MATH 113: 2/28 WORKSHEET

Using truth tables we were able to define *semantic* notions—consistency, tautology, entailment, etc. (See Chapter 12.) Using the technology of formal proofs we can make similar definitions except *syntactic*. In general, semantic refers to notions about truth tables and when things are true, while syntactic refers to notions defined using proofs. The idea is, truth tables are about the meaning (semantics) of things while proofs are just words (syntax).

Chapter 20 introduces the following notions.

$$P_1, \dots, P_n \vdash C$$

means that $P_1, \dots P_n$ *syntactically entail* C. Namely, this happens when there is a proof with conclusion C from premises $P_1, \dots, P_n$.

(1) Write down the definition of symantic entailment ($P_1, \dots, P_n \vDash C$) from before, and compare.
(2) Justify the following two facts:

$$A \to B, A \vdash A \wedge B \qquad \text{and} \qquad A \to B, A \vDash A \wedge B$$

If C is syntactically entailed from zero premises (written $\vdash C$ for short), then we call C a *theorem*.

(1) Write down the definition of tautology from before, and compare.
(2) Is $A \to A \vee B$ a tautology? Is it a theorem?

Two sentences A and B are *provably equivalent* if $A \vdash B$ and $B \vdash A$.

(1) Unroll the definition of $\vdash$ to say what provable equivalence means directly in terms of the existence of formal proofs.
(2) Write down the definition of logical equivalence from earlier and compare.
(3) Show that $A \to B$ and $\neg B \to \neg A$ are both provably equivalent and logically equivalent.

Sentences $P_1, \dots, P_n$ are *jointly inconsistent* if $P_1, \dots, P_n \vdash \bot$, i.e. if you can prove a contradiction from them. Otherwise they are *jointly consistent*.

(1) Write down what it means for P by itself to be inconsistent.
(2) Write down the truth table-based definition of contradiction from earlier, and compare.
(3) Show that $A \wedge \neg A$ is both inconsistent and a contradiction.

Note! All of these definitions (except consistency) are about the existence of a certain proof or two. So to check them you only have to exhibit a proof that works. But checking that they don't hold is harder, since you have to say something about all possible proofs.

For example, to see that $A \vdash B$ you just have to write down a formal proof of B from premise A. To see that $A \nvdash B$ you have to say why none of the infinitely many possible proofs will work.

For each of these new syntactic notions there was a corresponding semantic notion. Based on the work earlier, how do you think the syntatic and semantic notions relate? Make a prediction.