MATH 013: 3/19 WORKSHEET
SETS OF SETS

Set is mathematician speak for a collection of objects. Any kind of object can be an element of a set, and that includes other sets.

Why would this ever be useful?

In probability, we represent *events*—possible results of a random trial—as a set of atomic outcomes. For example, if your random trial is rolling two six-sided dice, you might be interested in the event where the two dices' values sum to 11. This event is the set $\{(5,6),(6,5)\}$ of the two outcomes which sum to 11.

If you want to talk about collections of events, you want to talk about a set whose elements are other sets. For example, if you want to talk about the collection of the events of the eleven possible sums of the dice, that would be a set whose elements are other sets.

In probability, events are all subsets of the same sample space. This is a common pattern in mathematics.

Subsets and powersets

If S is a set then the *powerset* of S, written $\mathcal{P}(S)$, is the set of all subsets of S. In symbols,

$$\mathcal{P}(S) = \{A : A \subseteq S\}.$$

Note that every element of $\mathcal{P}(S)$ is itself a set.

(1) List all elements of $\mathcal{P}(\emptyset)$.
(2) List all elements of $\mathcal{P}(A)$, where $A = \{1\}$.
(3) List all elements of $\mathcal{P}(B)$, where $B = \{1,2\}$.

Counting powersets

If $|S| = n$, then

$$|\mathcal{P}(S)| = 2^n 2^{|S|}.$$

To see why, think about how you specify a subset A of S. For each element of S, you either say yes it's in A or no it's not. That's a binary choice, and there are n many of them. So that's 2^n possible ways to make the choice.

(1) How many elements are in $V_0 = \emptyset$?
(2) How many elements are in $V_1 = \mathcal{P}(V_0)$?
(3) How many elements are in $V_2 = \mathcal{P}(V_1)$?
(4) How many elements are in $V_3 = \mathcal{P}(V_2)$?
(5) How many elements are in $V_4 = \mathcal{P}(V_3)$?
(6) How many elements are in $V_5 = \mathcal{P}(V_4)$?

(7) How many elements are in $V_6 = \mathcal{P}(V_5)$?

More counting

Let $|S| = n$. The number of subsets of S with k many elements is

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

The reason is, what you are counting is the number of ways to choose k different things from n options.

(1) A set S has 5 elements. How many subsets of S have 2 elements?
(2) A sample space S has 6 atomic outcomes. Assume the probability distribution is uniform. How many events over S have probability $\frac{1}{2}$? Probability $\frac{2}{3}$?