# MATH 211
# RULES FOR HYPERREALS

## Extending the reals to the hyperreals

(1) **The Extension Principle**
- The reals are a subset of the hyperreals;
- There is a nonzero infinitesimal;
- Any function $f$ on the reals has a *natural extension* to a function on the hyperreals, with the same number of variables.

(2) **The Transfer Principle.** Any *real statement* about functions on the reals holds for their natural extensions on the hyperreals.

A *real statement* is a finite combination of equalities, inequalities, or statements about whether a function is defined or undefined.

Examples:
- (Commutativity of addition) $x + y = y + x$;
- (Rules for $<$) If $0 < x < y$ then $0 < 1/y < 1/x$;
- (Domains) $\sqrt[3]{x}$ is defined everywhere;
- (No division by 0) $x/0$ is never defined;
- (Algebraic identities) $x^2 - y^2 = (x + y)(x - y)$;
- (Trig identities) $\sin^2 x + \cos^2 x = 1$.

Nonexamples:
- There are no nonzero infinitesimals.
- The domain of sin only consists of real numbers;
- Every input to the exponential function is finite.

## The algebra of infinitesimal, finite, and infinite numbers

For these, $\varepsilon$ and $\delta$ (epsilon and delta) are infinitesimal, $b$ and $c$ are finite but non-infinitesimal, and $H$ and $K$ are infinite.

(1) **Real Numbers**
- (a) 0 is the only infinitesimal real number;
- (b) Every real number is finite.

(2) **Negatives**
- (a) $-\varepsilon$ is infinitesimal;
- (b) $-b$ is finite and non-infinitesimal;
- (c) $-H$ is infinite.

(3) **Reciprocals**
- (a) $1/\varepsilon$ is infinite (if $\varepsilon \neq 0$);
- (b) $1/b$ is finite;
- (c) $1/H$ is infinitesimal.

(4) **Sums**
- (a) $\varepsilon + \delta$ is infinitesimal;
- (b) $b + \varepsilon$ is finite and non-infinitesimal;
- (c) $b + c$ is finite (possibly infinitesimal);
- (d) $H + \varepsilon$ and $H + b$ are infinite.

(5) **Products**
- (a) $\varepsilon \cdot \delta$ and $\varepsilon \cdot b$ are infinitesimal;
- (b) $b \cdot c$ is finite and non-infinitesimal;
- (c) $b \cdot H$ and $H \cdot K$ are infinite.

(6) **Quotients**
- (a) $\varepsilon/b$, $\varepsilon/H$, and $b/H$ are infinitesimal;
- (b) $b/c$ is finite and non-infinitesimal;
- (c) $b/\varepsilon$, $H/\varepsilon$, and $H/B$ are infinite.

(7) **Powers**
- (a) $\varepsilon^c$ is infinitesimal;
- (b) $b^c$ is finite and non-infinitesimal;
- (c) $H^c$ is infinite.

(8) **Indeterminate forms**
- (a) $\varepsilon/\delta$, $H/K$, $\varepsilon \cdot H$ and $H + K$ could all be either infinitesimal, finite but non-infinitesimal, or infinite. It depends on the specific values.

  Remember: $0/0$, $\infty/$, $0 \cdot \infty$, and $\infty - \infty$ are indeterminate.

## Properties of $\approx$

Two numbers $a$ and $b$ are *infinitely close*, $a \approx b$, if their difference $a - b$ is infinitesimal.

(1) **Basic properties.**
  (a) $a$ is infinitesimal if and only if $a \approx 0$;
  (b) If $a$ and $b$ are real and $a \approx b$ then $a = b$.
(2) **Equality-like properties.**
  (a) $a \approx a$;
  (b) If $a \approx b$ then $b \approx a$;
  (c) If $a \approx b$ and $b \approx c$ then $a \approx c$.
(3) **Size properties.** Assume $a \approx b$.
  (a) If $a$ is infinitesimal then so is $b$;
  (b) If $a$ is finite then so is $b$;
  (c) If $a$ is infinite then so is $b$.

## Standard parts

**Standard part principle.** Every finite hyperreal number $a$ is infinitely close to exactly one real number. We call this number the *standard part* of $a$, and denote it $\mathrm{st}(a)$.

(1) **Basic properties.** Let $a$ be finite.
  (a) $\mathrm{st}(a)$ is a real number;
  (b) $a = \mathrm{st}(a) + \varepsilon$ for some infinitesimal $\varepsilon$;
  (c) If $a$ is real then $a = \mathrm{st}(a)$.
(2) **Arithmetic properties.** Let $a$ and $b$ be finite.
  (a) $\mathrm{st}(-a) = -\mathrm{st}(a)$;

$$\lim_{x\to c} -f(x) = -\lim_{x\to c} f(x)$$

  (b) $\mathrm{st}(a+b) = \mathrm{st}(a) + \mathrm{st}(b)$;

$$\lim_{x\to c} f(x) + g(x) = \lim_{x\to c} f(x) + \lim_{x\to c} g(x)$$

  (c) $\mathrm{st}(a-b) = \mathrm{st}(a) - \mathrm{st}(b)$;

$$\lim_{x\to c} f(x) - g(x) = \lim_{x\to c} f(x) - \lim_{x\to c} g(x)$$

  (d) $\mathrm{st}(a \cdot b) = \mathrm{st}(a) \cdot \mathrm{st}(b)$;

$$\lim_{x\to c} f(x)g(x) = \left(\lim_{x\to c} f(x)\right)\left(\lim_{x\to c} g(x)\right)$$

  (e) $\mathrm{st}(a/b) = \mathrm{st}(a)/\mathrm{st}(b)$, if $\mathrm{st}(b) \neq 0$;

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)}$$

  (f) $\mathrm{st}(a^n) = \mathrm{st}(a)^n$;

$$\lim_{x\to c}(f(x))^n = \left(\lim_{x\to c} f(x)\right)^n$$

  (g) $\mathrm{st}(\sqrt[n]{a}) = \sqrt[n]{\mathrm{st}(a)}$, if $a \geq 0$;

$$\lim_{x\to c} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to c} f(x)}$$

  (h) If $a \leq b$ then $\mathrm{st}(a) \leq \mathrm{st}(b)$.

If $f(x) \leq g(x)$ then $\lim_{x\to c} f(x) \leq \lim_{xtoc} g(x)$.