MATH 113: 4/4 WORKSHEET

One of the uses mathematicians put to first-order logic is to understand how structures differ. Do they behave the same except things are renamed, or are there fundamental differences in their behavior? A *structure* is the same as what we have been calling an interpretation—a choice of domain and the meaning of the names and predicates.

Orders.

An *order* is a domain L with a binary predicate $\leq$ which satisfies the Reflexivity, Antisymmetry, and Transitivity axioms:

- (Reflexivity) $\forall x\ x \leq x$
- (Antisymmetry) $\forall x \forall y\ [(x \leq y) \wedge y \leq x) \rightarrow x = y]$
- (Transitivity) $\forall x \forall y \forall z [(x \leq y \wedge y \leq z) \rightarrow x \leq z]$

When working with orders we adopt the following notation:

- $x < y$ is an abbreviation for $x \leq y \wedge x \neq y$;
- $x \geq y$ is an abbreviation for $y \leq x$; and
- $x > y$ is an abbreviation for $y < x$.

The familiar number systems—natural numbers, integers, rational numbers, real numbers—are all examples of orders. In effect, we don't care about their arithmetic properties and we just focus on $\leq$: what number is bigger than what. So when we talk about them being different, we mean different in ways we can express just using $\leq$. For example, one way the rational numbers differ from the integers is that you can divide any two nonzero rationals and get a rational, whereas dividing integers can take you outside the structure. That difference wouldn't count for our purposes.

Finite orders.

Draw pictures to convince yourself that for any finite whole number n there is an order with n objects in it.

The natural numbers versus the integers.

The integers $\mathbb{Z}$ are the whole numbers, positive or negative, while the natural numbers $\mathbb{N}$ are the whole numbers ≥ 0.

(1) Draw pictures to illustrate $\mathbb{Z}$ and $\mathbb{N}$. Do they look to have the same structure? If not, can you say how they differ?
(2) In an order, a *minimum* is an object m so that $m \leq x$ for every other object x. Can you write a FOL sentence which expresses "the order has a minimum"? [Hint: if you first write a sentence $\varphi(m)$ which expresses "m is a minimum", then you simply need to write $\exists y\ \varphi(y)$.]
(3) Explain how this lets you distinguish $\mathbb{Z}$ from $\mathbb{N}$ as orders.

The natural numbers versus the positive integers.

As an ad hoc name, let $\mathbb{P}$ refer to the positive integers, i.e. those > 0.

(1) Draw a picture to illustrate $\mathbb{P}$ and compare it to your picture for $\mathbb{N}$. Do they look to have the same structure?
(2) Show they have the same structure by explaining how to label the picture either with the objects in $\mathbb{P}$ or with the objects in $\mathbb{N}$.

The integers versus the rationals.

The rationals $\mathbb{Q}$ are numbers that are fractions of two integers. They include numbers like 7, $\frac{1}{2}$, or $-\frac{19}{3}$, but don't include $\sqrt{2}$ or π.

(1) One property of $\mathbb{Q}$ is that between any two distinct numbers you can find a number in the middle. Can you explain why this is? Even better, can you come up with a formula for a number that's between x and y?
(2) Can you write a FOL sentence which expresses "if $x < y$ then there is an object z between x and y"?
(3) Explain how this lets you distinguish $\mathbb{Q}$ from $\mathbb{Z}$ as orders.

The rationals versus the reals.

The real numbers $\mathbb{R}$ include irrational numbers like $\sqrt{3}$ or π.

(1) Repeat what you did with $\mathbb{Q}$ but for $\mathbb{R}$. That is, come up with a formula that gives you a number between x and y and write a FOL sentence which expresses "if $x < y$ then there is an object z between x and y".
(2) Do you think you can distinguish $\mathbb{Q}$ and $\mathbb{R}$ as orders? Give an intuition for why or why not.