MATH 013: 1/29 WORKSHEET
MORE COUNTING: PERMUTATION AND COMBINATION

Last time we looked at how arithmetic can be used for counting. For things to line up, we needed the choices to be independent. What do we do when choices are not independent?

A motivating example.
A race has 6 people in it. How many possible outcomes are there, if all you care about is who came in first, second, . . . , and sixth?

- There are 6 possible positions and 6 racers, but there are not 6^6 possible outcomes. This is because exponentiation counts *independent* choices but these choices are not independent—if racer A gets first no one else can.
- Instead, think of it like this. There are 6 choices for first place. Once that is fixed, there are 5 choices for second place. Then 4 choices for third place, and so on down to 1 choice for sixth place.
- In total, there are $6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$ possible outcomes.

Factorial and permutations.
A *permutation* is mathematician-speak for an ordering of a set. That is, a permutation is a way to order a collection of things. The *factorial* is the mathematician operation which counts permutations.

- $n! = n(n-1)(n-2)\cdots 1$ counts the number of permutations of a set with n elements.
- Alternatively, you can define $n!$ by the rules

$$0! = 1$$
$$(n+1)! = (n+1) \cdot n!$$

(1) Write out the values of $n!$ for the first several n, until it gets too large to be reasonable.
(2) You are arranging books on a shelf. If you have 12 books, how many ways are there to order them on your shelf?
(3) You are again arranging books on a shelf. This time, however, you want books by the same author to be together. If you have books from 4 authors, with 3 books from each, how many ways are there to order them on your shelf?
(4) A qualifying race is ran for a marathon, where to qualify a racer must beat a certain time. If there are 6 entrants how many possible outcomes are there? In this context, an outcome is the information of the order of the sixth racers plus the info of who qualified.

Falling factorial.

Sometimes what you are counting isn't a whole permutation, but just how to order some of the things. For this we use a *falling factorial* to count them.

- $(n)_k = \underbrace{n(n-1)\cdots(n-k+1)}_{k \text{ many}}$.
- $(n)_k$ counts the number of ways to order k objects from a selection of n choices.
- $(n)_k$ only makes sense when $k \le n$.

(1) A race with 100 racers is run. You only care about who the top 3 racers are. How many possible outcomes are there?

(2) Explain why $(n)_n = n!$.

(3) You are arranging books on a shelf. You have 20 books, but only room for 16 on the shelf. How many ways are there to order them?

(4) One way to think of ordering k things from n choices is to order all n things, then throw away all but the first k. Use this way of thinking to explain why

$$(n)_k = \frac{n!}{(n-k)!}.$$

Combinations.

With permutations order matters. A then B then C is different from A then C then B. If you are counting things where order doesn't matter, you need a different mathematical device than a factorial.

- $\binom{n}{k}$ (pronounced "n choose k") is the number of ways to select k objects from n choices, where order doesn't matter.
- As a formula,

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{(n)_k}{k!}.$$

(1) A school is sending a team to a robotics competition. There are 10 students in robotics club but there is only space to send 3 of them. How many possible teams are there?

(2) You are a member of a large polycule consisting of 6 other people. You want to take 2 of them on a movie date. How many possible ways are there to do this?

(3) You and your friends are big into board games. You need to select games for a party. You have 20 different board games (you're *big* into board games) and will have time to play 4 of them. How many ways are there to pick them? What if you care about the order the games are played?

(4) One way to think about counting how many ways to select k things from n where order doesn't matter is this: Order k things from n. This overcounts, and to get the correct count you dived by the number of ways to order those k things. Use this way of thinking to explain the formula for $\binom{n}{k}$.