MATH 113: 3/26 WORKSHEET

Interpretations[1]

Informally speaking, an *interpretation* is a possible way of assigning meaning to sentences in formal logic. In truth-functional logic, this amounts to assigning a truth value for each propositional variable. In this way, a truth table for a sentence is a listing of how it behaves for any interpretation. For example, the truth table for $\neg A \vee B$ lists its truth value for each of the four possible interpretations for its variables.

With first-order logic things are more complicated and we can't package all interpretations nicely in a single table.

Interpretations in first-order logic.
An *interpretation* gives meaning to object names $a, b, \ldots$ and predicates $F(x), \ldots$. It assigns three things.

- A domain, a collection of objects to be talked about.
- Each object name is assigned an object in the domain which it names.
- Each predicate is assigned an extension—the set of objects in the domain for which the predicate holds. Unary predicates $F(x)$ have their extensions be sets of objects, binary predicates $G(x, y)$ have their extensions be sets of ordered pairs of objects, etc.

With truth-functional logic, given an interpretation (assignment of true or false to each variable) you could then say whether a sentence is true or false in that interpretation. We can do the same for first-order logic, and the process is exactly the same for the truth-functional connectives.

Satisfaction and connectives. Here φ and ψ refer to sentences in first-order logic. We define what it means for an interpretation to *satisfy* a sentence, meaning the sentence is true in that interpretation.

- $\varphi \wedge \psi$ is true in an interpretation if and only if both φ and ψ are true in that interpretation.
- $\varphi \vee \psi$ is true in an interpretation if and only if either φ or ψ is true in that interpretation.
- $\neg\varphi$ is true in an interpretation if and only if φ is false (= not true) in that interpretation.
- $\varphi \to \psi$ is true in an interpretation if and only if either φ is false or ψ is true in that interpretation.
- $\varphi \leftrightarrow \psi$ is true in an interpretation if and only if φ and ψ have the same truth value in that interpretation.

Compare these rules to the truth tables for the five connectives.

[1] This material corresponds to chapter 31 of the textbook.

Quantifiers are trickier to get right. At base the idea is this: $\forall x\ P(x)$ should be true when $P(a)$ is true for every object a and $\exists x\ Q(x)$ should be true when there is some object a for which $Q(a)$ is true, where $Q(a)$ is the sentence you get by replacing every x with a name for a. Getting this right takes some care.

Free versus bound variables.

An instance of a variable in a sentence is *bound* if its scope is fixed by a quantifier. A variable is *free* if it is not bound.

(1) In $\forall x\ M(x, y)$, x is bound but y is free.
(2) In $\exists x\ \forall y\ \neg M(x, y)$ both x and y are bound.
(3) In $P(y) \wedge \forall x\ \neg P(x)$, x is bound but y is free.

Technically it is allowed to have a variable appear both free and bound, e.g. $P(x) \wedge \exists x\ Q(x, x)$. However that is confusing so we will assume it doesn't happen. That offending formula could be rewritten $P(x) \wedge \exists y\ Q(y, y)$, since it doesn't matter what we call the variable in a quantifier.

Substitution.

Write $\phi(x)$ to mean that ϕ is a sentence with x as a free variable (maybe there are others. Let c be an unused name for an object. Write $\phi[x/c]$ to mean the formula you obtain by replacing every x with c.

With this technology we can now define satisfaction for quantifiers.

Satisfaction and quantifiers.

Here $\varphi(x)$ is a sentence in first-order logic with free variable x, and c is an unused name for an object.

- $\forall x\ \phi(x)$ is true in an interpretation if and only if for any object d in the domain if we extend the interpretation to interpret c to be d then $\phi(c)$ is true.
- $\exists x\ \phi(x)$ is true in an interpretation if and only if there is an object d in the domain so that if we extend the interpretation to interpret c to be d then $\phi(c)$ is true.