MATH 355 LECTURE NOTES
CHAPTER ω: LARGE CARDINALS

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1. A problem from category theory

At the beginning of chapter 3 we ran into some size issues: some collections, such as the collection of all sets or the collection of all ordinals, are too big to be sets. If they were sets subject to the rules for sets then we'd get a contradiction. Another area of mathematics where size issues arise is *category theory.*

In category theory, you want to understand a type of mathematical structure by looking at the *category* of all structures of that type—the category of groups, the category of graphs, the category of partial orders, etc. These are usually proper classes. For similar reasons as we saw in chapter 3, this presents some obstacles to some work with categories.

These are the problems that French algebraist Alexander Grothendieck faced when he was rewriting algebraic geometry to use category theoretic methods. He solved them by rediscovering a set theoretic idea originally due to Zermelo. In brief, his key idea was to ask, what if we had a domain that was rich enough to be closed under all the mathematical operations we might want to do with it?

Definition 1. A *Grothendieck universe* is a set U satisfying the following properties:

(1) If $y \in x \in U$ then $y \in U$. (That is, U is a transitive set.)
(2) If $x, y \in U$ then $\{x, y\} \in U$.
(3) If $x \in U$ then $\mathcal{P}(x) \in U$.
(4) If $I \in U$ and x_i for each $i \in I$ is in U, then their union $\bigcup_{i \in I} x_i \in U$.

In short, a Grothendieck universe is a set U closed under element, pairs, powerset, and unions indexed by a set in U.

It may look like these four properties don't give you all the mathematical operations you want, but you can prove they give you a lot more.

Lemma 2. *Let U be a Grothendieck universe. Then U is closed under singletons, cartesian products, disjoint unions, intersections, functions, and subsets.*

I won't give a proof of this. It amounts to similar arguments we did in chapter 3 when we looked at the axioms.

The idea is, a Grothendieck universe U is a rich enough structure to do all of the basic operations of mathematics. So we can imagine that U were the entire universe of mathematics. So when doing category theory, instead of looking at the category of all groups, we look at the category of all groups in U. While the former is a proper class, the latter is a set. Thus we can avoid size issues

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and still have a domain rich enough to do all the operations of mathematics. (What if we need to look at a group too big to be in U? Then we can use an even larger Grothendieck universe.)

Whenever you see a new definition, you want to know what examples look like. So let's see a couple example of Grothendieck universes.

Example 3. $\emptyset$ is vacuously a Grothendieck universe.

Example 4. V_ω is a Grothendieck universe.

These examples are of limited use, especially the first one. We want to do mathematics with infinite sets, so we want Grothendieck universes which have infinite sets as elements.

Let's see what it takes to get useful Grothendieck universes.

Definition 5. A cardinal κ is *inaccessible* if κ is uncountable, regular, and strong limit. Recall that κ being regular means that if $\lambda < \kappa$ and $f : \lambda \to \kappa$ then the range of f is bounded in κ, and κ being strong limit means that if $\lambda < \kappa$ then $2^\lambda < \kappa$.

Theorem 6. *U is a Grothendieck universe which contains an infinite set iff $U = \mathrm{V}_\kappa$ for some inaccessible κ.*

A lemma will be helpful to prove this.

Lemma 7. *Suppose κ is inaccessible and suppose $x \subseteq \mathrm{V}_\kappa$. Then, $x \in \mathrm{V}_\kappa$ iff $|x| < \kappa$.*

Proof. It suffices to prove that $|\mathrm{V}_\alpha| < \kappa$ for all $\alpha < \kappa$, since $x \in \mathrm{V}_\kappa$ iff $x \in \mathrm{V}_\alpha$ for some $\alpha < \kappa$. We prove this by induction. The zero case is true because $|\mathrm{V}_\alpha| = 0 < \kappa$ because κ is inaccessible. For the successor case, suppose $|\mathrm{V}_\alpha| = \lambda < \kappa$. Then $|\mathrm{V}_{\alpha+1}| = 2^\lambda < \kappa$ by strong limitness. For the limit case, suppose $|\mathrm{V}_\alpha| = \lambda_\alpha < \kappa$ for all $\alpha < \gamma$. Then $|\mathrm{V}_\gamma| = \sup \lambda_\alpha < \kappa$ by regularity. □

Proof of Theorem. ($\Leftarrow$) We check that V_κ satisfies the definition of being a Grothendieck universe. The first three properties are things we already checked in chapter 3. For the last: by the axiom of choice plus the lemma it's enough to consider families of sets indexed by a cardinal $\lambda < \kappa$. So suppose that $x_\alpha \in \mathrm{V}_\kappa$ for each $\alpha < \lambda$. For each α, let γ_α be least such that $x_\alpha \in \mathrm{V}_{\gamma_\alpha}$. Then the map $\alpha \mapsto \gamma_\alpha$ gives a function $\lambda \to \kappa$. By regularity, this map is bounded, so there is $\gamma < \kappa$ so that $\gamma_\alpha < \gamma$ for all α. Then, $\{x_\alpha : \alpha \in \lambda\} \subseteq \mathrm{V}_\gamma$ and so $\{x_\alpha : \alpha \in \lambda\} \in \mathrm{V}_{\gamma+1} \subseteq \mathrm{V}_\kappa$.

($\Rightarrow$) Let U be a Grothendieck universe and suppose there is an infinite set in U. Let $\kappa = \sup_{x \in U} |x|$. Then κ is uncountable, because U contains the powerset of an infinite set. And κ is strong limit, because if $\lambda < \kappa$ there is $x \in U$ with $\lambda \leq |x|$ and so $2^\lambda \leq 2^{|x|} < 2^{2^{|x|}} \leq \kappa$, by closure under powerset. Finally, κ is regular: To see this, suppose $f : \lambda \to \kappa$ for some $\lambda < \kappa$. Without loss of generality we may assume that each $f(\alpha)$ is a cardinal; this can be done because κ is a limit cardinal and so the cardinals are unbounded below κ For each $\alpha < \lambda$, pick $x_\alpha \in U$ so that $f(\alpha) = |x|_\alpha$. Then by closure under unions $x = \bigcup_{\alpha<\lambda} x_\alpha \in U$. Thus $|x| = \sup_{\alpha<\lambda} |x|_\alpha = \sup_{\alpha<\lambda} f(\alpha) \leq \kappa$. And we get $< \kappa$ by the same trick of taking an extra powerset. All in all, we have seen that κ is inaccessible.

To finish the proof, let's see that $U = \mathrm{V}_\kappa$. For the $\supseteq$ direction, we prove by induction that $\mathrm{V}_\alpha \in U$ for all $\alpha < \kappa$. This is true for $\alpha = 0$ because if $x \in U$ then $\emptyset \in \mathcal{P}(x) \in U$ and so $\emptyset = \mathrm{V}_0 \in U$. For the successor case, if $\mathrm{V}_\alpha \in U$ then $\mathcal{P}(\mathrm{V}_\alpha) = \mathrm{V}_{\alpha+1} \in U$. For the limit case, suppose $\mathrm{V}_\alpha \in U$ for all $\alpha < \gamma < \kappa$. By closure under unions we get that $\bigcup_{\alpha<\gamma} \mathrm{V}_\alpha = \mathrm{V}_\gamma \in U$.

For the $\subseteq$ direction, suppose toward a contradiction that $x \in U \setminus \mathrm{V}_\kappa$. While it may be that $|x|$ is small (x could even be a singleton!), by transitivity we have that elements of x are in U, elements of elements of x are in U, and so on. In all, if t is the *transitive closure* of x, the smallest transitive set $\supseteq x$, then $t \subseteq U$. Indeed, closure under powersets and unions gives that $t \in U$. I claim that $\kappa \leq |t|$. The reason is, in order for t to not be $\in \mathrm{V}_\kappa$ it must be that for cofinally many $\alpha < \kappa$ there is an element of t of rank α. Then $\mathcal{P}(t) \in U$. But then $2^\kappa \leq \kappa$, since κ was defined to be the supremum of cardinalities of sets in U. □

Corollary 8. ZFC *does not prove the existence of a Grothendieck universe with an infinite set as an element.*

Proof. Because the existence of an inaccessible cardinal is independent of ZFC. □

Definition 9. *Tarski–Grothendieck* set theory TG is ZFC + $\forall\lambda$ there is an inaccessible $> \lambda$. Equivalently, TG may be formulated as ZFC + "every set is an element of a Grothendieck universe.

By the corollary, TG is a strict extension of ZFC. So while it's a useful background theory to use for category theoretic purposes, it does take us beyond ZFC in strength.

This is to be expected. Informally, ZFC is a bunch of axioms saying we can do all the basic operations of mathematics. And a Grothendieck universe is a set closed under all the basic operations of mathematics. We should expect that positing the existence of Grothendieck universes puts us beyond ZFC in strength, since it's in effect positing the existence of a set which satisfies all the axioms of ZFC. And by the second incompleteness theorem, ZFC cannot prove the existence of a set which satisfies all the axioms of ZFC.

More generally, this is where large cardinals get used. Sometimes you're doing something that's too strong for ZFC to prove. It requires objects that are too large for the basic axioms to guarantee their existence. So if you want to do that bit of mathematics, you need to bring in large cardinals.

We saw an example of this in chapter 4. If you want to talk about the tree property holding at $\aleph_2$, ZFC isn't powerful enough for that. You need large cardinals (a so-called weakly compact cardinal).

2. Going below an inaccessible

A reasonable question is, did we actually need that inaccessible?

For example, Andrew Wiles famously proved Fermat's last conjecture that $a^n + b^n = c^n$ has no positive integer solutions when $n > 2$. His proof built on Grothendieck's machinery. So if you formalized the proof, it was a proof from TG, not a proof from ZFC. Could you redo Wiles's proof to work in ZFC?

It turns out the answer is yes. A careful readthrough of Wiles's proof reveals that everywhere he used Grothendieck universes to do a construction, you could replace it with a more careful construction that didn't use that strength. Indeed, you need much less than the full power of ZFC to prove Wiles's theorem.

We won't investigate how to do that with his proof, since that would first require we learn his proof. But let's talk a little bit about how you can do similar reductions.

Recall that the Grothendieck universe characterization of inaccessible cardinals yields that a V_κ for κ inaccessible is a set closed under all the basic operations of mathematics. But when we said all the basic operations, we were including a lot. More than we need for any one application.

The main culprit is the last property: if $I \in U$ and $x_i \in U$ for all $i \in I$ then $\bigcup_{i \in I} x_i \in U$. This is supposed to work for any choice of x_i from U. Any construction will probably not use all choices, just using a few. So we should be able to get away with less. Let's see how to formalize this, to get a notion of a "partial" Grothendieck universe.

We're going to need a little bit of logic. Whereas before we got away with a more informal approach, now we need some more precise tools. To start, let's be more clear about what is meant when we say "blah blah is true in a set V_α.

Definition 10 (Tarski)**.** Let M be a transitive set. The *satisfaction relation* for M, the relation $(M, E) \models \varphi(a_1, \ldots, a_n)$ is recursively defined as follows, where φ is a logical formula in the language of set theory and $a_1, \ldots, a_n \in M$:

- $M \models a = b$ if $a = b$;
- $M \models a \in b$ if $a \in b$;
- $M \models \varphi(a_1, \ldots, a_n) \wedge \psi(a_1, \ldots, a_n)$ if $M \models \varphi(a_1, \ldots, a_n)$ and $M \models \psi(a_1, \ldots, a_n)$;
- $M \models \varphi(a_1, \ldots, a_n) \vee \psi(a_1, \ldots, a_n)$ if $M \models \varphi(a_1, \ldots, a_n)$ or $M \models \psi(a_1, \ldots, a_n)$;
- $M \models \neg\varphi(a_1, \ldots, a_n)$ if $M \not\models \varphi(a_1, \ldots, a_n)$;
- $M \models \forall x \varphi(x, a_1, \ldots, a_n)$ if $M \models \varphi(b, a_1, \ldots, a_n)$ for all $b \in M$.
- $M \models \exists x \varphi(x, a_1, \ldots, a_n)$ if $M \models \varphi(b, a_1, \ldots, a_n)$ for some $b \in M$.

If T is a set of logical formulae then $M \models T$ means $M \models \varphi$ for every $\varphi \in T$.

An important point is that this definition takes place within ZFC, our ordinary mathematical context. Namely, we can formalize notions like "φ is a logical formula" inside set theory, because this essentially is just checking that φ is a finite sequence satisfying certain properties. So transfinite recursion applies and we get that the satisfaction relation for (M, E) is a set.

This is how we formalize the intuitive notion of "φ is true in a structure".

Example 11.

- $\mathrm{V}_{\omega+\omega} \models \mathsf{Powerset}$
- $\mathrm{V}_\omega \models \neg\mathsf{Infinity}$
- $\mathbb{R} \models \neg\exists x\ x \cdot x = -1$
- $\mathbb{C} \models \exists x\ x \cdot x = -1$
- If κ is inaccessible then $\mathrm{V}_\kappa \models \mathsf{ZFC}$.

With this definition in hand we can state an important property of the V_α hierarchy.

Theorem 12 (Montague Reflection Principle)**.** *Fix a formula $\varphi(x_1, \ldots, x_n)$ in the language of set theory, with free variables $x_1, \ldots, x_n$. Then there are unboundedly many ordinals α so that for all $a_1, \ldots, a_n \in \mathrm{V}_\alpha$ we have that $\mathrm{V}_\alpha \models \varphi(a_1, \ldots, a_n)$ iff $\varphi(a_1, \ldots, a_n)$.*

Like we saw a few times in chapter 3, this is really a schema for infinitely many theorems, one for each formula φ.

For time reasons we won't prove this. Briefly, the main idea is to do a recursive construction in ω many steps to "catch your tail", similar to how we constructed aleph fixed points. But as a lemma in this proof you need some more precise results from logic. (Namely a version of the *Tarski–Vaught test.*)

Corollary 13. *Fix finitely many axioms T of* ZFC*. Then there are unboundedly many α so that $\mathrm{V}_\alpha \models T$.*

Proof. Apply the Montague reflection principle for the formula you get by taking the conjunction of the axioms in T. □

Let me again emphasize that this is really a schema for infinitely many corollaries, one for each possible T.

Let's relate this back to Grothendieck universes. What says that $\bigcup i \in I x_i$ exists is the Replacement axiom—that's why $\{x_i : i \in I\}$ is a set, and then you take its union. For the connection to inaccessible cardinals, regularity is what we used. But this amounts to saying that Replacement is true in V_κ.

Most proofs will only use finitely many constructions and only need finitely many closures. Thus, they can be carried out with only finitely many uses of the Replacement axiom. We can imagine those proofs being carried out in some V_α satisfying a large enough finite fragment of ZFC. And so proofs like Wiles's can be recast as occuring within ZFC. The Grothendieck universe/inaccessible cardinal was just a convenience.

Of course, this sketch papers over a lot of bookkeeping. It's work to check that the translation to a weaker base theory goes through fine in the details. It's known that Fermat's last conjecture can be proven in (a fragment of) ZFC. It is nontrivial to check this.[1]

This corollary raises a question: can you have a V_α so that $\mathrm{V}_\alpha \models$ ZFC? We know the answer is yes when α is an inaccessible cardinal. Is that the only time that happens?

Definition 14. A cardinal κ is *worldly* if $\mathrm{V}_\kappa \models$ ZFC.

The following proposition is why we can say "cardinal" in this definition, rather than ordinal.

Proposition 15. *If $\mathrm{V}_\alpha \models$ ZFC then α is a cardinal. More specifically, if $\mathrm{V}_\alpha \models$ ZFC then the cardinals $< \alpha$ are cofinal in α.*

Proof sketch. If α were an ordinal there'd be a largest cardinal $< \alpha$. This is why the "more specifically" is actually more specific.

Suppose $\lambda < \alpha$ is a cardinal. Because $\mathrm{V}_\alpha \models$ ZFC we have that $\mathrm{V}_\alpha \models \exists \kappa > \lambda$ so that $2^\lambda = \kappa$. Because V_α is closed under subsets, the cardinal it thinks is is 2^λ really is the true 2^λ (i.e. as computed in V). So $\lambda < 2^\lambda < \alpha$. Since λ was arbitrary we get that the cardinals are cofinal below α. □

[1] See Colin McLarty's "What does it take to prove Fermat's last theorem? Grothendieck and the logic of number theory" `https://www.jstor.org/stable/20749620`.

Note that we don't need every axiom of ZFC to be true in V_α for this. We only need enough axioms to prove that 2^λ always exists.

It turns out there are more worldly cardinals than inaccessible cardinals. First let me clarify give a new bit of notation. Suppose we have two transitive sets $M \subseteq N$, e.g. $M = \mathrm{V}_\lambda$ and $N = \mathrm{V}_\kappa$. We say that M is an *elementary submodel* of N, written $M \prec N$, if for all formulae $\varphi(\bar{x})$ and all $\bar{a} \in M$ we have $M \models \varphi(a)$ iff $N \models \varphi(a)$. That is, an elementary submodel is one that agrees with the larger model about what is true. Of course, the smaller model can only know about its own elements, so those are the only one's allowed.

A recursive argument similar to the one for the Montague Reflection principle yields the following.

Theorem 16. *Let κ be a worldly cardinal. Then there is a cardinal $\lambda \leq \kappa$ where λ has countable cofinality so that $\mathrm{V}_\lambda \prec \mathrm{V}_\kappa$. In particular, $\mathrm{V}_\lambda \models$ ZFC.*

Corollary 17. *If κ is inaccessible then there is a worldly cardinal $\lambda < \kappa$.*

Proof. By the theorem there is worldly $\lambda \leq \kappa$ of countable cofinality. But κ is regular, so it must be that $\lambda < \kappa$. □

Indeed, it turns out that if κ is inaccessible then the $\lambda < \kappa$ which are worldly are unbounded in κ.

Back to the comparison with Grothendieck universes. For the definition of those, the part corresponding to the Replacement Axiom was the requirement that if each $x_i \in U$ for $i \in I \in U$, then $\bigcup_{i \in I} x_i \in U$. The connection to Replacement is, Replacement guarantees that $\{x_i : i \in I\}$ is a set, and then you use the Union axiom to get the union.

But this is actually saying something a little bit stronger than the Replacement axiom. Remember that for the Replacement axiom you had to have a definable way to associate each $i \in I$ to x_i. For Grothendieck universes/inaccessible cardinals, you get a stronger property, when you can do it even if you don't have a way to define which x_i corresponding to which i. What we see with worldly cardinals is, if we only want to guarantee $\bigcup_{i \in I} x_i \in U$ exists when we can definably associate i's to x_i's, then that costs less. We can do it with a smaller cardinal. For arguments like the category theoretic ones Grothendieck was making, an inaccessible is overkill.

This raises the question, is there anything for which an inaccessible is necessary? To answer this, let me mention a fun theorem from analysis.

Theorem 18 (Banach–Tarski theorem)**.** *Consider a solid ball S in $\mathbb{R}^3$. You can partition S into five pieces so that by independently rotating and translating the pieces you produce two disjoint copies of S.*

This theorem is sometimes termed a paradox, for it looks like you're making volume out of nowhere! The reason it works is that while S itself can be given a volume (namely $\frac{4}{3}\pi$ times the radius), the pieces you break S into cannot be given a volume. There is no way to consistently assign them a volume. What the theorem illustrates is that when you apply volume-preserving transformations to *non-measurable* sets like these you can end up with something at the end with more volume. That is, if you want to not create volume out of nothing, your pieces have to all be *measurable*.

This theorem uses the axiom of choice for its proof. Some mathematicians didn't like it and similar 'paradoxical' results, and wanted to do analysis without the axiom of choice. There's a problem though: the axiom of choice is used all over analysis. In particular, in order to even be able to define the *Lebesgue measure*—the formal notion of volume/area/measure, you need a fragment of AC. A more careful look at the construction reveals that you are usitnf the fragment of AC known as *dependent choice* (DC).

The question then is, can you have DC—enough of AC to do analysis—without pathological objects like nonmeasurable sets slipping in? Robert Solovay answered this affirmatively.

Theorem 19 (Solovay 1970)**.** *Assuming there is an inaccessible cardinal, there is a model of* ZF+DC *in which every set of reals is measurable.*

Solovay's proof uses an inaccessible. Is it necessary? Saharon Shelah proved the answer is yes.

Theorem 20 (Shelah 1984)**.** *If every set of reals is measurable then by thinning out the universe of sets we get an inner model with an inaccessible cardinal.*

Solovay's model thus gives a necessary use of an inaccessible cardinal. If we want to be able to do analysis without non-measurable sets, then there needs to be an inaccessible cardinal lurking around. We can't escape it, like we could with proving Fermat's last conjecture.

In line with Shelah's theorem, a theme that has emerged in contemporary set theory is that there is a zoo of mathematical theorems which need large cardinals for their proofs. Rather than being fanciful creatures of the higher infinite, they are intimately connected to concrete statements of mathematics. Perhaps the most prominent here are *determinacy* principles. These game theoretic principles are about sets of reals. But we know that proving them requires large cardinals. For example, the large cardinal strength of *projective determinacy* is infinitely many *Woodin* cardinals. (The definition of a Woodin cardinal is technical. Let me just mention that it is well above an inaccessible. Intermediate between inaccessibles and Woodins is a host of large cardinals: such as Mahlos, weakly compacts, ineffables, Ramseys, measurables, strongs.)

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