MATH 211: 3/31 WORKSHEET
CONVERGENCE TESTS I

Today we focus on series whose terms are all positive. We will have various tests you can use to check if a series converges or diverges. Most tests are one way; if they apply they tell you the convergence status, but if they don't apply they say nothing.

A useful fact: Any positive-term series either converges to a positive number or else diverges to $+\infty$.

Divergence test.
If $\lim_{n\to\infty} a_n \neq 0$ then $\sum_{n=1}^{\infty} a_n$ diverges.

Comparison test.
Suppose $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are positive term series where $a_n \leq b_n$ for all n.

- If $\sum_{n=1}^{\infty} b_n$ converges then $\sum_{n=1}^{\infty} a_n$ converges.
- If $\sum_{n=1}^{\infty} a_n$ diverges then $\sum_{n=1}^{\infty} b_n$ diverges.

Limit comparison test.
Consider $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ positive term series. Suppose that $\lim_{n\to\infty} \frac{a_n}{b_n} \leq c$, for c a constant (equivalently: for all infinite N, $\mathrm{st}(\frac{a_N}{b_N}) \leq c$). Then

- If $\sum_{n=1}^{\infty} b_n$ converges then $\sum_{n=1}^{\infty} a_n$ converges.
- If $\sum_{n=1}^{\infty} a_n$ diverges then $\sum_{n=1}^{\infty} b_n$ diverges.

Integral test.
Suppose $f(x)$ is a continuous, decreasing, always positive function. Then $\int_1^{\infty} f(x)\,\mathrm{d}x$ converges if and only if $\sum_{n=1}^{\infty} f(n)$ converges.

p-series.
The *p-series* is a series of the form $\sum_{n=1}^{\infty} \frac{1}{n^p}$, where p is a constant.
The p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if and only if $p > 1$.

For each series determine whether it converges or diverges. Which test did you use? If more than one can be applied, try giving multiple explanations.

(1) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n+4}$

(2) $\displaystyle\sum_{n=1}^{\infty} \frac{3}{n^2-1}$

(3) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+1}{n^3-1}$

(4) $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{3^n+4}$

(5) $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n}$

(6) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{1+\ln n}$

(7) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{2^n}$

(8) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n-n^2}$

(9) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n!}$

Determine whether each improper integral converges by checking convergence of a series.

(1) $\displaystyle\int_2^{\infty} \frac{\mathrm{d}x}{\ln x}$

(2) $\displaystyle\int_2^{\infty} x^{-x}\,\mathrm{d}x$