MATH 113: 4/16 WORKSHEET

Modal logic (ML) extends truth functional logic by adding two modal operators.

- $\Box P$ means that P is necessarily true, it could not have been otherwise.
- $\Diamond P$ means that P is possible, it could be different.

In introducing this new grammar the first question is, how does it behave? There are multiple answers here, for two reasons. First, ML can be used to model many different things, which don't all behave the same. Second, even if you look at a specific modeling there is disagreement as to how it should behave.

There's a few ways to give the meaning of the new symbols. The textbook goes by deduction rules. This is great if you're doing proofs with them, but we aren't so we'll state them in terms of entailment $\vdash$. We retain all the deduction rules of TFL, and add on to them.

Below, φ and ψ are used to stand for an arbitrary sentence of ML.

The system K.

K is the basic system of rules for ML. The other systems will add on extra rules to K.

- (Necessitation Rule) If φ is a theorem (that is, $\vdash \varphi$) then $\Box\varphi$ is also a theorem ($\vdash \Box\varphi$).
- (Distribution Axiom) $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$ is always a theorem.
- (Duality Axioms) $\neg\Box\varphi \to \Diamond\neg\varphi$ and $\neg\Diamond\varphi \to \Box\neg\varphi$ are always theorems.

The system T.

T has all the rules of K plus one extra rule.

- (M Axiom) $\Box\varphi \to \varphi$ is always a theorem.

This axiom expresses the idea—not provable in K!—that if something is necessarily true then it is true.

The system $S4$.

$S4$ has all the rules of T plus one extra rule.

- (4 Axiom) $\Box\varphi \to \Box\Box\varphi$ is always a theorem.

A consequence of this rule is you can always add extra boxes in front of a sentence. It has the opposite effect applies to diamonds: you can always delete extra diamonds from the front of a sentence.

The system $S5$.

$S5$ has all the rules of $S4$ plus one extra rule. It can be either of the following, which are equivalent over $S4$.

- (5 Axiom) $\Diamond\varphi \to \Box\Diamond\varphi$ is always a theorem.
- (B Axiom) $\varphi \to \Box\Diamond\varphi$ is always an axiom.

This rule has the consequence that any long string of diamonds and boxes can be shortened down to just the last in the list.

There are yet more exotic and poorly named modal logics—$S4.2$! $S4.3$! $Dm.2$! GL!—but we will not talk about them.