MATH 013: 2/5 WORKSHEET
COMPOSITIONS AND KHAYYAM'S TRIANGLE

Combinations.

With permutations order matters. A then B then C is different from A then C then B. If you are counting things where order doesn't matter, you need a different mathematical device than a factorial.

- $\binom{n}{k}$ (pronounced "n choose k") is the number of ways to select k objects from n choices, where order doesn't matter.
- As a formula,
$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{(n)_k}{k!}.$$

(1) A school is sending a team to a robotics competition. There are 10 students in robotics club but there is only space to send 3 of them. How many possible teams are there?

(2) You are a member of a large polycule consisting of 6 other people. You want to take 2 of them on a movie date. How many possible ways are there to do this?

(3) You and your friends are big into board games. You need to select games for a party. You have 20 different board games (you're *big* into board games) and will have time to play 4 of them. How many ways are there to pick them? What if you care about the order the games are played?

(4) One way to think about counting how many ways to select k things from n where order doesn't matter is this: Order k things from n. This overcounts, and to get the correct count you dived by the number of ways to order those k things. Use this way of thinking to explain the formula for $\binom{n}{k}$.

There is a pattern of numbers that has a connection to $\binom{n}{k}$.

Khayyam's triangle. This is a triangle of numbers. It starts, at row 0, with a single 1. Each new row has one more number than the previous, so row n has $n+1$ many numbers. The outer numbers are always 1s, and the inner numbers are calculated by adding the two numbers above them.

Here are the first few rows of Khayyam's triangle:

$$\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & & 1 & & & & & \\
 & & & & 1 & & 2 & & 1 & & & & \\
 & & & 1 & & 3 & & 3 & & 1 & & & \\
 & & 1 & & 4 & & 6 & & 4 & & 1 & & \\
 & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\
1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1
\end{array}$$

A little history.

This pattern was independently noticed by mathematicians in different areas—India (10th century), Persia (10th century), China (11th century), Italy (13th century).

Blaise Pascal, a French mathematician, studied it in 1665, and many western sources thereby name it Pascal's triangle. I follow convention elsewhere and name it Khayyam's triangle, in honor of a Persian mathematician who studied it five centuries before Pascal. Note, however, that Omar Khayyam was not the first to study the triangle. Like many old discoveries in mathematics, it is impossible to attribute it to a single person.

(1) Fill out the next three rows of the triangle.
(2) Observe that, after row 0, the second number in each row is the row number. Explain why that is.

Calculate $\binom{n}{k}$ for small values of n and k and you might notice a connection to the triangle.

Khayyam's triangle and combinations.

$$
\begin{array}{ccccccccc}
 & & & & \binom{0}{0} & & & & \\
 & & & \binom{1}{0} & & \binom{1}{1} & & & \\
 & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & \\
 & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & \\
\binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4}
\end{array}
$$

The kth number in the nth row of the triangle (where we count starting at $k = 0$ and $n = 0$) is $\binom{n}{k}$. To confirm this is a pattern and not just a coincidence for the first few rows, we need to separately explain why it works for the outer numbers and the inner numbers. They are generated by different rules so they need different explanations.

- (Outer numbers) The outer numbers are always 1, so we need to explain why $\binom{n}{0}$ and $\binom{n}{n}$ are always 1. For this, answer the following two questions: How many ways are there to pick n things from n? To pick 0 things from n?

- (Inner numbers) The inner numbers are the sum of the two numbers above, which amounts to saying that
$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$
To see why that's true, here's a way to think about choosing k things from n.
Break your collection up into the first $n-1$ things and the last thing. To pick k things, you either pick the last thing plus $k-1$ things from the first $n-1$, or you pick all k from the $n-1$ things. These two values are counted by $\binom{n-1}{k-1}$ and $\binom{n-1}{k}$, so you add them to get $\binom{n}{k}$.

(1) Pick a few different values of n and k and confirm $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$ by using the factorial formula to compute these three values.

(2) Which method do you find easier for computing $\binom{n}{k}$, the factorial formula or using the triangle? Why?

(3) Observe that the triangle is symmetric—if you flip it horizontally you get the same pattern. Combined with the fact that the entries in the triangle are the $\binom{n}{k}$'s, this tells you that
$$\binom{n}{k} = \binom{n}{n-k}.$$
Can you use the definition of $\binom{n}{k}$—that is, how many ways there are to select k things from n—to explain this equality?

Khayyam's triangle and binomial coefficients.

A useful algebra fact is that $(x+y)^2 = x^2 + 2xy + y^2$. Is there a pattern for $(x+y)^3$, $(x+y)^4$, and so on expansions of binomial powers?

- Let's think about $(x+y)^4$. When you expand the product into a sum of terms, each term comes from picking either x or y in the 4 copies of $(x+y)$. Some terms appear more than once; for example, x^3y can occur by picking the y first, second, third, or fourth. The *coefficient*—the number in front of the term after you combine like terms—is the number of ways to pick x's and y's to get that term.
- How many ways are there to pick 1 y's from 4 $(x+y)$'s? This is picking 1 thing from 4, so it is counted by $\binom{4}{1}$.
- In general, the number of $x^{4-k}y^k$ terms is counted by $\binom{4}{k}$.
- Even more generally, the number of $x^{n-k}y^k$ terms in the expansion of $(x+y)^n$ is $\binom{n}{k}$.

In all, what this tells us is that the n-th row of Khayyam's triangle gives the coefficients of the expansion of $(x+y)^n$. For this reason the values $\binom{n}{k}$ are also called *binomial coefficients*.

(1) Use Khayyam's triangle to write out the expansion for $(x+y)^3$.
(2) Use Khayyam's triangle to write out the expansion for $(A+B)^4$.
(3) Use Khayyam's triangle to write out the expansion for $(x-1)^3$.
(4) Use Khayyam's triangle to write out the expansion for $(x+2)^3$.