# MATH 013: 1/27 WORKSHEET
# COUNTING AND EXPONENTIATION

A basic use of arithmetic is to count things. You can always count a collection by pointing at each object in turn and counting up. But that's slow and we want shortcuts. The meaning of different arithmetic operations tells us how they can be used to count.

**The meaning of addition.**

Addition corresponds to *or* or bringing together disjoint collections. If you can choose one of $n$ objects or one of $m$ other objects, you have $n + m$ objects in total.

Examples.

- A farmer has 12 cows and 7 horses. She has $12 + 7$ animals in total.
- A fast-food restaurant offers 6 different burgers and 5 different sandwiches. In total it offers $6 + 5$ options.

Warning! Your collections must be *disjoint*—no overlap—for addition to give the correct count. If there is an overlap adding will overcount and you need to subtract the count of the overlap to get the correct count.

- A high school is looking for a qualified teacher for a general science class. Either a physics certificate or a chemistry certificate is required. If there are 8 teachers with a physics certificate, 11 teachers with a chemistry certificate, and 3 teachers with both, then there are $8 + 11 - 3$ teachers who can teach the class.

For all questions, write as a formula, then simplify to a single number. The formula is the more important part.

(1) A video game has you collect items for crafting. If you have 13 sticks, 12 pieces of coal, and 4 pieces of leather, how many crafting items do you have?

(2) Between 1 and 20 there are 10 odd numbers, 8 prime numbers, and 7 numbers that are both prime and odd. How many numbers between 1 and 20 are there which are either odd or prime? How many are neither odd nor prime?

(3) In a survey of a group of students, 22 reported vaping, 30 reported drinking alcohol, and 41 reported doing at least one of the two. How many students both vape and drink?

**The meaning of multiplication.**

Multiplication corresponds to making independent choices. If you have $n$ choices for option A and $m$ choices for option B, then you have $n \cdot m$ choices for the A/B combination.

Examples:

- A restaurant has 4 options for side and 11 options for sandwiches. In total that's $4 \cdot 11$ possible lunches.
- Suppose the restaurant also has 6 options for a drink. Then they'd have $4 \cdot 11 \cdot 6$ possible lunches.
- A theater has 12 rows, each with 20 chairs in it. That is $12 \cdot 20$ chairs in total, because you can think of choosing a chair as first choosing a row then choosing a chair inside it.

For all questions, write as a formula, then simplify to a single number. The formula is the more important part.

(1) A college offers 4 sections of a certain math course, each with a capacity of 32 students. How many students in total can they accommodate taking this course?

(2) What if the college adds 2 more sections which can each fit 120 students. Then how many students can they accommodate taking this course?

(3) To choose an outfit you pick a bottom, a top, and a jacket. If you have 10 bottoms, 8 tops, and 3 jackets, how many outfits can you put together? What if you also include a choice of one of your 4 shoes as a component?

(4) A fancy restaurant has a three course meal. The first course is either soup or salad, with 3 soup options and 2 salad options. The second course is an entree, with 12 options. The final course is dessert, with 8 options. How many possible meals are there?

**Exponentiation.**

Exponentiation $a^n$ can be defined as repeated multiplication. We call $a$ the *base* and $n$ the *exponent*. If $n$ is a positive integer, then

$$a^n = \underbrace{a \cdot a \cdot \cdots \cdot a}_{n \text{ times}}.$$

Zero 0 as an exponent can be thought of as multiplying together zero copies of a number. Thus,

$$a^0 = 1,$$

beacuse 1 is the starting point before doing any multiplying. (Compare to how 0 is the starting point before doing any adding.)

Negative exponents reverse the direction. Instead of repeated multiplication, they represent repeated division.

$$a^{-n} = \frac{1}{a^n} = \frac{1}{a \cdot a \cdot \cdots \cdot a}$$

(1) Check that the definitions of exponentiation for 0 and negative exponents ensures that $a^{n+1} = a^n \cdot a$ for any integer $n$.
(2) Use the definition of exponentiation to explain why $a^{n+m} = a^n \cdot a^m$.
(3) Check that $a^n \cdot a^{-n} = a^0$.
(4) Explain why $a^{n \cdot m} = (a^n)^m$.
(5) Explain why $a^{n-m} = \frac{a^n}{a^m}$.
(6) What is $0^n$ for positive $n$? For negative $n$? For $n = 0$?
(7) What is $1^n$? Does your answer depend on $n$?

Exponentiation could instead have been defined by counting.

**The meaning of exponentiation.**

Exponentiation counts the number of associations or functions. If you have $n$ many inputs which must each be independently assigned to one of $a$ many outputs, there are $a^n$ many possible associations/functions. Note that the exponent is the number of inputs and the base is the number of outputs.

Examples:

- A friend group is talking about their favorite color. If there are 6 colors to choose from and 3 friends, that is $6^3$ possible patterns for their favorite colors. You can connect this to the repeated multiplication idea by thinking of this as, 6 choices for friend A's favorite, 6 choices for friend B's favorite, and 6 choices for friend $C$'s favorite for $6 \cdot 6 \cdot 6$ in total.
- A company is organizing a lunch for its employees. There are 19 employees each of whom has their choice of 4 different lunch options. Thus there are $4^{19}$ many possible patterns for the employee lunches.
- A special case of using exponentiation to count is counting subsets. A *set* is mathematician-speak for a collection of things, and a *subset* of a set is a selection of some of its things. (Possibly all, possible none.) You can think of a subset of a set as assigning either In or Out to each object. This gives two choices per object, so if a set has $n$ objects then it has $2^n$ subsets.

For all questions, write as a formula. If you have a calculator then also simplify to a single number.

(1) There are 12 students in a class. At the end of the semester they will each be assigned one of 5 grades ($A$ through $F$). How many possible patterns of final grades are there?
(2) Consider the previous example, but now letter grades come in 3 varieties—plus, minus, or neither. Now how many possible patterns of final grades are there?
(3) A project manager buys coffee for her team of 10 people. (For some of them it's actually tea.) If the coffee shop has 8 different coffee options and 3 different tea options, how many possible drink orders are there the project manager might buy?
(4) List all $8 = 2^3$ subsets of the set $\{A, B, C\}$. (Don't forget the empty set with no objects in it!) Can you list all $16 = 2^4$ subsets of the set $\{A, B, C, D\}$?
(5) This example is to help you think about why it matters for counting with exponentiation that the choices be independent. There are two ways to order two people $A$ and $B$ in a line—either $A$ first or else $B$ first. If you think of ordering people as assigning positions, $A$ can be in first or last place, and $B$ can be in first or last place. Using exponentiation this would give $2^2 = 4$ many ways to order the two people. Explain why exponentiation is the wrong choice to count this.