MATH 355 PROBLEM SET
CHAPTER 2: CARDINALS

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Problem 1. *Show that all of these sets have the same cardinality:*

- $\mathcal{P}(\mathbb{N})$;
- $\mathbb{R}$;
- *Any open interval* (a, b), *i.e. with* $a < b$; *and*
- *Any closed interval* $[a, b]$, *i.e. with* $a < b$.

Problem 2. *Prove that there are* $2^{\aleph_0}$ *many continuous functions* $\mathbb{R} \to \mathbb{R}$. *[Hint: first calculate how many functions* $\mathbb{Q} \to \mathbb{R}$ *there are.] How many functions* $\mathbb{R} \to \mathbb{R}$ *are there?*

Problem 3. *Prove there are* $2^{\aleph_0}$ *many open sets in* $\mathbb{R}$, *where an open subset of* $\mathbb{R}$ *is a union of open intervals. A* G_δ*-set is a set of reals which is a countable intersection of open sets. How many* G_δ*-sets are there? A* $G_{\delta\sigma}$*-set is a set of reals which is a countable union of* G_δ*-sets. How many* $G_{\delta\sigma}$*-sets are there?*

Problem 4. *Show that if* κ *is an uncountable cardinal there is no order embedding* $\kappa \to \mathbb{R}$.

Problem 5. *Prove the Cantor–Schroeder–Bernstein theorem without using Zermelo's well-ordering theorem, by following this outline.*

1. *Explain why it's enough to prove the special case that if* $X \subseteq Y \subseteq Z$ *and* $|X| = |Z|$ *then* $|X| = |Y|$.
2. *Show that if a function* $F : \mathcal{P}(Z) \to \mathcal{P}(Z)$ *is monotone, meaning that* $A \subseteq B$ *implies* $F(A) \subseteq F(B)$, *then it has a fixed point—a set* $P \subseteq Z$ *so that* $F(P) = P$.
3. *Given a bijection* $f : Z \to X$ *define* $F : \mathcal{P}(Z) \to \mathcal{P}(Z)$ *as* $F(A) = (Z \setminus Y) \cup f''A$. *Use a fixed point for* F *to get a bijection* $Z \to Y$.

Problem 6. *Prove that cardinal trichotomy, the statement that the order relation on cardinals has trichotomy, implies Zermelo's well-ordering theorem.* [Warning! Since you can't assume Zermelo's well-ordering theorem this means you can't use all of the stuff in section 2 and onward that builds on Zermelo.]

Problem 7. *Say that a set* X *is* D-finite *if every injection* $X \to X$ *is a bijection, and* X *is* D-infinite *otherwise. Say that a set* X *is* I-finite *if for any* $\mathcal{A} \subseteq \mathcal{P}(X)$ *there is* $M \in \mathcal{A}$ *so that there is no* $A \in \mathcal{A}$ *with* $A \supsetneq M$. *Otherwise,* X *is* I-infinite.

Prove that for a set X *the following are equivalent.*

1. X *is infinite;*
2. X *is* D*-infinite;*
3. X *is* I*-infinite.*

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Problem 8. *Prove the following basic exponentation facts hold for cardinal exponentiation.*

$$\kappa^{\lambda+\mu} = \kappa^\lambda \cdot \kappa^\mu$$
$$\kappa^{\lambda\cdot\mu} = \left(\kappa^\lambda\right)^\mu$$
$$\kappa^\lambda \cdot \mu^\lambda = (\kappa \cdot \mu)^\lambda$$

Problem 9. *A* beth fixed point *is a cardinal κ so that $\kappa = \beth_\kappa$. Prove that for any cardinal λ there is $\kappa > \lambda$ a beth fixed point. Prove that for any cardinals λ and μ there is $\kappa > \lambda$ so that $\operatorname{cof} \kappa = \mu$ and κ is a beth fixed point.*

Problem 10. *Generalizing the combinatorial definition of factorial for finite cardinals, define $\kappa!$ to be the cardinality of the set of bijections $\kappa \to \kappa$. Show that $\aleph_0! = 2^{\aleph_0}$. More generally, show that $\kappa! = 2^\kappa$ for any infinite κ.*

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