MATH 211: 4/11 WORKSHEET
TAYLOR SERIES

Taylor series.

The power series $\sum_{n=0}^{\infty} a_n(x-c)^n$ with coefficient
$$a_n = \frac{f^{(n)}(c)}{n!}$$
is called a *Taylor series*. If it is centered at $c = 0$ it is also called a *Maclaurin series*. (They are so named because Maclaurin studied them after Taylor did in the early 18th century, who in turn studied them well after Madhava first looked at examples coming from trig.)

Here's some useful Taylor series:
$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$
$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$
$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$
$$\ln(1-x) = -\sum_{n=1}^{\infty} \frac{x^n}{n}$$
$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$
$$\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

(1) What are the radii of convergence of these series? Some we've already done, others you have to calculate.

Adding and subtracting power series.
If $f(x) = \sum_{n=0}^{\infty} a_n x^n$ and $g(x) = \sum_{n=0} b_n x^n$ are given by power series with the same center, you can work out power series for their sum and difference, with radius of convergence the smaller of the two:

$$f(x) + g(x) = \sum_{n=0}^{\infty} (a_n + b_n) x^n$$

$$f(x) - g(x) = \sum_{n=0}^{\infty} (a_n - b_n) x^n$$

$$cf(x) = \sum_{n=0}^{\infty} c a_n x^n \qquad (c \text{ constant})$$

(1) Write the Maclaurin series for $\sin x - \cos x$.
(2) Write the Maclaurin series for $3 \cos x$.
(3) Write the Maclaurin series for $e^x + e^{2x}$.

Taylor polynomials.

If instead of an infinite sum you take a finite sum you get a polynomial approximation for the function $f(x) = \sum_{n=0}^{\infty} a_n(x-c)^n$. Call

$$f_n(x) = \sum_{n=0}^{k} a_n(x-c)^n$$

the *k-th Taylor polynomial* for $f(x)$. Note that $k = 1$ gives the tangent line at $x = c$, so you can think of these as giving better approximations than the tangent line.

(1) A plane currently has a velocity of 420 miles per hour. Taking its starting position as 0, determine the tangent line approximation for its position $x(t)$ as a function of time. Use this to estimate its position after 3 hours.
(2) Now take into account acceleration. If the plane's acceleration is currently -20 miles per hour, determine the degree 2 approximation for $x(t)$, and use it to estimate its position after 3 hours.
(3) Calculate the first few Taylor polynomials for $\sin x$. Use a graphing calculator to graph $\sin x$ and these Taylor polynomials. What do you observe?
(4) Do the same for $\cos x$.
(5) Do the same for e^x.
(6) Pick a different series we looked at and do the same.

Differential equations.

Recall that a differential equation is an equation involving derivatives of an equation. You can use power series to solve, or approximately solve, differential equations.

(1) Consider the differential equation $y' - y = 0$ with initial condition $y(0) = 1$. Explain why $y = e^x$ is the solution. Imagining you didn't know the solution, find it using power series.
   (a) Assume that the solution $y = \sum_{n=0}^{\infty} a_n x^n$ is given by a power series.
   (b) Calculate the power series for y' by differentiating term by term.
   (c) Now substitute the two series into the formula $y' - y = 0$ to get an equation involving a single power series.
   (d) The initial condition $y(0) = 1$ tells you what a_0 is. Substitute that into the equation and you can find a_1. Then you can find a_2. Repeat to find a general formula for a_n. This gives you the power series for y.

(2) Do the same process to find the solution to $y'' + y = 0$ satisfying the initial condition $y(0) = 0$ and $y'(0) = 1$.