MATH 113: 4/18 WORKSHEET
THE SEMANTICS OF MODAL LOGIC

(This material corresponds to chapter 44 of the textbook.)

> "The world is everything that is the case."
> Ludwig Wittgenstein, *Tractatus Logico Philosphicus*, 1.

In TFL, each row of a truth table corresponded to a possible world—a possible state of affairs for the variables in question. With modal logic we treat the same basic notion. Formally, a *possible world* is an assignment of true or false to every variable we are concerned with. Where modal logic differs is we want to understand how these possible worlds relate.

For many applications, we don't want every possible world to be accessible from any other. For this reason, we need to say not just what the possible worlds are but how they are connected. The *accessibility relation*, usually called R, denotes which worlds are possible from others. That is, $R(w_1, w_2)$ means that world w_2 is possible relative to world w_1. An *interpretation* of modal logic is then a collection of possible worlds with the accessibility relation between them.

Given an interpretation and a world w in that interpretation, we can define truth in that world.

- The truth value of variables P is given by the assignment for w.
- The truth value of sentences built up with connectives $\wedge, \vee, \neg, \rightarrow, \leftrightarrow$ follows the familiar rules.
- $\Box\varphi$ is true at w if φ is true at every world accessible to w; that is, if φ is true at every v so that $R(w, v)$.
- $\Diamond\varphi$ is true at w if φ is true at some world accessible to w; that is, if φ is true at some v so that $R(w, v)$.

If we just talk about the accessibility relation and not what is true at each world we call this a *frame*. That is, a frame is a collection of objects with a binary relation between them. So an interpretation is a frame equipped with an assignment of truth values of variables at each world in the frame.

Once you have a notion of interpretation you get a notion of entailment $\vDash$. With modal logic we will subscript our $\vDash$ to state which system it corresponds to.

$P \vDash_K C$ (P *semantically entails* C over K) means that in any interpretation, in any world where P is true we have C is true. That is, if you take any frame and any assignment of truth values for the worlds, then if P is true at a world it must be that C is also true at that world.

And with $\vDash$ and interpretations we get the usual host of semantic concepts.

- An argument $P_1, \ldots, P_n, \therefore C$ is *modally valid* if $P_1, \ldots, P_n \vDash_K C$.
- A is a *modal truth* if it is entailed by no premises, i.e. $\vDash_K A$. That is, a modal truth is true at any world in any interpretation.
- A is a *modal contradiction* if $\vDash_K \neg A$. That is, a modal contradiction is false in any world in any interpretation.
- A is *modally satisfiable* if A is true at some world in some interpretation.
- A and B are *modally equivalent* if A is true at a world in an interpretation if and only if B is true at that world in that interpretation.