MATH 113: 4/2 WORKSHEET

Doing something with interpretations[1]

On Monday we saw how we could use *interpretations* to define semantic concepts, akin to what we did with truth-functional logic. Unlike with TFL's truth tables, interpretations are more complicated objects and hence trickier to deal with. In particular, it is much harder to check any property of the form "every interpretation is blah blah", since there are infinitely many interpretations.

Today let's talk about some cases where we nonetheless can do this.

Equivalences from TFL.

Any equivalence from TFL gives equivalences in FOL. For example, in TFL P and $\neg\neg P$ are equivalent. So if you replace P with any sentence of FOL, you get an equivalence in FOL. More, if you have a sentence of FOL in which P appears you can replace it with $\neg\neg P$ to get an equivalent sentence.

Explain why the following sentences of FOL are equivalent.

(1) $\exists x\ P(x)$ and $\neg\neg\exists x\ P(x)$
(2) $\forall x\ P(x)$ and $\forall x\ \neg\neg P(x)$
(3) $\forall x\ P(x) \to Q(x)$ and $\forall x\ [\neg P(x) \vee Q(x)]$

DeMorgan laws for quantifiers.

The following sentences of FOL are equivalent:

$$\neg\forall x\ \varphi(x) \qquad \text{and} \qquad \exists x\ \neg\varphi(x)$$
$$\neg\exists x\ \varphi(x) \qquad \text{and} \qquad \forall x\ \neg\varphi(x)$$

To see why we can think about how you check a quantified statement is false. Checking that $\forall x\ \varphi(x)$ is false in an interpretation is done by finding an object a for which $\varphi(a)$ is false. But that is also how you check that $\exists x\ \neg\varphi(x)$ is true. And checking that $\exists x\ \varphi(x)$ is false is done by showing that every object a has $\varphi(a)$ is false. But that is how you check that $\forall x\ \neg\varphi(x)$ is true.

Explain why the following sentences of FOL are equivalent.

(1) $\neg\forall x\ P(x,x)$ and $\exists x\ \neg P(x,x)$
(2) $\exists x\ Q(x)$ and $\neg\forall x\ \neg Q(x)$
(3) $\neg\exists x\forall y\ P(x,y)$ and $\forall x\exists y\ \neg P(x,y)$

[1] This material corresponds to chapters 33 and 34 of the textbook.

Validities, contradictions, etc. from TFL.

Like with equivalences, we can use semantic notions for TFL to get for free some notions in FOL. For example, in TFL $P \wedge \neg P$ is a contradiction. Thus, in FOL $\varphi \wedge \neg\varphi$ is a contradiction. We can reason as follows: in an arbitrary interpretation, φ is either true or false. If it's true, then $\neg\varphi$ is false, so the conjunction is false. If φ is false so the conjunction is false. Note that we never needed to know the meaning of φ to carry out this reasoning.

Practice problems:

(1) Are $\exists x\ P(x)$ and $P(a)$ equivalent?
(2) Is the argument $P(a), \therefore \exists x\ P(x)$ valid? Say why or why not.
(3) Is $\forall x\ P(x) \vee \neg P(x)$ a validity, contradiction, or neither?
(4) Does $\exists x\ [A(x) \wedge B(x)]$ entail $\exists x\ A(x)$?
(5) Are $\exists x \forall y\ P(x,y)$ and $\forall x \exists y\ P(x,y)$ jointly satisfiable?
(6) Are $\neg\exists x\ [P(x) \vee Q(x)]$ and $\forall x\ [\neg P(x) \wedge \neg Q(x)]$ equivalent?