MATH 113: 3/28 WORKSHEET

Some special relations[1]

Some *binary relations*—that is, predicates $P(x, y)$ with two variables—are useful and interesting in mathematics. Let's look at some of them, and along the way see how some of the logical ideas we've been learning relate to things you've seen in other math classes.

Equality.
The equality relation $=$, also called the identity relation, makes sense over any domain. Here let's use the domain D consisting of all whole numbers between 0 and 9.

(1) List out the extension of $=$. (That is, list out all ordered pairs $\langle x, y\rangle$ from D so that $x = y$.)
(2) Explain why $=$ satisfies the following properties.
  (a) $\forall x\ x = x$
  (b) $\forall x \forall y\ (x = y \to y = x)$
  (c) $\forall x \forall y \forall z\ (x = y \land y = z \to x = z)$

Other relations also satisfy those three properties.

Parity.
Again use the domain D consting of whole numbers from 0 to 9. Let $E_2(x, y)$ mean x and y have the same parity. That is, either both are even or both are odd.

(1) Count how many ordered pairs $\langle x, y\rangle$ are in the extension of E_2. List a few of them.
(2) Explain why E_2 satisfies the following properties.
  (a) $\forall x\ E_2(x, x)$
  (b) $\forall x \forall y\ [E_2(x, y) \to E_2(y, x)]$
  (c) $\forall x \forall y \forall z\ [E_2(x, y) \land E_2(y, z) \to E_2(x, z)]$

These three properties have names. If $E(x, y)$ is a relation on a domain D, say that:

- E is *reflexive* if it satisfies $\forall x\ E(x, x)$.
- E is *symmetric* if it satisfies $\forall x \forall y\ [E(x, y) \to E(y, x)]$.
- E is *transitive* if it satisfies $\forall x \forall y\ \forall z\ [E(x, y) \land E(y, z) \to E(x, z)]$.

If $E(x, y)$ is reflexive, symmetric, and transitive we call it an *equivalence relation*, as it means that objects are equivalent or the same in some respect.

[1] Part of this corresponds to chapter 35 of the textbook.

Equivalence modulo 3.

Again use the domain D consting of whole numbers from 0 to 9. Let $E_3(x, y)$ mean x and y have the same remainder when you divide by 3. For example, $E_3(0, 6)$ holds because both have remainder 0, while $E_3(1, 5)$ does not hold because 1 has remainder 1 and 5 has remainder 2. Mathematicians call this relation *equivalence modulo* 3 and write $x \equiv y \mod 3$.

Is E_3 an equivalence relation? Explain.

Equivalence modulo 10.

Use the domain D consisting of whole numbers from 0 to 99 Let $E_{10}(x, y)$ mean that x and y have the same remainder when you dived by 10. In other words, they have the same digit in the units place. Mathematicians call this relation *equivalence modulo* 10 and write $x \equiv y \mod 10$.

Is E_{10} an equivalence relation? Explain. Come up with a definition for equivalence modulo n for any whole number n. Do you think it is an an equivalence relation?

Order.

Use the domain D consisting of whole numbers from 0 to 9 and consider the relation $\leq$ that is the usual less-than-or-equal order on numbers.

(1) Explain why $\leq$ is reflexive.
(2) Explain why $\leq$ is not symmetric.
(3) Explain why $\leq$ is transitive.
(4) Explain why $\leq$ satisfies the following property, which we call *anti-symmetry*: $\forall x \, \forall y \, [(x \leq y \wedge y \leq x) \rightarrow x = y]$.

Orders, in general. If $\leq$ is a binary relation on a domain D which is reflexive, transitive, and anti-symmetric, then mathematicians call $\leq$ an *order*. (Some call it a *partial order*.) It represents a sense in which some objects are larger than others.

(1) Earlier you looked at $\leq$ on a limited domain. If you look at $\leq$ on all of the integers, is it still an order? What if you look at $\leq$ on all of the real numbers?
(2) Here's a different example of an order: Consider the set $S = \{a, b, c\}$ with three distinct elements. Let the domain D consist of all *subsets of* S—all sets x which are contained in S.
   (a) List all the objects in D. [Hint: there are 8 of them. Don't forget the empty set which has zero elements!]
   (b) Consider the *subset relation* on D: $x \subseteq y$ means that every object in x is in y. Explain why $\subseteq$ is an order.
   (c) Can you find objects x and y in D so that neither $x \subseteq y$ nor $y \subseteq x$ is true? Compare to the $\leq$ order from before.