MATH 113: 4/23 WORKSHEET
FRAMES FOR MODAL LOGIC

(This material corresponds to chapter 44 of the textbook.)

Interpretations for modal logic.

An *interpretation* for modal logic is a collection of possible worlds, each with an assignment of truth values to the variables, with an accessibility relation saying which worlds are possible from which. We can think of the *frame* as arrows between worlds.

Given an interpretation you can determine the truth of any statement at a world w in the frame:

- The truth value of variables P is given by the assignment for w.
- The truth value of sentences built up with connectives $\wedge, \vee, \neg, \rightarrow, \leftrightarrow$ follows the familiar rules.
- $\Box\varphi$ is true at w if φ is true at every world accessible to w; that is, if φ is true at every v so that $R(w, v)$.
- $\Diamond\varphi$ is true at w if φ is true at some world accessible to w; that is, if φ is true at some v so that $R(w, v)$.

Today we're trying to get an idea of how the structure of a frame affects modal truth. As a tool, let's consider a few kinds of statements.

Control statements.

- A *button* is a statement B so that $B \rightarrow \Box B$ is true at every world. The idea is, once you push a button it stays pushed.
- A *switch* is a statement S so that $\Diamond S \wedge \Diamond\neg S$ is true in every world. The idea is, a switch can be freely toggled on and off.
- A *recurrence* is a statement R so that $R \rightarrow \Box\Diamond R$ is true at every world. The idea is, R will always recur again at a future world.

Three different week frames.

Consider three different frames, each attempting to model the days in the week.

- Frame A, the "once and done frame" represents a single week. There are seven worlds, Sunday through Saturday, and world w_1 accesses world w_2 if w_2 is the same or later in the week. For example, Monday accesses Monday and Monday accesses Tuesday, but Tuesday doesn't access Monday.
- Frame B, the "repeated week frame" represents a repeated weekly cycle. There are seven worlds, Sunday through Saturday, and world w_1 accesses world w_2 if at some point in the future it is the day for world w_2. For example, in this frame Tuesday accesses Monday.
- Frame C, the "endless march of time frame" represents an endless succession of weeks. There are infinitely many frames, Sunday through Saturday for weeks 0 up to infinity. Each world accesses itself and every future world.

- Draw a picture of each frame to illustrate it.
- For each frame, try to come up with a button.
- For each frame, try to come up with a switch.
- For each frame, try to come up with a recurrence.
- The repeated week frame has the curious property that any statement is a recurrence for it. Can you explain why? Is the same true for the other two frames?

Two buttons or switches are *independent* if you can trigger them independently. For example two buttons B_1 and B_2 are independent if $\Box\neg(B_1 \to B_2)$ and $\Box\neg(B_2 \to B_1)$ are true at every world. In other words, it's impossible that pushing B_1 forces you to push B_2 and vice versa.

- Pick a frame and come up with two buttons which are not. Can you come up with two independent buttons?
- Can you do the same for switches?
- What about with a button and a switch?