# MATH 113: 3/3 WORKSHEET

Last Friday you compared *semantic* concepts (based on truth tables) to *syntactic* concepts (based on proofs). Hopefully you observed that in every example you looked at the parallel concepts lined up exactly. This might have made you wonder: are the syntactic concepts equivalent to their semantic counterparts?

To answer this it suffices to look at one pair of concepts.

$$P_1, \ldots, P_n \vDash C$$

means that $P_1, \ldots, P_n$ *semantically entail* $C$: in any row of the truth table where each $P_i$ is true the conclusion $C$ is also true.

$$P_1, \ldots, P_n \vdash C$$

means that $P_1, \ldots, P_n$ *syntactically entail* $C$: there is a formal proof with premises $P_1, \ldots, P_n$ and conclusion $C$.

It turns out that these two kinds of entailment are equivalent. That is, $P_1, \ldots P_n \vDash C$ if and only if $P_1, \ldots, P_n \vdash C$. Both directions of this equivalence are important properties of truth functional logic.

**Soundness.**

If $P_1, \ldots, P_n \vdash C$ then $P_1, \ldots, P_n \vDash C$. This is called soundness because it says that the deduction rules are all *sound*—if you start out with true premises then your conclusion must also be true.

The idea behind this property is simple. Namely, a formal proof is just a combination of a bunch of deduction rules. So we only have to check that each deduction rule is sound. To do this, we can write truth tables which correspond to the reasoning encapsulated in each deduction rule. It's a little tedious to go through and check this for every rule, but there is a straightforward goal in mind.

(1) Check that the $\wedge$I and $\wedge$E rules are sound.
(2) Check that the $\to$I and $\to$E rules are sound.

**Completeness.**

If $P_1, \ldots, P_n \vDash C$ then $P_1, \ldots, P_n \vdash C$. This is called completeness beacuse it says that the deduction rules are *complete*—any possible valid argument can be realized as a formal proof using the deduction rules.

Checking this property is more complex. What we would need to do is to show that any truth table demonstrating that an argument is valid can be turned into a formal proof. It' snot so straightforward to see how to do this.

Knowing that the two notions of entailment are equivalent we can show that the other pairs of parallel concepts are equivalent.

**Equivalence.**
$A$ and $B$ are semantically equivalent if $A \vDash B$ and $B \vDash A$. They are syntactically equivalent if $A \vdash B$ and $B \vdash A$.

**Tautology/Theorem.**
$A$ is a tautology can be phrased as $\vDash A$: $A$ is semantically entailed from no premises. And $A$ is a theorem means $\vdash A$.

**Consistency.**
Sentences $P_1, \ldots, P_n$ are semantically inconsistent if there is no row in the truth table where they are all true. This can be rephrased as $P_1, \ldots, P_n \vDash \bot$. Thus it is equivalent to them being syntactically inconsistent: $P_1, \ldots, P_n \vdash \bot$. And since consistent means not inconsistent, semantic consistency and syntactic consistency are equivalent.

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Next week we will start looking at a new logical system, first-order logic, which expands truth-functional logic to have extra expressiveness. It will turn out that this new logical system also has soundness and completeness. In general, if you look at any logical system you can ask if it has these two properties. While they are important and powerful tools if you have them, not every logical system does. There are many interesting and useful logical systems which are not complete. So it turns out that these properties are something that makes truth-functional logic quite special.