MATH 355 PROBLEM SET
CHAPTER 3: THE CUMULATIVE HIERARCHY

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Let t be a transitive set. When we say that an axiom is *true in* t we mean that if you restrict the quantifiers in the axiom to only quantify over elements of t then it comes out as true. The idea is, you imagine that t were the whole universe you're quantifying over.

Problem 1. *Check that the axioms of Extensionality, Union, and Infinity are true in* ω_1*, but the axioms of Pairing and Powerset are not true in* ω_1*.*

Problem 2. *Suppose* $x \in \mathrm{V}_\alpha$*. Prove that any choice function on* x *is in* $\mathrm{V}_{\alpha+n}$ *for some finite* n*. What is the optimal value of* n *you can achieve? Conclude that if* γ *is limit then* AC *is true in* V_γ*.*

Problem 3. *Prove that if* $\gamma > \omega$ *is limit then Foundation plus every axiom of* Z *is true in* V_γ*.*

The last two problems together imply that the only sticking point to V_γ satisfying all of ZFC is Replacement.

Problem 4. *Show that Replacement is not true in* V_{ω_1} *nor in* $\mathrm{V}_{\aleph_\omega}$*. [Hint: For each index* α *define a function from a set* $\in \mathrm{V}_\alpha$ *which is onto* α*. Explain why its image can't be a set in* V_α*.]*

The next two problems ask you to prove the full Mostowski collapse theorem.

Problem 5. *Prove the following: Suppose* (X, E) *is a well-founded relation on a set* X*. Then there is a transitive set* t *and an onto map* $\pi : X \to t$ *so that* $x \mathrel{E} y$ *if and only if* $\pi(x) \in \pi(y)$*.*

Problem 6. *[This problem builds on the previous one.] Say that a relation* (X, E) *is* extensional *if it satisfies the axiom of Extensionality. That is,* $x, y \in X$ *are equal if and only if for all* $z \in X$ *we have* $z \mathrel{E} x$ *iff* $z \mathrel{E} y$*. Prove that if* (X, E) *is extensional and well-founded then the* π *from the previous problem is a bijection.*

Problem 7. *Prove that Replacement is equivalent to transfinite recursion. One direction we did in class, so what you need to prove is: working over the axioms* $\mathsf{Z} + \mathsf{AC}$*, prove that if transfinite recursion is valid then the image of a set under a class function is a set. [Hint: construct the image by transfinite recursion, using* AC *to have a well-order to work over.]*

The next two problems ask you to prove that Zorn's lemma is equivalent to the axiom of Choice.

Problem 8. *Prove over* ZFC *that*

- *(Zorn) Suppose* $(P, \leq)$ *is a partially ordered set with the property that if* $C \subseteq P$ *is linearly ordered by* $\leq$ *then there is* $b \in P$ *so that* $c \leq b$ *for all* $c \in C$*. Then there is a maximal element in* P*. Namely, there is* $m \in P$ *so that there is no* $x \in P$ *with* $m < x$*.*

Problem 9. *Prove over* ZF *that (Zorn) implies* AC*.*

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Problem 10. *Prove that, over* ZF*,* AC *is equivalent to*

- *There are no sets A, B so that if $\kappa < |A|$ then $\kappa < |B|$ but A and B are incomparable in cardinality. (Meaning that there is no injection $A \to B$ and also no injection $B \to A$.*

[Hint: This problem is a sequel to problem 6 from Chapter 2, which in turn was a sequel to problem 1 from Chapter 1. Do them first, or at least understand what they are saying.]

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