

a. P(green, green)
b. P(exactly one blue marble)
c. P(at least one white marble)

Other Applications

Consider a championship competition where the best two teams compete in a series of games and the team that wins the most games in the series wins the title. The baseball World Series is an example of a best-of-seven series where the team to win four games out of a possible seven games wins the championship title. In hockey, the Stanley Cup is another example of a best-of-seven series where two teams compete until one team wins four games out of a possible seven to become the league champion. During the series, the game location changes every one or two games. For the purposes of this text, it is assumed that the location will switch each game. Typically, playing on a team's home field is considered an advantage. Because of this, the probability of winning may change from location to location.

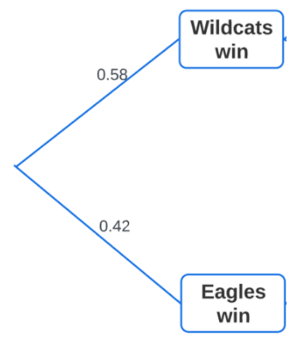
➢ **EXAMPLE 4.8.4:** Create a probability tree for the following best-of-three series and then find the following probabilities.

The Wildcats and the Eagles are in a best-of-three baseball competition. When the Wildcats play on their home field, their probability of winning is 0.58. When the Eagles play on their home field, their probability of winning is 0.67. The Wildcats had the best season record and will host the first competition on their home field and then the location will alternate for each game.

a. $\mathrm{P(Eagles\ win\ the\ series\ in\ two\ games)}$
b. $\mathrm{P(Wildcats\ win\ the\ series)}$
c. $\mathrm{P(Eagles\ win\ the\ series)}$

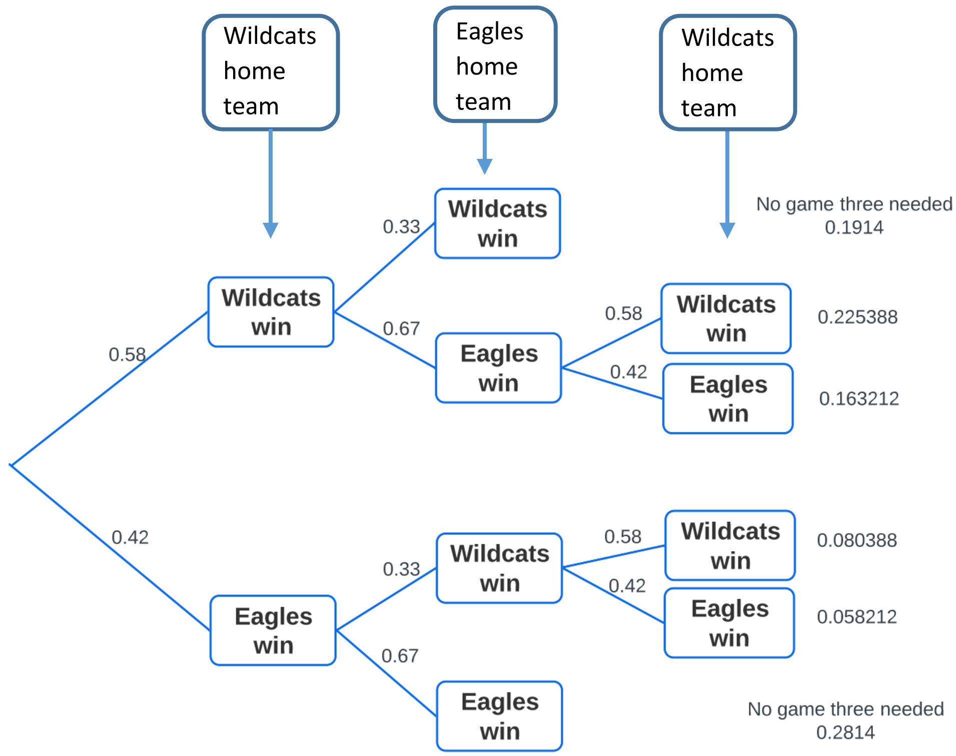
SOLUTION:

First, create the probability tree to represent the competition. In the first game, either the Wildcats win or the Eagles win. Since the Wildcats earned the first home game, $\mathrm{P(Wildcats\ win)} = 0.58$ (given) and the probability the Eagles win is the complement $\mathrm{P(Eagles\ win)} = 1 - \mathrm{P(Wildcats\ win)} = 1 - 0.58 = 0.42.$
Recall that the probabilities on a set of branches must total one.



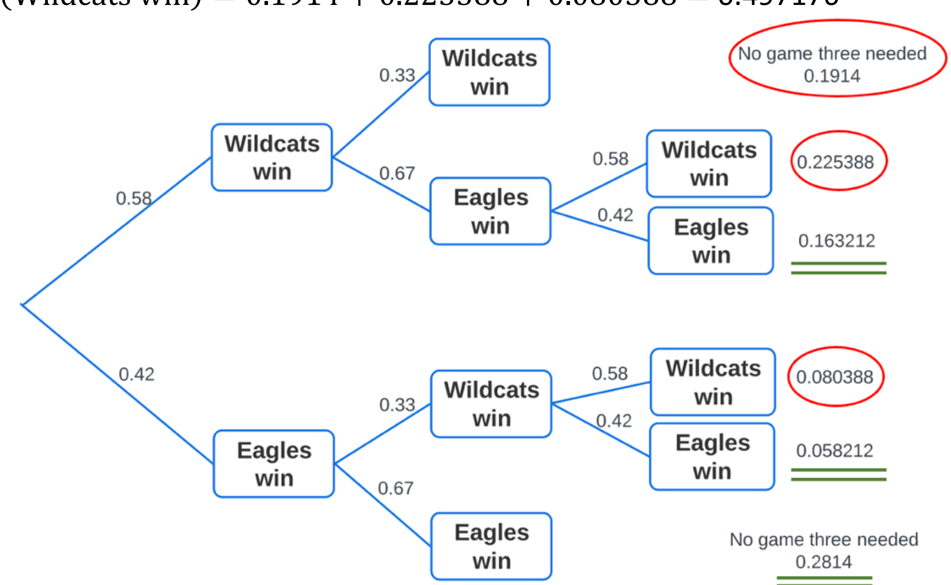
In this text, it is assumed that the location will change for each game making the Eagles the home team for the second game. When the Eagles are the home team, $\mathrm{P(Eagles\ win)} = 0.68$ (given) and the probability the Wildcats win is the complement $\mathrm{P(Wildcats\ win)} = 1 - \mathrm{P(Eagles\ win)} = 1 - 0.67 = 0.33.$

Finally, if a third game is necessary, the location changes back to the Wildcat's home field where $\mathrm{P(Wildcats\ win)} = 0.58$ and $\mathrm{P(Eagles\ win)} = 0.42$. When one teams wins a majority of games, two games in a three games series, the tournament ends. Notice in the uppermost branch in the following probability tree, the Wildcats won the first two games and the tournament ended. In the lowermost branch, the Eagles won the first two games and the tournament ended.



a. There is one possible result where the Eagles win the series in two games. Multiply the successive probabilities to get
$\text{P(Eagles win in two games)} = (0.42)(0.67) = 0.2814.$

b. To calculate the probability of the Wildcats winning the series, find the probability of each complete branch where the Wildcats won two games and then find the sum of the probabilities. Branches where the Wildcats win the tournament are highlighted with a red oval.
$\text{P(Wildcats win)} = 0.1914 + 0.225388 + 0.080388 = 0.497176$



c. To calculate the probability of the Eagles winning the series, find the probability of each complete branch where the Eagles won two games and then find the sum of the probabilities. Branches where the Eagles win the tournament are highlighted with a green underline in the previous probability tree for part b.
$\mathrm{P(Eagles\ win)} = 0.163212 + 0.058212 + 0.2814 = 0.502824$

Therefore, even though the Wildcats had home field advantage, it is more likely that the Eagles will win the series.

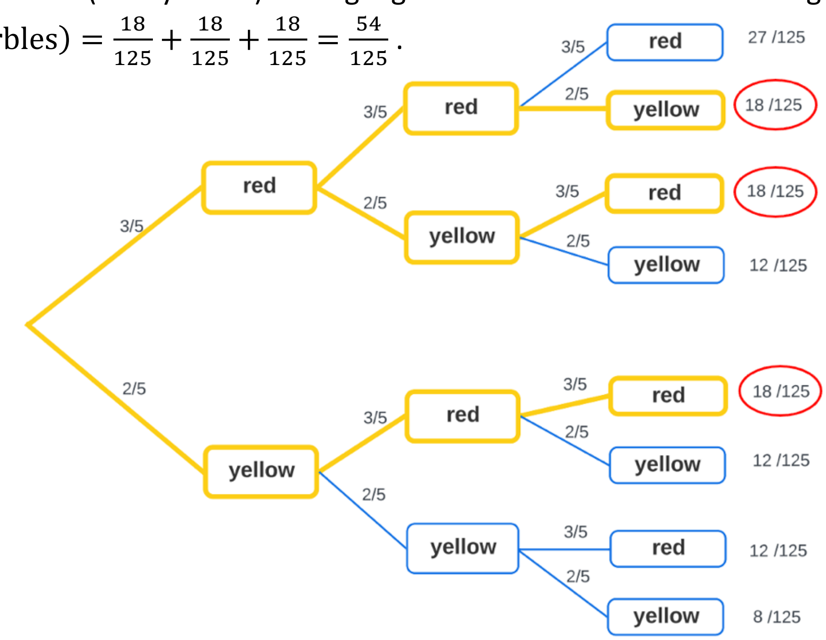
➢ **YOU TRY IT 4.8.C:** Create a probability tree for the following best-of-three series and then find the following probabilities.

The Cubs and White Sox are in a best-of-three crosstown Chicago baseball competition. When the White Sox play on their home field, their probability of winning is 0.7. When the Cubs play on their home field, their probability of winning is 0.65. The White Sox will host the first competition on their home field and the location will alternate for each game.

a. $\mathrm{P(White\ Sox\ win\ the\ series\ in\ two\ games)}$
b. $\mathrm{P(Cubs\ win\ the\ series)}$

Binomial Experiment

Consider the problem from part b of Exercise 4.8.2 where a bag contains three red and two yellow marbles. One marble is selected from the bag, the color is noted, and the marble is replaced back into the bag. This experiment is repeated three times, replacing the marble each time. To find the probability of choosing two red marbles, complete branches where two red marbles were chosen (in any order) are highlighted with a red oval. The sum gives
$\mathrm{P(two\ red\ marbles)} = \frac{18}{125} + \frac{18}{125} + \frac{18}{125} = \frac{54}{125}$.



Notice that in each complete branch containing exactly two red marbles, there are two factors of $\frac{3}{5}$ when a red marble was chosen and one factor of $\frac{2}{5}$ when a marble of another color was chosen. Since multiplication is commutative, the factors of $\frac{3}{5}, \frac{3}{5}$, and $\frac{2}{5}$ can be multiplied in any order for the product $\frac{18}{125}$. Three different complete branches contained exactly two red marbles. This is a combination of choosing two red marbles from a group of three marbles, or $C(3,2) = 3$, representing the three ways in which exactly two red marbles can be chosen. So, an alternate method for calculating the probability of two red marbles is shown below.

Number of combinations (branches) containing two red marbles and one marble of another color

Probability of choosing red twice

Probability of choosing yellow once

$$\text{P(two red marbles)} = 3 \cdot \left(\frac{3}{5} \cdot \frac{3}{5}\right) \cdot \frac{2}{5} = \frac{54}{125}$$

$$\text{P(two red marbles)} = \text{C}(3,2) \cdot \left(\frac{3}{5}\right)^2 \cdot \left(\frac{2}{5}\right)^1 = \frac{54}{125}$$

The calculation for this probability can be generalized as follows:

$$\text{P(two red marbles)} = \text{C}(3,2) \cdot \text{P(red)}^{\#\text{ times red chosen}} \cdot \text{P(other color)}^{\#\text{ times other color chosen}}$$

In an experiment where three marbles are chosen from a bag, finding the probability of picking exactly two red marbles is one specific example of a binomial experiment.

DEFINITION: A **binomial experiment** is an experiment with two distinct outcomes. The experiment is repeated a finite number of times with each trial independent of the other trials.

Four necessary criteria must be met in order to be able to calculate probability using the binomial experiment calculation.

Binomial Experiment Criteria

(1) The experiment must have two distinct, mutually exclusive outcomes. These outcomes can be categorized as success and failure.

(2) The experiment is repeated for a finite number of trials.

(3) Within each trial, the probability of success remains the same.

(4) Each trial is independent.

Returning to the example of selecting three marbles with replacement from a bag containing three red and two yellow marbles. All four binomial experiment criteria are met. First, there are two distinct, mutually exclave outcomes: the marble is red or it is not red. Success is defined as choosing a red marble. Second, the experiment is repeated three times which is a finite number. With replacement, the probability of choosing a red marble remains $\frac{3}{5}$ and the probability of not choosing a red marble remains $\frac{2}{5}$ each time a marble is chosen, confirming criteria three. And finally, each trial is independent. Specifically, knowing the result of the first marble color does not affect the results of the second or third marble color since the marble is replaced each time. As previously shown, the probability of choosing exactly two red marbles is calculated as follows:

$$\mathrm{P(two\ red\ marbles)} = \mathrm{C}(3,2) \cdot \left(\frac{3}{5}\right)^2 \cdot \left(\frac{2}{5}\right)^1 = \frac{54}{125}$$

$$\mathrm{P(two\ red\ marbles)} = \mathrm{C}(3,2) \cdot \mathrm{P(red)}^{\#\ \mathrm{times\ red\ chosen}} \cdot \mathrm{P(other\ color)}^{\#\ \mathrm{times\ other\ color\ chosen}}$$

This probability calculation can be generalized using r as the number of times a success occurs out of n trials. Let $\mathrm{p} = \mathrm{P(success)}$ and $\mathrm{q} = \mathrm{P(failure)}$. Because the two distinct outcomes are mutually exclusive, $\mathrm{q} = 1 - \mathrm{p}$. The probability of a binomial experiment is given by:

$$\mathrm{P(r\ successes\ from\ n\ trials)} = \mathrm{C(n,r)} \cdot (\mathrm{p})^{\mathrm{r}} \cdot (\mathrm{q})^{\mathrm{n-r}}$$

Calculating the Probability of a Binomial Experiment

Given an experiment meets the four binomial experiment criteria with n repeated trials and $\mathrm{P(success)} = \mathrm{p}$ and $\mathrm{P(failure)} = \mathrm{q}$, the probability of the binomial experiment is given by:

$$\mathrm{P(r\ successes\ )} = \mathrm{C(n,r)} \cdot \mathrm{p}^{\mathrm{r}} \cdot \mathrm{q}^{\mathrm{n-r}}$$

- **EXAMPLE 4.8.5:** A single die is rolled four times. Find the probability of rolling a six three times.

SOLUTION:

Confirm all four binomial experiment criteria are met. (1) There are two distinct outcomes where success is defined as rolling a six and failure is defined as not rolling a six. (2) The experiment is repeated four times which is a finite number. (3) The probability of success is $\text{P(roll a six)} = \frac{1}{6}$ and the probability of failure is $\text{P(not roll a six)} = \frac{5}{6}$. (4) Each trial is independent since the result of a particular roll will not affect the result of the following roll.

To find the probability of rolling a six three times, it might be helpful to restate the event of interest as the probability of rolling a six three times and a number other than six one time.

$$\text{P(roll a six three times)} = C(4,3) \cdot \left(\frac{1}{6}\right)^3 \cdot \left(\frac{5}{6}\right)^1$$

$$\text{P(roll a six three times)} = 4 \cdot \frac{1}{216} \cdot \frac{5}{6} = \frac{20}{1{,}296} = \frac{5}{324} \approx 0.015 \text{ or } 1.5\%.$$

➢ **EXAMPLE 4.8.6:** Jose's history quiz has 10 multiple choice questions. Each question has four choices. If Jose is guessing, find the following probabilities.

a. Find the probability that Jose answers eight questions correctly.
b. Find the probability that Jose answers at least eight questions correctly.

SOLUTION:

Confirm all four binomial experiment criteria are met. (1) There are two distinct outcomes and success is defined as answering the question correctly. (2) The experiment is repeated ten times which is a finite number. (3) The probability of success is $\text{P(correct)} = \frac{1}{4}$ and the probability of failure is $\text{P(not correct)} = \frac{3}{4}$. (4) Each trial is independent since the result of a particular question will not affect the result of the following question.

a. To find the probability of answering eight questions correctly, it might be helpful to restate the event of interest as the probability of answering eight questions correctly and two questions incorrectly.

$$\text{P(answer eight correctly)} = C(10,8) \cdot \left(\frac{1}{4}\right)^8 \cdot \left(\frac{3}{4}\right)^2$$

$$\text{P(answer eight correctly)} = 45 \cdot \frac{1}{65{,}536} \cdot \frac{9}{16} = \frac{405}{1{,}048{,}576} \approx 0.000386238 \text{ or } 0.04\%.$$

b. To find the probability of answering at least eight questions correctly, Jose can answer eight, nine, or all ten questions correctly. Therefore, a binomial experiment probability calculation is completed for each scenario, and then the sum of the resulting probabilities is found.

$$\text{P(answer eight correctly)} = \text{C}(10,8) \cdot \left(\frac{1}{4}\right)^8 \cdot \left(\frac{3}{4}\right)^2 = \frac{405}{1{,}048{,}576} \approx 0.000386238$$

$$\text{P(answer nine correctly)} = \text{C}(10,9) \cdot \left(\frac{1}{4}\right)^9 \cdot \left(\frac{3}{4}\right)^1 = \frac{30}{1{,}048{,}576} \approx 0.00002861$$

$$\text{P(answer ten correctly)} = \text{C}(10,10) \cdot \left(\frac{1}{4}\right)^{10} \cdot \left(\frac{3}{4}\right)^0 = \frac{1}{1{,}048{,}576} \approx 0.000000954$$

Thus, the probability of answering at least eight questions correctly is $0.000386238 + 0.00002861 + 0.000000954 = 0.000415802$ or 0.04%.

- **EXAMPLE 4.8.7:** A card is drawn from a standard 52-card deck five times without replacement. Find the probability of getting two aces.

SOLUTION:

Confirm all four binomial experiment criteria are met. (1) There are two distinct outcomes and success is defined as getting an ace. (2) The experiment is repeated five times which is a finite number. (3) The probability of success is $\text{P(ace)} = \frac{4}{52}$ on the first card drawn but changes to $\text{P(ace)} = \frac{3}{51}$ if an ace was drawn first or $\text{P(ace)} = \frac{4}{51}$ if an ace was not drawn first. Since the probability of success is not the same for all trials, this experiment is not a binomial experiment and must be calculated using another method.

Recall from Section 4.5, the probability can be calculated using combinations.

$$\text{P(two aces and three other cards)} = \frac{C(4,2) \cdot C(48,3)}{C(52,5)} = \frac{6 \cdot 17{,}296}{2{,}598{,}960} = \frac{103{,}776}{2{,}598{,}960} \approx 0.0399 \text{ or } 3.99\%.$$

- **YOU TRY IT 4.7.D:** Kiara is taking a 20-question true/false science test. If she is guessing, find the following probabilities:
  - **a.** Find the probability that Kiara answers 17 questions correctly.
  - **b.** Find the probability that Kiara answers at least 19 questions correctly.

Quick Review

- Use the probability for the intersection of events to multiply successive branches to calculate the probability of a complete branch in a probability tree.

  When more than one complete branch meets the description, find the probability of each complete branch that meets the description and then find the sum of these probabilities.

- A **binomial experiment** is an experiment with two distinct outcomes that are mutually exclusive. The experiment is repeated a finite number of times with each trial independent of the other trials.

  If the probability of success is p and the probability of failure is q, then the probability of r successes out of n trials is given by:

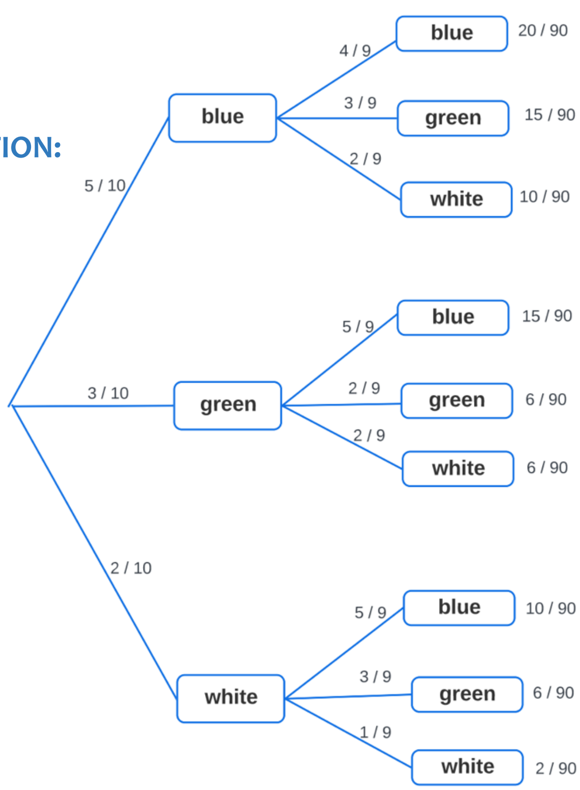
$$\mathrm{P}(\mathrm{r\ successes}) = \mathrm{C}(\mathrm{n},\mathrm{r}) \cdot \mathrm{p}^{\mathrm{r}} \cdot \mathrm{q}^{\mathrm{n-r}}$$

YOU TRY IT 4.8.A SOLUTION:

a. $\frac{1}{12}$
b. $\frac{7}{12}$
c. $\frac{1}{2}$

YOU TRY IT 4.8.B SOLUTION:

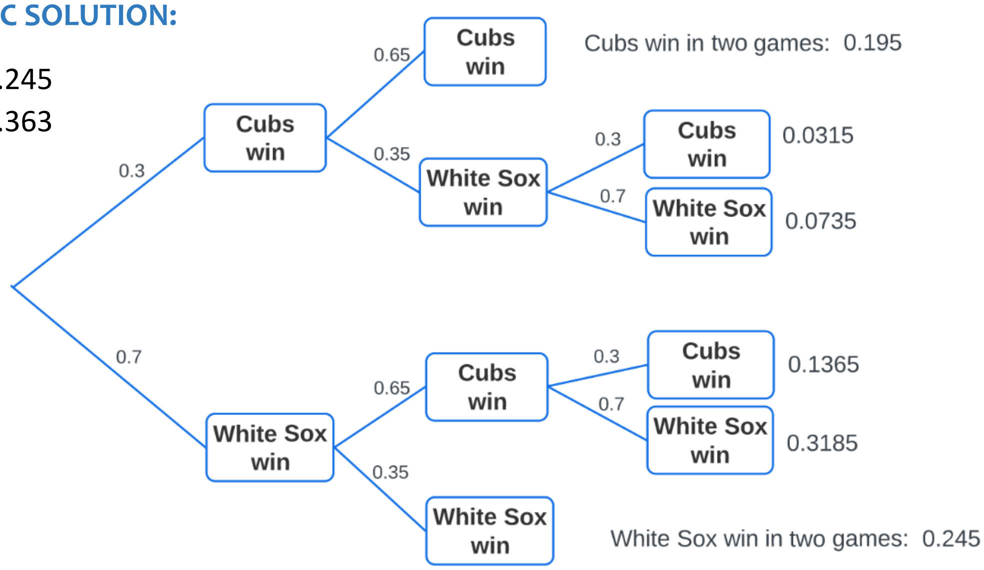
a. $\frac{1}{15}$
b. $\frac{5}{9}$
c. $\frac{17}{45}$



YOU TRY IT 4.8.C SOLUTION:

a. 0.245

b. 0.363



YOU TRY IT 4.8.D SOLUTION:

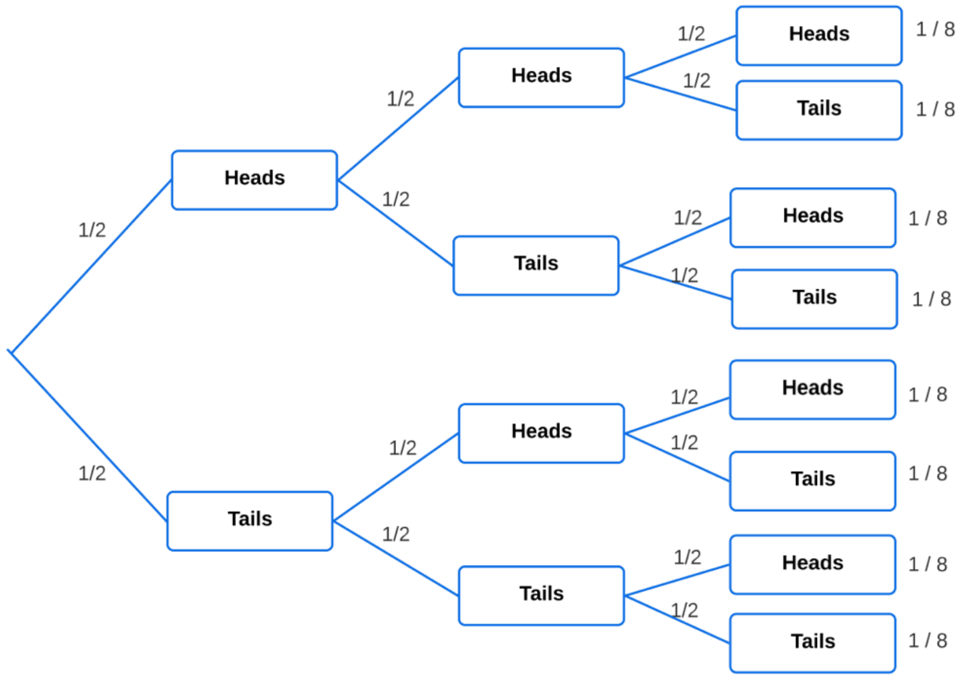
a. $\frac{285}{262{,}144} \approx 0.0011 = 0.11\%$

b. $\frac{21}{1{,}048{,}576} \approx 0.00002 = 0.002\%$

Section 4.8 | Exercises

In Exercises 1 – 7, use the probability tree below to answer the following questions. The probability tree represents flipping three coins. Find the following probabilities:

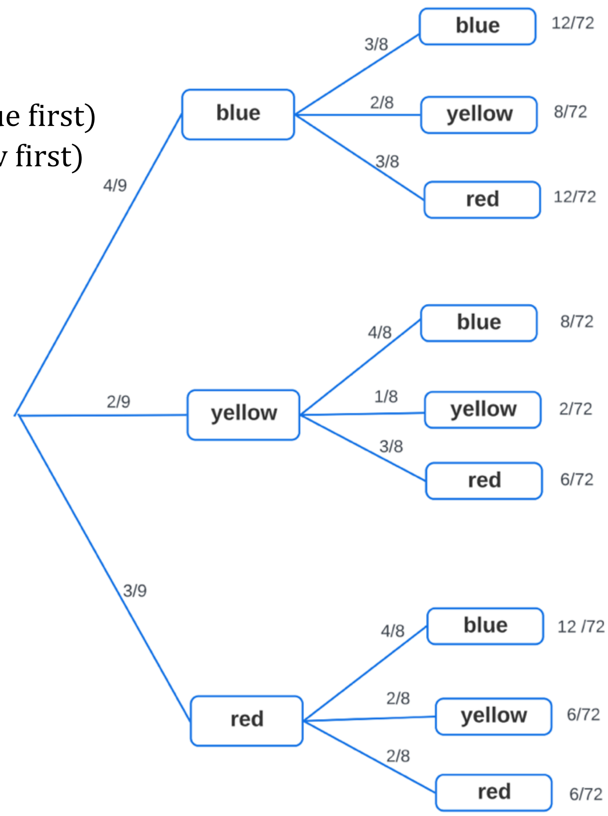
1. P(heads, tails, tails)
2. P(tails, heads, tails)
3. P(no tails)
4. P(one heads and two tails)
5. P(at least one heads)
6. P(heads third | heads first and tails second)
7. P(heads third | heads first)



In Exercises 8 – 14, use the probability tree below to answer the following questions. The probability tree represents choosing two marbles (without replacement) from a bag that contains four blue marbles, two yellow marbles, and three red marbles.

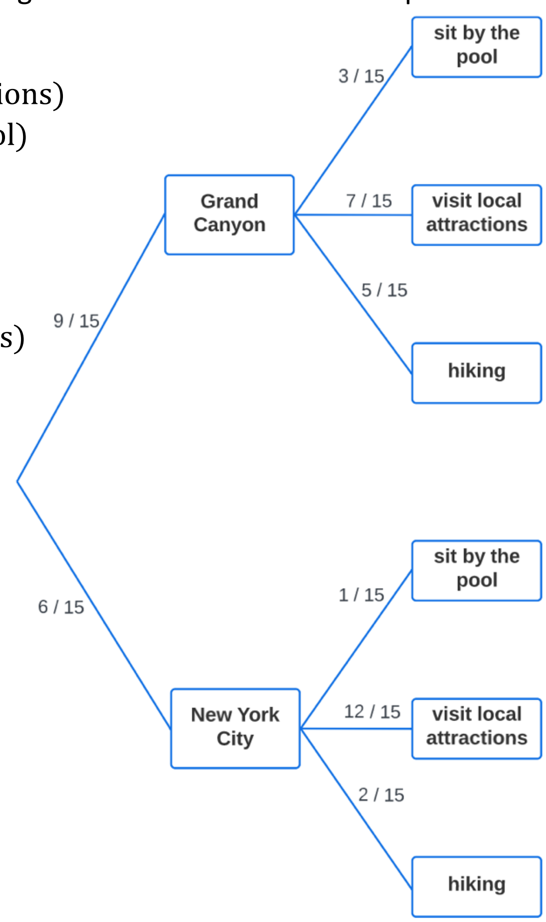
8. P(yellow, red)
9. P(blue, blue)
10. P(one blue and one red)
11. P(two marbles of the same color)

12. P(at least one blue)
13. P(yellow second | blue first)
14. P(red second | yellow first)



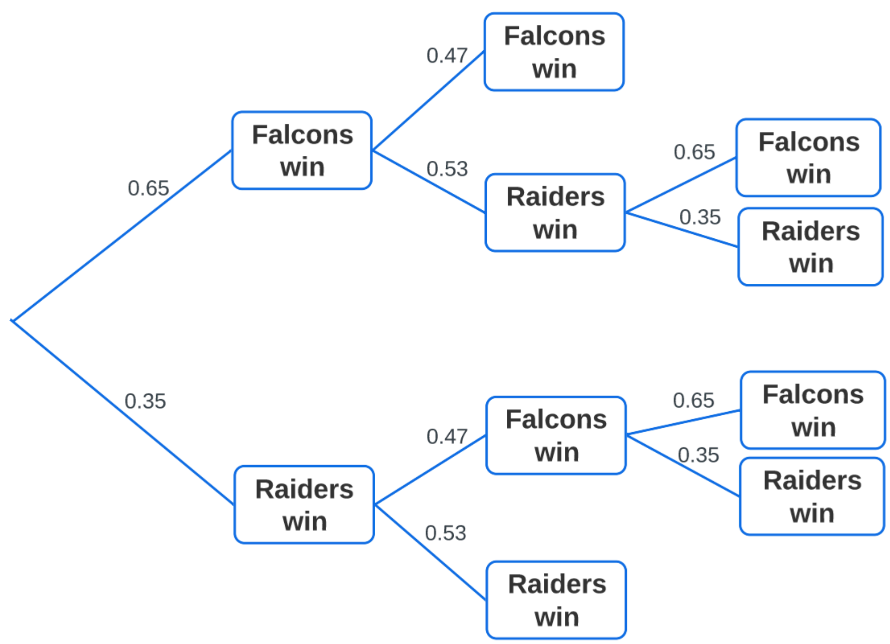
In Exercises 15 – 21, use the probability tree below to answer the following questions. The probability tree summarizes the results of a survey asking 15 travelers to choose their preferred activity vacation spot and activity.

15. P(Grand Canyon and visit local attractions)
16. P(New York City and sitting by the pool)
17. P(sit by the pool)
18. P(visit local attractions)
19. P(not hiking)
20. P(hiking | Grand Canyon)
21. P(New York City | visit local attractions)



In Exercises 22 – 30, use the probability tree to answer the following questions. The probability tree represents a best-of-three tournament.

22. P(Falcons win, Raiders win, Falcons win)
23. P(Raiders win, Falcons win, Raiders win)
24. P(Falcons win in two games)
25. P(Raiders win the tournament)
26. P(Falcons win the tournament)
27. P(three games played)
28. P(Raiders win the tournament | Falcons win the first game)
29. P(Falcons win the tournament | three games were played)
30. P(Falcons win the first game | Falcons win the tournament)



In Exercises 31 – 35, create a probability tree and answer the questions using the tree. For questions 31 – 34, express answers as simplified fractions.

31. Create a probability tree for the following experiment and then find the following probabilities.
    Flip one coin and roll one die.
    a. P(flip tails, roll a six)
    b. P(roll a three)
    c. P(flip a heads and roll an even number)
    d. P(roll a five | heads first)
    e. P(roll a prime number | tails first)
    f. P(flip heads | a two was rolled)

32. Create a probability tree for the experiment and then find the following probabilities. **Flip four coins.**

    a. P(tails, heads, tails, tails)
    b. P(flip exactly three tails)
    c. P(flips two heads and two tails)
    d. P(flip at least one heads)
    e. P(flip two or more tails)

33. Create a probability tree for the experiment and then find the following probabilities. **A bag of marbles contains three blue, two white, and five red marbles. Create a probability tree for the experiment of choosing two marbles (with replacement).**

    a. P(blue, white)
    b. P(blue, blue)
    c. P(a blue marble and a red marble)
    d. P(two marbles of the same color)
    e. P(two marbles that are not the same color)
    f. P(red second | blue first)

34. Create a probability tree for the experiment and then find the following probabilities. **A bag of marbles contains six green and four white marbles. Create a probability tree for the experiment of choosing three marbles (without replacement).**

    a. P(green, white, white)
    b. P(white, green, green)
    c. P(exactly one green marble)
    d. P(three marbles of the same color)
    e. P(two white marbles | green marble first)

35. Create a probability tree for the experiment and then find the following probabilities. **The Bears and the Giants are in a best-of-three basketball competition. When the Bears play on their home court, their probability of winning is 0.74. When the Giants play on their home court, their probability of winning is 0.70. The Bears will host the first competition on their home court and then the location will alternate for each game.**

    a. P(Bears, Giants, Bears)
    b. P(Giants win the competition in two games)
    c. P(a team wins the competition in two games)
    d. P(Bears win the competition)
    e. P(Giants win the competition)
    f. P(Bears win the competition | the competition was decided in two games)

In Exercises 36 – 60, if possible, calculate using the probability using binomial experiment.

36. List the four criteria that must be met to calculate probability using the binomial experiment.
37. A coin is flipped six times. Find the probability of getting heads four times.
38. A coin is flipped seven times. Find the probability of getting tails three times.
39. Boris has five cats. Find the probability of him having three female cats.
40. Yuriy and Svetlana have seven children. Find the probability that they have five boys
41. A single die is rolled 10 times. Find the probability of rolling a number less than 3 nine times.
42. A single die is rolled five times. Find the probability of rolling one two times.
43. Shannon has a ten-question true/false test. If Shannon guesses on each question, find the probability that he answers eight questions correctly.
44. Marcus is just learning to play darts. He hits the bullseye 35% of the time. On Friday, Marcus threw 40 practice darts. What is the probability that he hit the bullseye 10 times?
45. Dr. Gray administers a medication with a success rate of 81% to nine of her patients. What is the probability that seven of her patients are cured?
46. A baseball player has a batting average of 0.250, which means that the player gets a hit 25% of the time. If the player bats 18 times in one week, what is the probability that the player gets 10 hits that week?
47. Jaymes is taking a 20-question science test.
    a. If Jaymes guesses on each question, and the test contains 20 true-false questions, find the probability that he answers 15 questions correctly.
    b. If Jaymes guesses on each question and the test contains 20 multiple-choice questions, find the probability that he answers 15 questions correctly when each question has four answer options.
    c. In which scenario does Jaymes have a better chance of answering 15 questions correctly?
48. Samuel is taking a 15-question multiple choice test.
    a. If Samuel guesses on each question, find the probability that he answers 12 questions correctly when each question has four answer options.
    b. If Samuel guesses on each question, find the probability that he answers 12 questions correctly when each question has five answer options.
    c. In which scenario does Samuel have a better chance of answering 12 questions correctly?
49. A coin is flipped five times and then a single die is rolled three times. Find the probability of getting heads two times and rolling a six two times.

50. Mila's psychology test contains 10 true/false questions and eight multiple choice questions. If Mila guesses on each question, find the probability that she will answer eight true/false and six multiple-choice questions correctly. Each multiple-choice question has four answer choices.
51. JR makes 89% of all three-point shots and 79% of all free-throw shots while playing basketball. Suppose she shoots 20 three-point shots and nine free-throw shows. What is the probability that she makes 18 three-point shots and eight free-throw shots?
52. Suppose MJ has a 73% free-throw average when playing basketball.
    a. If MJ attempts 17 free-throws, find the probability that he makes exactly 14 shots.
    b. If MJ attempts 17 free-throws, find the probability that he makes at least 15 shots.
    c. In which scenario (a or b) does MJ have a better chance of making free-throws?
    d. What is the difference between the probabilities?
53. A coin is flipped 10 times. Find the probability that the coin landed on heads at least eight times.
54. Bryson is pitching and throws a strike 70% of the time. If he throws 12 pitches in one inning, find the probability that at least 10 pitches were strikes.
55. At a local bags tournament, 43% of bags thrown went into the hole. In a game, Maron threw 20 bags. Find the probability that at least 18 of her bags went into the hole.
56. A new drug has shown to be effective in 72% of participants during trials. In a group of 50 patients, what is the probability that the drug is not effective for at most 5 patients.
57. If a student guesses at each question, is the student more likely to get 10 out of 12 true/false questions correct or 16 out of 20 multiple-choice questions containing four-answer choices for each question?
58. Macy typically makes 75% of her free-throws. What is the minimum number of free-throws she should attempt in practice in order for the probability of making exactly 15 shots to be at least 20%?
59. Beckham enjoys playing darts. He determined that he makes a bullseye is 10% of the time. On Tuesday night, he threw 12 darts. What is the greatest number of bullseyes he needs to make in order for the probability to be at least 20%?
60. A smart phone manufacturer has determined that there is a 5% chance that a phone is defective after manufacturing.
    a. If the manufacturer produces 50 phones in a batch, find the probability that exactly two are defective.
    b. If the manufacturer produces 50 phones in a batch, find the probability that at most two are defective.
    c. What is the minimum number of phones that must be included in the batch in order for the probability of exactly two being defective to be 5% or less?
    d. What is the minimum number of phones that must be included in the batch in order for the probability of at most two being defective to be 5% or less?

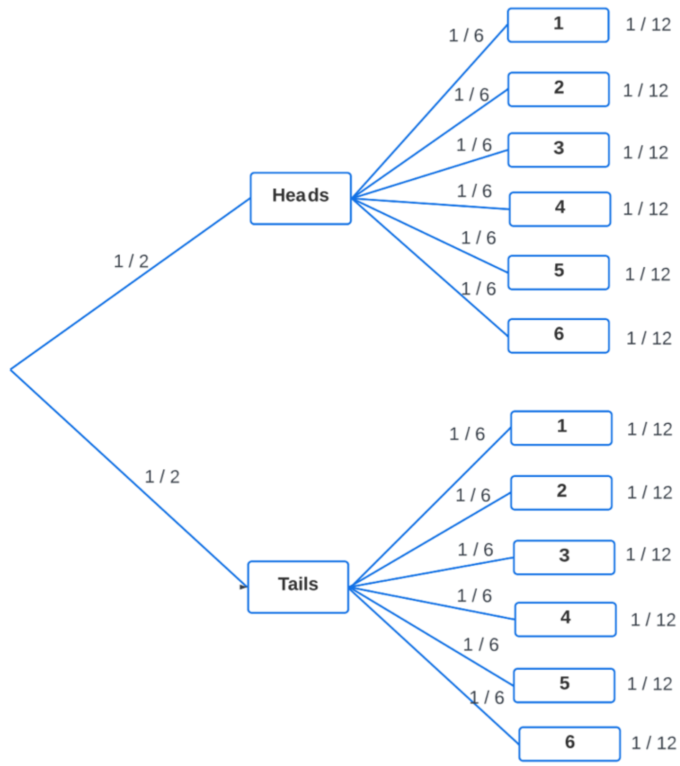
Section 4.8 Exercise Solutions

1. $\frac{1}{8}$
2. $\frac{1}{8}$
3. $\frac{1}{8}$
4. $\frac{3}{8}$
5. $\frac{7}{8}$
6. $\frac{1}{2}$
7. $\frac{1}{2}$
8. $\frac{1}{12}$
9. $\frac{1}{6}$
10. $\frac{1}{3}$
11. $\frac{5}{18}$
12. $\frac{13}{18}$
13. $\frac{1}{4}$
14. $\frac{3}{8}$
15. $\frac{7}{25}$
16. $\frac{2}{75}$
17. $\frac{11}{75}$
18. $\frac{3}{5}$
19. $\frac{56}{75}$
20. $\frac{1}{3}$
21. $\frac{8}{15}$
22. 0.223925
23. 0.057575
24. 0.3055
25. 0.36365
26. 0.63635
27. 0.509
28. $0.53 \cdot 0.35 = 0.1855$
29. $\frac{0.33085}{0.509} = 0.65$
30. $\frac{0.529425}{0.63635} = 0.8319713994$

31. a. $\frac{1}{12}$ b. $\frac{1}{6}$

c. $\frac{1}{4}$ d. $\frac{1}{6}$

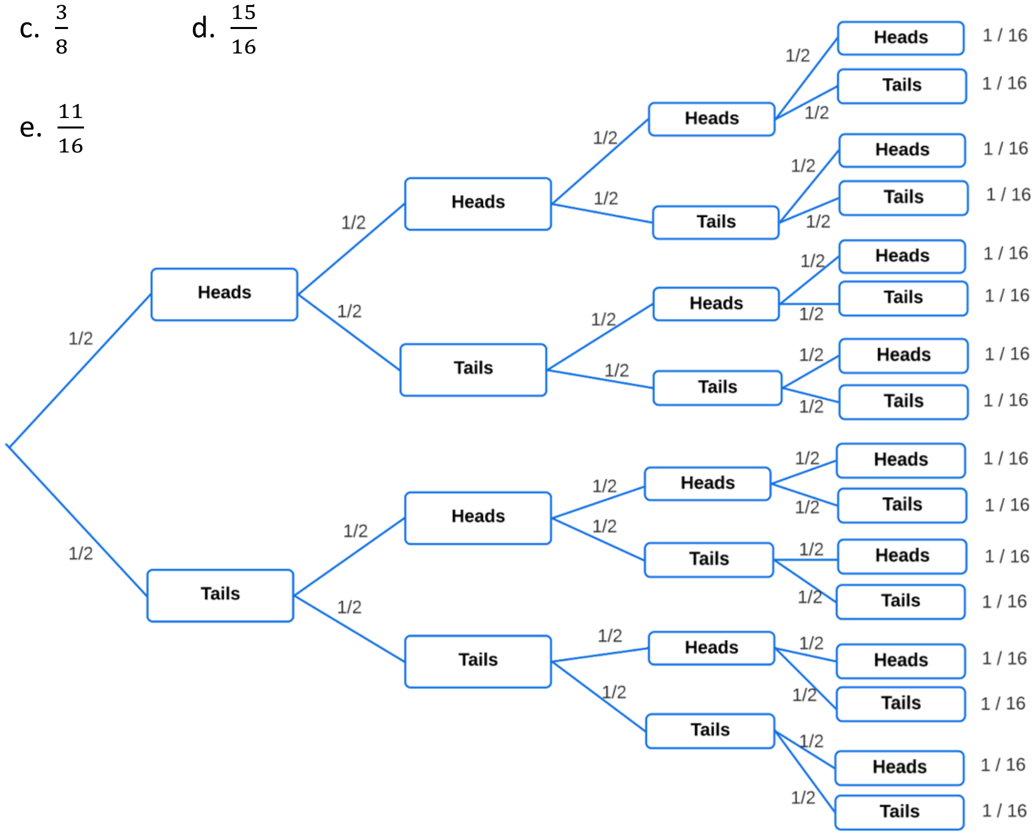
e. $\frac{1}{2}$ f. $\frac{1}{2}$



32. a. $\frac{1}{16}$ b. $\frac{1}{4}$

c. $\frac{3}{8}$ d. $\frac{15}{16}$

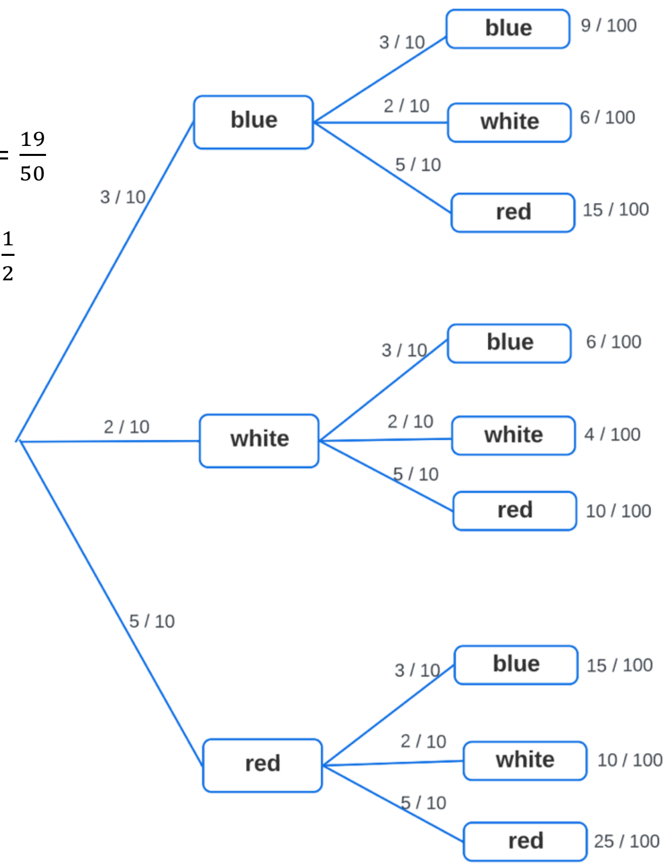
e. $\frac{11}{16}$



33. a. $\frac{6}{100} = \frac{3}{50}$ b. $\frac{9}{100}$

c. $\frac{30}{100} = \frac{3}{10}$ d. $\frac{38}{100} = \frac{19}{50}$

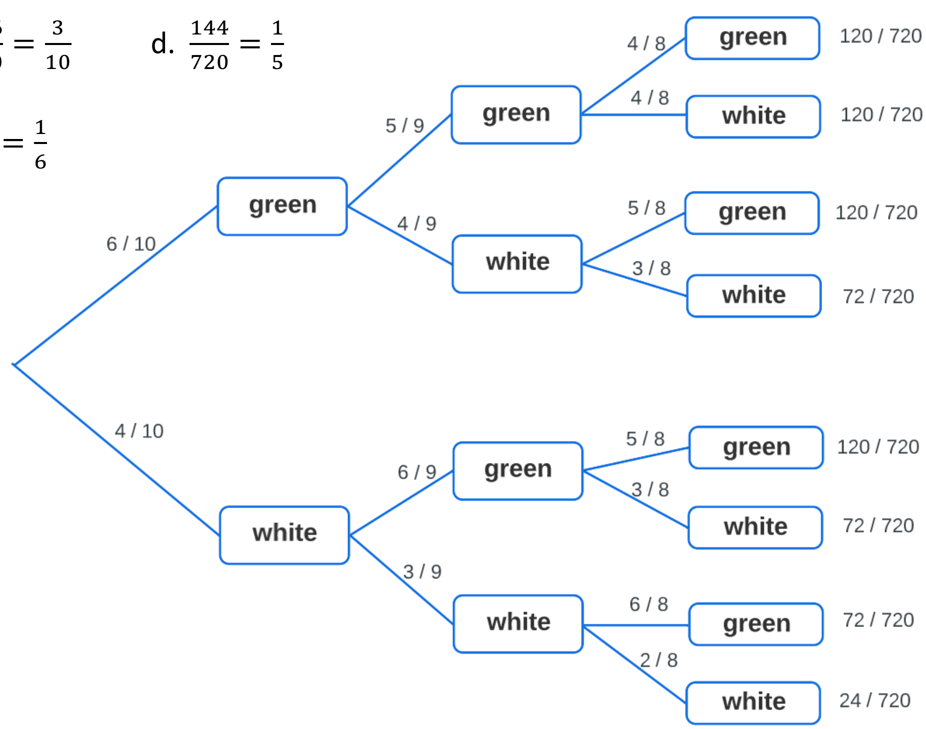
e. $\frac{62}{100} = \frac{31}{50}$ f. $\frac{5}{10} = \frac{1}{2}$



34. a. $\frac{72}{720} = \frac{1}{10}$ b. $\frac{120}{720} = \frac{1}{6}$

c. $\frac{216}{720} = \frac{3}{10}$ d. $\frac{144}{720} = \frac{1}{5}$

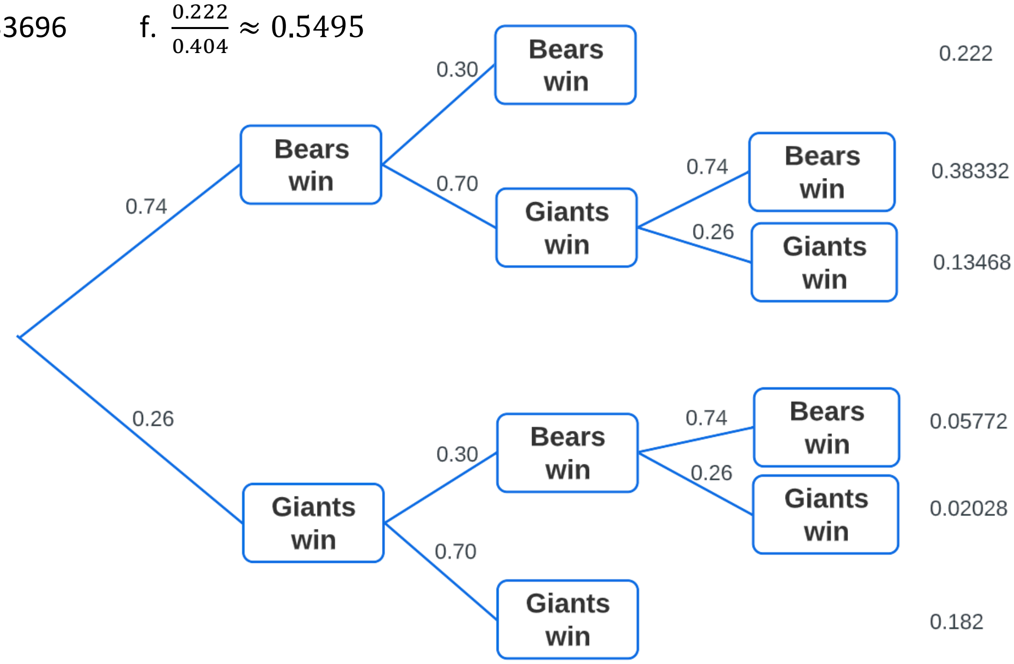
e. $\frac{12}{72} = \frac{1}{6}$



35. a. 0.38332 b. 0.1820

c. 0.404 d. 0.66304

e. 0.33696 f. $\frac{0.222}{0.404} \approx 0.5495$



36. Four criteria: (1) experiment must have two distinct outcomes, (2) experiment is repeated a finite number of times, (3) the probability of success remains the same for every trial, (4) each trial is independent.

37. $C(6,4) \cdot \left(\frac{1}{2}\right)^4 \cdot \left(\frac{1}{2}\right)^2 = \frac{15}{64} = 0.234375$

38. $C(7,3) \cdot \left(\frac{1}{2}\right)^3 \cdot \left(\frac{1}{2}\right)^4 = \frac{35}{128} = 0.2734375$

39. $C(5,3) \cdot \left(\frac{1}{2}\right)^3 \cdot \left(\frac{1}{2}\right)^2 = \frac{5}{16} = 0.3125$

40. $C(7,5) \cdot \left(\frac{1}{2}\right)^5 \cdot \left(\frac{1}{2}\right)^2 = \frac{21}{128} = 0.1640625$

41. $C(10,9) \cdot \left(\frac{1}{3}\right)^9 \cdot \left(\frac{2}{3}\right)^1 = \frac{20}{59{,}049} \approx 0.00033870176$

42. $C(5,2) \cdot \left(\frac{1}{6}\right)^2 \cdot \left(\frac{5}{6}\right)^3 = \frac{625}{3{,}888} \approx 0.1607510288$

43. $C(10,8) \cdot \left(\frac{1}{2}\right)^8 \cdot \left(\frac{1}{2}\right)^2 = \frac{45}{1024} \approx 0.04394453125$

44. $C(40,10) \cdot (0.35)^{10} \cdot (0.65)^{30} \approx 0.0570562341$

45. $C(9,7) \cdot (0.81)^7 \cdot (0.19)^2 \approx 0.2973067947$

46. $C(18,10) \cdot (0.25)^{10} \cdot (0.75)^8 \approx 0.0041778001$

47. a. $C(20,15) \cdot \left(\frac{1}{2}\right)^{15} \cdot \left(\frac{1}{2}\right)^{5} = \frac{15{,}504}{1{,}048{,}576} \approx 0.0147857666$
    b. $C(20,15) \cdot \left(\frac{1}{4}\right)^{15} \cdot \left(\frac{3}{4}\right)^{5} \approx 0.000003426$
    c. True/false test is more likely.
48. a. $C(15,12) \cdot \left(\frac{1}{4}\right)^{12} \cdot \left(\frac{3}{4}\right)^{3} \approx 0.000011441$
    b. $C(15,12) \cdot \left(\frac{1}{5}\right)^{12} \cdot \left(\frac{4}{5}\right)^{3} \approx 0.000000954$
    c. Four-answer choice test is more likely.
49. $C(5,2) \cdot \left(\frac{1}{2}\right)^{2} \cdot \left(\frac{1}{2}\right)^{3} \cdot\ C(3,2) \cdot \left(\frac{1}{6}\right)^{2} \cdot \left(\frac{5}{6}\right)^{1} \approx 0.0217013889$
50. $C(10,8) \cdot \left(\frac{1}{2}\right)^{8} \cdot \left(\frac{1}{2}\right)^{2} \cdot\ C(8,6) \cdot \left(\frac{1}{4}\right)^{6} \cdot \left(\frac{3}{4}\right)^{2} \approx 0.000168979$
51. $C(20,18) \cdot (0.89)^{18} \cdot (0.11)^{2} \cdot C(9,8) \cdot (0.79)^{8} \cdot (0.21)^{1} \approx 0.080917$
52. a. $C(17,14) \cdot (0.73)^{14} \cdot (0.27)^{3} \approx 0.16335$
    b. $C(17,15) \cdot (0.73)^{15} \cdot (0.27)^{2} + C(17,16) \cdot (0.73)^{16} \cdot (0.27)^{1} +$
    $C(17,17) \cdot (0.73)^{17} \cdot (0.27)^{0} \approx 0.12293$
    c. Exactly 14 shots is more likely.
    d. 0.04042
53. $C(10,8) \cdot \left(\frac{1}{2}\right)^{8} \cdot \left(\frac{1}{2}\right)^{2} + C(10,9) \cdot \left(\frac{1}{2}\right)^{9} \cdot \left(\frac{1}{2}\right)^{1} + C(10,10) \cdot \left(\frac{1}{2}\right)^{10} \cdot \left(\frac{1}{2}\right)^{0} = 0.0546875$
54. $C(12,10) \cdot (0.7)^{10} \cdot (0.3)^{2} + C(12,11) \cdot (0.7)^{11} \cdot (0.3)^{1} +$
    $C(12,12) \cdot (0.7)^{12} \cdot (0.3)^{0} \approx 0.2528153478 \approx 25.28\%$
55. $C(20,18) \cdot (0.43)^{18} \cdot (0.57)^{2} + C(20,19) \cdot (0.43)^{19} \cdot (0.57)^{1} +$
    $C(20,20) \cdot (0.43)^{20} \cdot (0.57)^{0} \approx 0.000016878 \approx 0.002\%$
56. $C(50,5) \cdot (0.28)^{5} \cdot (0.72)^{45} + C(50,4) \cdot (0.28)^{4} \cdot (0.72)^{46} +$
    $C(50,3) \cdot (0.28)^{3} \cdot (0.72)^{47} + C(50,2) \cdot (0.28)^{2} \cdot (0.72)^{48} +$
    $C(50,1) \cdot (0.28)^{1} \cdot (0.72)^{49} + C(50,0) \cdot (0.28)^{0} \cdot (0.72)^{50} \approx 0.001874 \approx 0.1874\%$
57. $C(12,10) \cdot \left(\frac{1}{2}\right)^{10} \cdot \left(\frac{1}{2}\right)^{2} \approx 0.0161132813$
    $C(20,16) \cdot \left(\frac{1}{4}\right)^{16} \cdot \left(\frac{3}{4}\right)^{4} \approx 0.000000357$
    The true/false test is more likely.
58. She must attempt 19 free-throws.
59. He must make 2 bullseyes.
60. a. $C(50,2) \cdot (0.05)^{2} \cdot (0.95)^{48} \approx 0.261101$
    b. $C(50,2) \cdot (0.05)^{2} \cdot (0.95)^{48} + C(50,1) \cdot (0.05)^{1} \cdot (0.95)^{49} +$
    $C(50,0) \cdot (0.05)^{0} \cdot (0.95)^{50} \approx 0.540533$
    c. 115 smart phones
    d. 124 smart phones