MATH 113: 5/2 WORKSHEET
STRICT IMPLICATION

When C. I. Lewis first developed what we now see as the first modal logics, his motivation wasn't to understand $\Box$ and $\Diamond$ (necessity and possibility). He was concerned with implication—if/then statements. Specifically, he disliked the material implication of TFL and wanted to do better. In the ordinary sense, "if A then B" means there is some sort of causal connection between A and B. But the meaning of $\to$ is not about causal connections. Anyone would say "if it is raining then I should bring an umbrella" but only a logician would say "if it is raining then $2 + 2 = 4$".

Lewis thought we could do better than that.

Paradoxes of the material implication.

Two theorems of material implication Lewis disliked were

$$A \to (\neg A \to B) \qquad \text{and} \qquad A \to (B \to A).$$

(1) Check, by truth tables or some other means, that these are tautologies.
(2) Try to write an ordinary English sentence giving the meaning of each.
(3) Why do you think Lewis might have found these paradoxical? Do you agree?

Lewis's solution was to introduce a new implication, "strict implication" and given by the symbol $⥽$, which doesn't allow the corresponding statements as tautologies. The contemporary approach is to define $A ⥽ B$ as $\Box(A \to B)$. That is, Lewis's strict implication says that A implies B provided in any possible world, if A is true then B is true.

Non-paradoxes(?) of strict implication.

Neither $A ⥽ (\neg A ⥽ B)$ nor $A ⥽ (B ⥽ A)$ are theorems of modal logic.

(1) Use the definition of $⥽$ to write out these sentences just using $\Box$ and $\to$.
(2) Confirm that $A ⥽ (\neg A ⥽ B)$ is not a theorem of the modal logic K by coming up with an interpretation wherein it is false. Can you come up with a counter-interpretation which validates a stronger modal logic? (For example, if your counter-interpretation is on a reflexive frame then you've shown it is not a theorem of T.)
(3) Do the same for $A ⥽ (B ⥽ A)$.

As is typical for them, philosophers don't uniformly agree that Lewis solved all the relevance problems of implication. Some think there are other, better ways to define implication so that $A \to B$ always implies that the antecedent A is relevant to the truth of the consequent B. (These are called, appropriately, relevance logics.) They think that there are cases where $A \strictif B$ is true but A isn't relevant to the truth of B.

The irrelevance of strict implication.

(1) Explain why $(A \wedge \neg A) \strictif B$ is a theorem of modal logic.
(2) Why do you think that relevance logicians would see this being a theorem as undesirable?
(3) Explain why $A \strictif (B \vee \neg B)$ is a theorem of modal logic.
(4) Why do you think that relevance logicians would see this being a theorem as undesirable?