MATH 355 PROBLEM SET
CHAPTER 4: TREES

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Problem 1. *Let R be a relation on a set X so that for $x \in X$ there is $y \in X$ so that $x \mathrel{R} y$. Prove that there is a branch through R, where a* branch *is a sequence $\langle x_n : n < \omega \rangle$ so that $x_n \mathrel{R} x_{n+1}$ for every n.*

A subset D of a tree T is called *dense* if for every $s \in T$ there is $t \sqsupseteq s$ so that $t \in D$. If $\mathcal{D}$ is a collection of dense subsets of T then a branch b through T is *$\mathcal{D}$-generic* if for every $D \in \mathcal{D}$ there is $\alpha <$ the height of T so that $b \upharpoonright \alpha \in D$.

Problem 2. *Prove that if $T = 2^{<\omega}$ and $\mathcal{D}$ is the collection of all dense subsets of T then there are no $\mathcal{D}$-generic branches.*

Problem 3. *Prove that if $T = 2^{<\omega}$ and $\mathcal{D}$ is any countable collection of dense subsets of T then there exists a $\mathcal{D}$-generic branch.*

Problem 4. *Let κ be a regular cardinal. Prove that if $T = 2^{<\kappa}$ and $\mathcal{D}$ is any collection of dense subsets of T so that $|\mathcal{D}| < \kappa$ then there exists a $\mathcal{D}$-generic branch.*

Let T be a tree. An *antichain* in T is a set $A \subseteq T$ so that if $s \neq t \in A$ then s and t are incomparable: $s \not\sqsubseteq t$ and $t \not\sqsubseteq s$.

Problem 5. *Prove that if $T = 2^{<\omega}$ then any antichain in T is countable.*

An antichain A in T is *maximal* if $A \cup \{s\}$ is not an antichain for any $s \in T \setminus A$.

Problem 6. *Let A be an antichain in T. Prove that A is maximal iff $D_A = \{s \in T : s \sqsupseteq a$ for some $a \in A\}$ is dense.*

The next few problems concerns two closely related results in Ramsey theory. For a set X, let $[X]^2$ denote the set of subsets of X with precisely 2 elements. A *coloring* of $[X]^2$ is a map which assigns each pair a color. And $H \subseteq X$ is *monochromatic* for a coloring χ if $\chi(\{x, y\})$ is the same for all $\{x, y\} \in [H]^2$.

Theorem (Finite Ramsey's Theorem)**.** *Fix natural numbers r and n. Then there is a natural number N so that for any coloring of $[N]^2$ into r colors there is monochromatic $H \subseteq X$ so that $|H| = n$.*

The smallest N for which this works is called the *Ramsey number* for n with r colors.

Theorem (Infinite Ramsey's Theorem)**.** *Fix a natural number r. Then for any coloring of $[\omega]^2$ into r colors there is an infinite monochromatic $H \subseteq \omega$*

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Problem 7. *Use Kőnig's lemma to prove the infinite Ramsey's theorem assuming the finite Ramsey's theorem.*

Problem 8. *Prove that the Ramsey number for* 3 *with* 2 *colors is* 6*. [Hint 1: there are two things to prove. (1) There is a coloring of* $[5]^2$ *into* 2 *colors with no monochromatic triple. (2) Any coloring of* $[6]^2$ *into* 2 *colors admits a monochromatic triple.] [Hint 2: You may find it clearest to draw* $[n]^2$ *as the complete graph on* n *vertices, and then the coloring colors the edges of the graph.]*

The next two problems relate Kőnig's lemma to computability theory. These are appropriate to attempt if you took computability theory last semester. When we say that a tree T is *computable* we mean that the order relation $\sqsubseteq_T$ is computable. (Use your favorite model of computation, e.g. Turing machines.) And similarly for computable from an oracle.

Problem 9. *Prove that there is a computable binary tree* T *so that* T *has no computable branch. [Hint: If you have a tree of partial attempts to build the halting set, it can't have a computable branch as that would mean that the halting set is computable. Can you do this in such a way that* T *is computable?]*

Problem 10. *Prove that if* T *is a computable binary tree then* T *has a branch which is computable using the halting set* $0'$ *as an oracle.*

The next problem concerns some ideas from logic. Consider a set T of logical formulae in a fixed formal language.[1] If φ is a sentence in the same language we say that T *proves* φ if there is a proof of φ using T as axioms.[2] And T is called *consistent* if T doesn't prove a contradiction $\varphi \wedge \neg\varphi$. A completion of T is consistent $S \supseteq T$ so that for any φ in the language of T we have that either S proves φ or S proves $\neg\varphi$.

Problem 11. *Use Kőnig's lemma to prove that any consistent set* T *of logical formulae in a countable language has a completion. [Hint: Build a tree of partial attempts to form a completion of* T*.]*

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[1] For example, the formal language of set theory has the single non-logical symbol $\in$.

[2] To make this a complete definition you need to formally define what a proof is. Let me leave that unstated, as the intuitive notion suffices.