# Math 211 Midterm 1

Friday, February 28

Name: Answer Key

This is the first midterm. There are seven questions, for a total of 100 points. **No electronic devices are permitted nor outside notes are permitted.** Carefully read each question and understand what is being asked before you start to solve the problem. Please show all your work and circle or mark in some way your final answers.

$$\frac{\mathrm{d}}{\mathrm{d}x}\arcsin x = \frac{1}{\sqrt{1-x^2}}$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\arccos x = -\frac{1}{\sqrt{1-x^2}}$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\arctan x = \frac{1}{1+x^2}$$

$$\int_a^b 2\pi x \,\mathrm{d}s$$
$$\int_a^b 2\pi f(x) \,\mathrm{d}s$$

1. (15 points) Find the mean value of the function $f(x) = -2xe^{-x^2}$ along the interval $-1 \leq x \leq 1$. Give an exact answer, fully simplified.

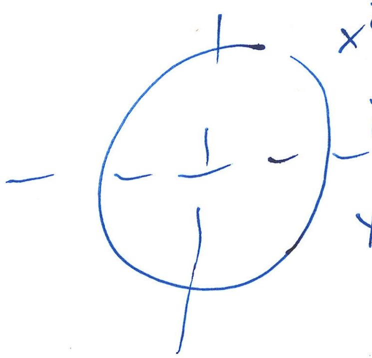
$$\text{Mean Value} = \frac{1}{1-(-1)} \int_{-1}^{1} -2x e^{-x^2} dx = \frac{1}{2} \int_{-1}^{-1} e^u du = \frac{1}{2} \cdot 0$$

$u = -x^2$
$du = -2x\,dx$

$x = 1 \to u = -1$
$x = -1 \to u = -1$

$$= 0$$

2. (15 points) Set up **but do not solve** an integral which gives the circumference of a circle of radius $r$. Your integral should be either a $dx$ integral or a $dy$ integral.



$x^2 + y^2 = r^2$

$y = \pm\sqrt{r^2 - x^2}$

$y' = \pm \frac{-x}{\sqrt{r^2-x^2}}$

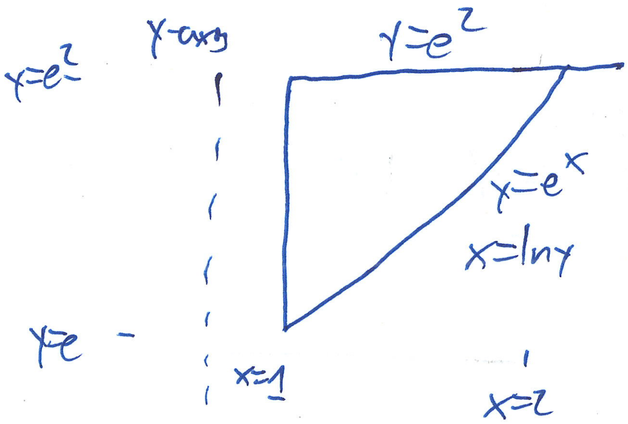
Circum $= 2 \cdot$ top arc length

$$= 2 \cdot \int_{-r}^{r} ds$$

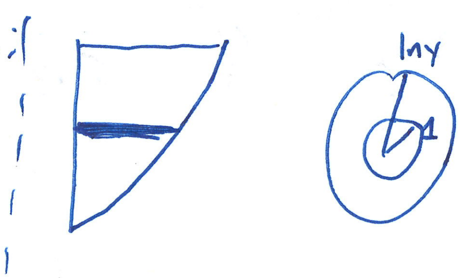
$$= 2 \cdot \int_{-r}^{r} \sqrt{1 + \frac{x^2}{r^2-x^2}}\, dx$$

$$= 2 \cdot \int_{-r}^{r} \sqrt{\frac{r^2}{r^2-x^2}}\, dx$$

3. (20 points) Consider the region bounded by the three curves $x = 1$, $y = e^2$, and $y = e^x$. This region is rotated around the $y$-axis to produce a three-dimensional solid. Set up **but do not solve** two different integrals to give the volume of this solid. One integral must use the disk/washer method, and the other must use the cylindrical shell method.

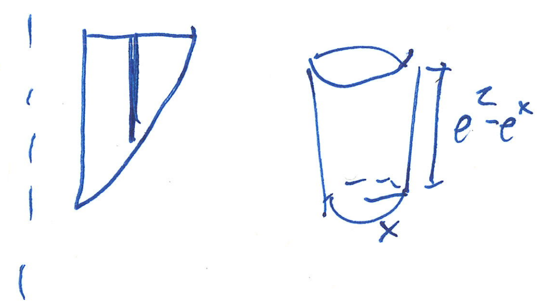



Washer:




$$Vol = \int_e^2 \pi\left([\ln y]^2 - 1\right) dy$$

Shell:




$$Vol = \int_1^2 2\pi x\left(e^2 - e^x\right) dx$$

4. (15 points) Your friend tells you that $\rho(x) = \frac{1}{2} - \frac{x}{8}$, where $0 \le x \le 4$, is a probability density function and asks you to find its mean. Confirm that this really is a probability density function and then find its mean. Give an exact answer, fully simplified.

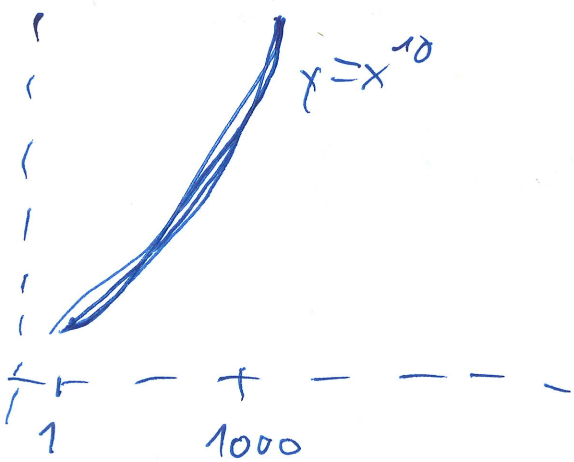
Need $\int \rho(x)\,dx = 1$:

$$\int_0^4 \frac{1}{2} - \frac{x}{8}\,dx = \frac{x}{2} - \frac{x^2}{16}\Big|_0^4 = \frac{4}{2} - \frac{16}{16} = \underline{1}$$

$$\text{Mean} = \int_0^4 \frac{x}{2} - \frac{x^2}{8}\,dx = \frac{x^2}{2\cdot 2} - \frac{x^3}{8\cdot 3}\Big|_0^4 = \frac{16}{4} - \frac{64}{8\cdot 3}$$

$$= 4 - \frac{8}{3} = \underline{\frac{4}{3}}$$

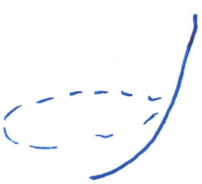
5. (15 points) Consider the curve $y = x^{10}$, where $1 \leq x \leq 1000$. This curve is rotated around the $y$-axis to make a surface and then around the $x$-axis to make another surface. Set up **but do not solve** integrals which give the surface area of these two surfaces formed by rotation. Both integrals should be $dx$ integrals. Without calculating them, say which surface area is larger and explain why.




$y' = 10x^9$

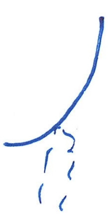
$ds = \sqrt{1+100x^{18}}\, dx$

Around y-axis:



$$S.A. = \int_1^{1000} 2\pi x \, ds = \int_1^{1000} 2\pi x \sqrt{1+100x^{18}}\, dx$$

Around x-axis:



$$S.A. = \int_1^{1000} 2\pi x^{10} \, ds = \int_1^{1000} 2\pi x^{10} \sqrt{1+100x^{18}}\, dx$$

Second is larger because $2\pi x^{10}\sqrt{1+100x^{18}} > 2\pi x\sqrt{1+100x^{18}}$ when $x \geq 1$.

6. (15 points) Calculate

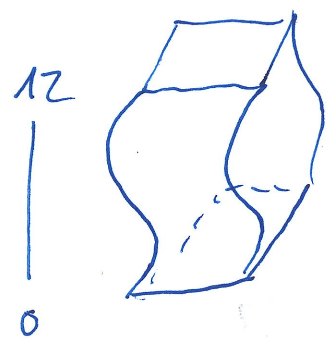
$$\int_{-\infty}^{\infty} \frac{2}{1+x^2}\,dx.$$

$$= \lim_{a\to-\infty} \int_a^0 \frac{2dx}{1+x^2} + \lim_{b\to\infty} \int_0^b \frac{2dx}{1+x^2} = \lim_{a\to-\infty} \left(2\arctan x\Big|_a^0\right) + \lim_{b\to\infty} \left(2\arctan x\Big|_0^b\right)$$

$$= \lim_{a\to-\infty} (0 - 2\arctan a) + \lim_{b\to\infty} (2\arctan b - 0)$$

$$= -2\left(-\frac{\pi}{2}\right) + 2\left(\frac{\pi}{2}\right) = \underline{2\pi}$$

7. (10 points) You are stacking sheets of thin square paper to make a three dimensional sculpture. The precision of your art is impeccable, and you make it so that the square of paper at height $y$ inches above the base has side length $1 + \sin(\pi y/6)$ inches. Your sculpture is 12 inches bottom to top. Set up **but do not solve** an integral which gives the (approximate) volume of your paper sculpture.




Area at $y$ is $(1+\sin(\pi y/6))^2$

$$\text{Vol} = \int_0^{12} (1+\sin(\pi y/6))^2\,dy$$