MATH 113: 3/24 WORKSHEET

The semantics of first-order logic[1]

When specifying the interpretation of a predicate, there's two ways we can do it. One is what philosophers call *intensional*—give a definition of what objects fall under it. For example, you might say that $P(x)$ is to be interpreted as "x is a prime number < 10". The other is what is called *extensional*—listing the objects it applies to. For example, you might say that $P(x)$ is true only of the objects $x = 2, 3, 5, 7$.

When are two predicates the same?
The meaning of an interpretation of a predicate is entirely in its *extension*—the collection of objects it applies to. So our two examples above define the same predicate, even though they were given in different ways. Put differently, first-order logic cannot see definitions. We would need to use other tools if that's what we want to study.

Extensions.
The *extension* of a predicate $P(x)$ is the set P of objects in the domain so that $P(x)$ is true. When we talk about extensions we'll use the letter for the predicate with no variables. Different logical connectives correspond to set theoretic operations. Here let D be the domain.

- Conjunction corresponds to intersection: the extension of $P(x) \wedge Q(x)$ is $P \cap Q = \{x \in D : x \in P \text{ and } x \in Q\}$.
- Disjunction corresponds to union: the extension of $P(x) \wedge Q(x)$ is $P \cup Q = \{x \in D : x \in P \text{ or } x \in Q\}$.
- Negation corresponds to complement: the extension of $\neg P$ is $D \setminus P = \{x \in D : x \notin P\}$.

If-then and iff.
These connectives don't correspond to set theoretic operations. Instead they are about the inclusion of sets.

- $\forall x\ P(x) \to Q(x)$ is equivalent to $P \subseteq Q$ (P is a *subset* of Q).
- $\forall x\ P(x) \leftrightarrow Q(x)$ is equivalent to $P = Q$ (P and Q are the same set).

[1]This material corresponds to chapter 30 of the textbook.

For predicates with more than one variable, their extension can't be merely a set of objects. For example, to understand the predicate $M(x, y)$ meaning "x's mother is y" it's not enough to list the x's or the y's by themselves, because what matters is their relationship. Instead, the extension of $M(x, y)$ consists of the *ordered pairs* $\langle x, y\rangle$ so that x's mother is y. The point is, we need to pair the two objects together, and order matters.

Diagrams are helpful to visualize the extension of a binary predicate. You can have a point for each object, and an arrow between objects represents they are related.

What does the extension of a trinary predicate $P(x, y, z)$ look like? Can you think of a way to draw/visualize it?