MATH 211: 2-17 WORKSHEET
SECTION 6.6: APPLICATIONS TO PHYSICS

Applications of integration to physics follow the pattern we've seen of the infinite sum theorem. To calculate some quantity we divide it into infinitesimal pieces then add them together by integration.

Density

Density is a measure of how tightly mass is packed into a space. The most commonly used is three dimensional (mass per volume) but you can also make sense of two dimensional (mass per area) and one dimensional (mass per length). These are useful when what you are looking at is best model as two or one dimensional. We will focus on the case of one dimensional density.

Imagine a sequence of metal beads on a chain, of varying length and material. The mass of each bead is its density ρ ("rho") multiplied by its length Δx. Thus, the total mass is $\sum \rho \Delta x$.

If instead of discrete beads you imagine a continually changing density $\rho(x)$ at location x, then the total mass is

$$\sum \rho(x)\,\mathrm{d}x \approx \int_a^b \rho(x)\,\mathrm{d}x.$$

(1) Use the integration formula to confirm that if a chain has a constant density ρ and length ℓ then its mass is $\rho\ell$.
(2) A chain has density $\rho(x) = \frac{e^x + e^{-x}}{2}$, where $-2 \leq x \leq 2$. Determine its total mass.
(3) A rope has density $\rho(x) = \sin x + 1$, where $0 \leq x \leq \pi$. Determine its total mass.

Moments (center of mass)

The *(first) moment* or *center of mass* of an object is the point around which it is balanced. The one dimensional case of this can be visualized as objects balanced on a beam. In order for the beam to not rotate on its fulcrum the two sides have to be balanced. This balancing is not simply in terms of mass; a mass farther away from the fulcrum will have a larger impact. This is because its *torque* (force times distance from the point of rotation) is larger. Thus, to have a balanced beam you need that

$$\sum m_i x_i = 0,$$

where m_i is the mass of the i-th object and x_i is the distance of the i-th object from the fulcrum.

This same idea can be used to define the center of mass of an object of continually varying density $\rho(x)$. Namely, the mass at position x is $\approx \rho(x)\,\mathrm{d}x$, which we multiply by the position x. So we add them up by integrating to get the moment:

$$M = \int_a^b x\rho(x)\,\mathrm{d}x.$$

(1) Use integration to confirm that if a chain has constant density ρ then its center of mass is the midpoint of its length.
(2) Calculate the moment of the chain from (2) in the previous section.
(3) A rope has density given by $\rho(x) = 1 + \sqrt{x}$, with $0 \leq x \leq 4$. Determine its mass and center of mass.

Probability

A *continuous random variable* is one which takes on a continuum of real values, rather than a discrete set of values. For example, the result of rolling a six-sided die is a discrete random variable, since there are six discrete options. On the other hand, the mass of a randomly selected rock is best modeled as a continuous random variable.

The *probability density function* of a continuous random variable X is a function $\rho(x)$ of real numbers where $\rho(x)$ represents the likelihood that $X = x$. To determine the probability that $a \leq X \leq b$ we add up the infinitesimal probability masses $\rho(x)\,\mathrm{d}x$ by integrating:

$$\Pr(a \leq X \leq b) = \int_a^b \rho(x)\,\mathrm{d}x.$$

For this to make sense it must be that $\rho(x) \geq 0$ everywhere and the total area under the curve is 1. This is because negative probability doesn't make sense and the total probability needs to be 1.

The choice of names is not coincidence. Probability mass and density behave much like physical mass and density. Under this analogy, the corresponding concept to center of mass is the *expected value* or *mean* of a random variable. Think of how you compute the mean of a discrete random variable: you sum up the possible values weighted by their probabilities. For example, the mean result from rolling a fair six-sided die is

$$\sum_{n=1}^{6} n \cdot \frac{1}{6} = 3.5.$$

The same idea works for continuous random variables, but you add up the infinitesimal masses $x \cdot \rho(x)\,\mathrm{d}x$. Thus, if X is a continuous random variable whose possible values lie in the interval $[a, b]$ its expected value is

$$E(X) = \int_a^b x\rho(x)\,\mathrm{d}x.$$

Note that this is identical to the center of mass formula.

(1) A random variable X takes on values between 0 and 1 given by the density function $\rho(x) = \frac{4}{3} - x^2$. Confirm that this really is a probability density function and compute its mean. Then compute the probability than the value of X is greater than its mean.
(2) The *uniform* probability distribution on an interval $[a, b]$ is the distribution where all points are equally likely. That is, the density function is constant. Write a formula for the uniform density function and confirm that its mean is the midpoint between a and b.

Note that many important probability density functions can take on values on an infinite interval. To make sense of this we will have to understand how to take an integral like

$$\int_{-\infty}^{\infty} \rho(x)\,\mathrm{d}x.$$

We will get to that topic soon.

Work

In physics, *work* is a measure of energy transfer due to the application of force along a distance. If the force F is constant, then the work is $W = Fx$, where x is the distance the objecct is moved. If the force $F(x)$ is not constant then we want to think of work as being composed of infinitesimal pieces. Namely, at position x the infinitesimal slice of work is $F(x)\,\mathrm{d}x$. Thus the total work is the integral

$$W = \int_a^b F(x)\,\mathrm{d}x.$$

(1) A spring exerts a force $F = cx$ when compressed a distance x from its equilibrium point ℓ, where c is a constant that depends on the spring. Determine the work done in compressing the spring from position $\ell - a$ to $\ell - b$.
(2) The force of gravity between particles of mass M and m is

$$F = \frac{gMm}{s^2},$$

where s is the distance between the particles. Determine the work needed to move the particle of mass m from distance a to distance b. When is this work negative? When is it positive?