MATH 013: 3/5 WORKSHEET
SETS

For probability it will be helpful to have a little bit of language to talk about collections of things. Specifically, we will be talking about collections of possible outcomes of random trials. But you can talk about collections of any kind of things—mathematical objects like numbers, physical objects like tables, mental objects like memories, and so on.

Sets.

Set is mathematician talk for a collection of objects, which we call the *elements* or *members* of the set. You hear many synonyms—collection, family, group, and so on. These all mean the same thing. We will use capital letters $A, B, C, \dots$ to refer to sets, and use lowercase letters $a, b, c, \dots$ to refer to objects inside of sets.

With sets, the basic question you can ask is, "is object x an element of set A"? We write $x \in A$ to mean x is an element of A and write $x \notin A$ to mean x is not an element of A.

Sometimes you can specify a set by verbally describing it, such as saying P is the set of all people currently in this room. Often you want to specify a set by listing its elements.

Notation for sets.

To specify a set by listing all of its elements, we list them enclosed in brackets $\{,\}$. For example, you might consider the following sets:

$$\{1, 2, 3, 4\} \qquad \text{or} \qquad \{\text{red}, \text{orange}, \text{yellow}, \text{green}, \text{blue}, \text{purple}\}.$$

The order you list the elements doesn't matter. So $\{1, 2, 3, 4\}$ and $\{2, 4, 3, 1\}$ are two different ways to write the same set. Remember that with sets the only basic question we can ask is whether some object is an element of the set. It doesn't make sense to ask, for instance, what the third element of a set is.

Set-builder notation.

Often it's impractical to list all the elements of a set, so we want a shorthand. *Set-buider notation* gives us that shorthand. Let's look at an example first.

- $E = \{x : x \text{ is an even integer }\}$ is the set $\{\dots, -4, -2, 0, 2, 4, \dots\}$. You can't list off all the elements of this set because there are infinitely many!
- In general, set-builder notation looks like

$$A = \{x : \text{some pattern to check } x \text{ against}\}.$$

  Here, x is a name for a possible element of A, so that we can use that name in the pattern for which objects should be in x.

(1) One way to write the set $A = \{0, 2, 4\}$ using set-builder notation is

$$A = \{n : \text{there are species of mammals with } n \text{ legs}\}.$$

Write A in two more ways using set-builder notation. At least one should use purely mathematical concepts.

(2) List out the elements of

$$B = \{x : x \text{ is a logical connective}\}.$$

In addition to talking about what objects are elements of what sets, we also want to be able to compare sets.

Set equality and subsets.

Sets A and B are *equal* if they have exactly the same elements. We can express this using language from logic: $A = B$ means that for any object x we have $x \in A$ if and only if $x \in B$. What this means is, sets are defined only by their elements. How you write a set doesn't matter, only what objects are in it.

One set A is contained inside another B if every element of A is an element of B. More formally, we say A is a *subset* of B and write $A \subseteq B$. We can express this using logical language: $A \subseteq B$ means that for any object x if $x \in A$ then $x \in B$. If A is not a subset of B we write $A \nsubseteq B$.

(1) Explain why $A = B$ is equivalent to $A \subseteq B$ and $B \subseteq A$.
(2) Explain why $A \subseteq A$ is true for any set A.
(3) Come up with two sets A and B so that $A \nsubseteq B$ and $B \nsubseteq A$.

Some sets in mathematics have special names because they are used so often. This is the case with *number classes*—sets of numbers of a certain type.

Number Classes

Mathematicians write number classes in a funky font (called "blackboard bold"), because when you run out the alphabet what's left is to write the same letters but weird.

- $\mathbb{N}$ is the set of *natural numbers*, also called counting numbers. These numbers $0, 1, 2, 3, \ldots$ are those that are used to count collections.
- $\mathbb{Z}$ is the set of *integers*, whole numbers $\ldots, -2, -1, 0, 1, 2, \ldots$ which may be positive or negative. The letter 'Z' is chosen to stand for the German word *Zahlen*, which means integers, because mathematicians hate you.
- $\mathbb{Q}$ is the set of *rational numbers*, those which can be written as a fraction of integers. Think, 'Q' = Quotient.
- $\mathbb{R}$ is the set of *real numbers*, any number you can express with a decimal, such as $\pi, \sqrt{2}, \ldots$ The name is not great, but you should think of these numbers as measuring continuous quantities.
- $\mathbb{C}$ is the set of *complex numbers*, an extension of the real numbers which is useful in advanced applications of mathematics.

These number classes are nested:

$$\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}.$$

Counting

Given a set—a collection of objects—you may want to count how many things are in it. We write $|A|$ for the number of objects in A. When A is finite, $|A|$ is a natural number.

A special set is the *empty set*, the unique set with no elements, which we write as $\emptyset$. Note that $|\emptyset| = 0$.

(1) Explain why if $A \subseteq B$ then $|A| \leq |B|$. Is it possible to have $A \subseteq B$ and $|A| = |B|$?
(2) Explain why $\emptyset \subseteq A$ for any set A.
(3) The empty set was said to be the *unique* set with no elements. Explain why it's unique. Why isn't it possible to have two sets $A \neq B$ that each have no elements?