MATH 218M TECHNICAL CONTENT
INTUITIONISTIC LOGIC

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1. Introduction

Intuitionistic logic attempts to capture a "constructive" approach to reasoning. To illustrate the distinction, let's consider the Goldbach conjecture. This conjecture, first stated in the 1700s, is that every even integer > 2 can be written as a sum of two prime numbers. In the nearly three hundred years since no one has been able to either prove or disprove this statement.

Imagine someone told you that they had reduced the Goldbach conjecture to an easier problem. Namely, they told you that they defined a certain integer g so that Goldbach is true iff g is even. So the problem reduces to checking whether g is even. Seems like progress, right? But then they tell you how they defined g: they say $g = 0$ if Goldbach is true and $g = 1$ if Goldbach is false. Seeing the definition you're probably not so interested anymore.

In classical logic, this is a perfectly fine definition. After all, the Goldbach conjecture, like any other statement, is either true or false. So it's a simple definition by cases. Intuitionists on the other hand would reject this definition. They would say that a proper definition would have to come along with an explanation of which case of the definition to use.

In general, intuitionists demand a higher standard of evidence, requiring you to be able to construct witnesses for your claims. This amounts to a restriction on the rules for reasoning with classical logic. Some argumentative moves that the classical logician accepts aren't deemed valid by the intuitionist.

2. The intuitionistic meaning of the connectives: the Brouwer–Heyting–Kolmogorov interpretation

Before discussing what rules of reasoning the intuitionist accepts, let's discuss their interpretation of the meaning of the connectives.

For boolean logic and fuzzy logic we gave the meaning of the connectives by defining them as truth functions. For boolean logic this took the form of truth tables, while for fuzzy logic we defined them using ideas from arithmetic. This won't work for intuitionist logic and we need a different approach.

The Brouwer–Heyting–Kolmogorov (BHK) interpretation is based on the the notion of a *construction*. A construction is a calculation or explicit demonstration to justify an assertion. For example, a construction for "there is a prime number" would consist of an explicit demonstration that shows a specific number p is prime; say, it could be be a calculation showing that the only numbers which divide 7 are 1 and 7 itself. More generally we can say what constitutes a construction for any statement built up using the language of logic.

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Intuitionistic logic uses the same language as boolean logic. Namely, we have two propositional constants 0 and 1, meaning sure falsity and sure truth; propositional variables $P, Q, \ldots$, and connectives $\wedge, \vee, \neg, \Rightarrow$. We also have parentheses to enforce order of operations.

- We will assume as a starting point that we know what constitutes a construction for a propositional variable. The idea is, a variable P stands in for a statement like $2 + 2 = 4$ or $5 - 3 = 3$ that we have a clear idea what a construction could look like.
- There is no construction for 0, and we take 1 to be a construction for itself. The idea is, sure falsity can never be demonstrated and sure truth needs no demonstration.
- A construction for $\varphi \wedge \psi$ is a construction for φ along with a construction for ψ.
- A construction for $\varphi \vee \psi$ is either the choice of φ along with a construction for φ, or else the choice of ψ along with a construction for ψ. The key point is, a construction for $\varphi \vee \psi$ has to tell you which of the two is true.
- A construction of $\varphi \Rightarrow \psi$ is a function that converts a construction for φ into a construction for ψ.
- $\neg\varphi$ is defined to be a synonym for $\varphi \Rightarrow 0$. Thus a construction for $\neg\varphi$ is a function which converts a construction of φ into a construction of 0.

A formula φ is *valid* in intuitionistic logic if there is a construction for φ and it is *invalid* in intuitionistic logic if there is a construction for $\neg\varphi$.

Example 1. Consider the formula $\neg 0$. A construction for $\neg 0$, synonymously $0 \Rightarrow 0$, would be a function which converts a construction of 0 into a construction of 0. Clearly the do-nothing function (the "identity function") works.

Example 2. Consider the formula $\varphi \Rightarrow \neg\neg\varphi$, where φ is any formula. A construction for this formula, synonymous with $\varphi \Rightarrow ((\varphi \Rightarrow 0) \Rightarrow 0)$, would be a function f which given a construction c for φ returns a construction $f(c)$ for $(\varphi \Rightarrow 0) \Rightarrow 0$. Note that the construction $f(c)$ should itself be a function whose input is a function g which converts a construction for φ into a construction for 0 and whose output is a construction for 0.

Having said all that the strategy for what to do should be clear: $f(c)$ has as input the function g and has output $g(c)$. Since c is a construction for φ and g sends a construction for φ to a construction for 0 we have that $g(c)$ is the desired construction for 0.

What these examples show is that $\neg 0$ and $\varphi \Rightarrow \neg\neg\varphi$ are both valid in intuitionistic logic.

Remark 3. There is no construction for 0, so what use is a function which converts a construction for φ into a construction for 0? The point is, such a function witnesses that there cannot be a construction for φ. For if there were, then it could be converted into an object which cannot exist.

The *law of excluded middle* (LEM) is the statement that $\varphi \vee \neg\varphi$ is true for any φ. Under the BHK interpertation, this would mean that for any φ either you have a construction for φ or else you have a function which converts a construction for φ into a construction for 0. For some choices of φ we have this, for example there is a construction for $0 \vee \neg 0$. But in general we don't have a construction for any instance of $\varphi \vee \neg\varphi$; there are φs, such as the Goldbach conjecture, for which we don't know how to either prove or refute.

In general, most interesting mathematical statements are about an infinite domain—integers, real numbers, etc.: "there is no largest prime number" or "every continous function is integrable". When talking about an infinite domain in you can't expect to always have either a construction for φ or a construction for $\neg\varphi$.

On the other hand, some mathematical statements are purely finitary. For example, consider the statement $413 \times 612 = 252,758$, call it φ. If φ is true, we can demonstrate such by the construction that is doing the multiplication of 413 by 612. And if φ is false we have a construction for $\neg\varphi$, namely the construction of doing the multiplication 413×612 and seeing the answer doesn't match.

Nonetheless, we always have a construction for $\neg\neg(\varphi \vee \neg\varphi)$. Showing this directly using the BHK interpretation is unwieldy. For a taste of why this is, think about what a construction for such a formula would even look like!

Instead, we want to build on this interpretation of the meaning of the connectives and produce a *deductive system* which more efficiently tells us what formulas are valid in intuitionism.

3. A deductive system

In general, a *deductive system* is a system of rules that tells you what deductions are valid: knowing blah, when can you conclude blah blah?

We will use the following syntax for deduction rules:

$$\frac{\varphi_1 \qquad \varphi_2 \qquad \cdots}{\psi}$$

means that knowing $\varphi_1, \varphi_2, \ldots$ you can conclude ψ. We call the formula(s) on the top the *antecedent(s)* and the formula on the bottom the *consequent*.

We can think of deduction rules as a different way of explaining the meaning of the logical vocabulary. And we can justify each rule by appealing to the BHK interpretation for the meaning of our logical vocabulary.

3.1. Deduction rules for $\wedge$.

$$\frac{\varphi \qquad \psi}{\varphi \wedge \psi} \qquad (\wedge \text{ introduction})$$

$$\frac{\varphi \wedge \psi}{\varphi} \qquad (\wedge \text{ elimination})$$

$$\frac{\varphi \wedge \psi}{\psi} \qquad (\wedge \text{ elimination})$$

The first rule is called an *introduction rule* because it introduces the connective into the consequent. The second two rules are *elimination rules* because they eliminate the connective for the consequent. Other connectives will also have introduction and elimination rules.

These rules can be justified by thinking about the meaning of $\wedge$. Recall that a construction for $\varphi \wedge \psi$ is a construction for φ along with a construction for ψ. The introduction rule thus says that if you have a construction for φ and you have a constructoin for ψ then you have a construction for $\varphi \wedge \psi$. The elimination rules say that if you have a construction for $\varphi \wedge \psi$ then you have a construction for φ and a construction for ψ.

3.2. Deduction rules for $\Rightarrow$.

For these rules we need to expand this notation. Write

$$\frac{\varphi \vdash \psi}{\theta}$$

to mean the following. The antecedent $\varphi \vdash \psi$ expresses "from the assumption of φ we can conclude ψ". So the whole rule expresses "if from the assumption of φ we can conclude ψ, then we can conclude θ without any assumption".

We can think of this expanded notation as encapsulating the simpler deduction rule notation. Namely, think of

$$\frac{\psi}{\theta}$$

as a shorthand for

$$\frac{A \vdash \psi}{\theta}$$

where A is some fixed background assumptions we don't want to write out every time. It could even be that A consists of zero assumptions, so that it is a shorthand for:

$$\frac{\vdash \psi}{\theta}$$

To illustrate the necessity of this expanded notation, look at the introduction rule for $\Rightarrow$.

$$\frac{\varphi \vdash \psi}{\varphi \Rightarrow \psi} \qquad (\Rightarrow \text{ introduction})$$

In English, this rule expresses "if from the assumption of φ we can conclude ψ, then we can conclude $\varphi \Rightarrow \psi$ without any assumption". This is expressing the meaning of the IF-THEN connective: a construction for $\varphi \Rightarrow \psi$ is a function which takes as input a construction of φ and associates to it a construction of ψ. The antecedent "$\varphi \vdash \psi$" expresses the existence of such a function; it expresses that if we assume we have a construction of φ then we know how to turn it into a construction for ψ.

The elimination rule doesn't need this expanded notation.

$$\frac{\varphi \Rightarrow \psi \qquad \varphi}{\psi} \qquad (\Rightarrow \text{ elimination})$$

This expresses the logical rule known as *modus ponens*. Phrased in the BHK-interpretation, it expresses: if we know how to transform a construction of φ into a construction of ψ and we have a construction of φ then we have a construction of ψ.

3.3. **Deduction rules for $\vee$.**

$$\frac{\varphi}{\varphi \vee \psi} \qquad (\vee \text{ introduction})$$

$$\frac{\psi}{\varphi \vee \psi} \qquad (\vee \text{ introduction})$$

Recall that a construction for $\varphi \vee \psi$ is either a choice of φ and a construction for φ or a choice of ψ and a construction for ψ. The introduction rules say that if you have a construction for either φ or ψ then you get a construction for $\varphi \vee \psi$, simply by saying what the choice is.

The elimination rule is more complicated. The reason is that if someone just tells you that $\varphi \vee \psi$ is true that isn't helpful unless they tell you which of the disjuncts is true. Instead, what the elimination rule expresses is that if you could reach the same conclusion using either disjunct then you don't need to know which of the two is true.

$$\frac{\varphi \vdash \theta \qquad \psi \vdash \theta \qquad \varphi \vee \psi}{\theta} \qquad (\vee \text{ elimination})$$

This one's complicated, so let's break it apart. There are three antecedents. The first expresses that you know how to turn a construction for φ into a construction for θ while the second expresses that you know how to turn a construction for ψ into a construction for θ. The third antecedent expresses that you have a construction for $\varphi \vee \psi$. Altogether this lets you conclude θ. The reason for this is, imagine someone gives you a construction for $\varphi \vee \psi$. If the construction consists of the choice of φ plus a construction for φ, then you use the $\varphi \vdash \theta$ antecedent to produce a construction for θ. Similarly, if the construction consists of the choice of ψ plus a construction for ψ, then you use the $\psi \vdash \theta$ antecedent.

If this looks a bit weird—but shouldn't I always know what the choice is?—let's consider when we apply this to mathematics and our formulas could refer to variables. For example, you could have a construction that works for an even integer n and a different construction that works for an odd integer n. Together these give you a construction that works for every integer, even though you can't guarantee which case you'll be in.

3.4. **Deduction rules for $\neg$.**

$$\frac{\varphi \vdash 0}{\neg\varphi} \qquad (\neg \text{ introduction})$$

Recall that $\neg\varphi$ is synonymous with $\varphi \Rightarrow 0$. What this rule expresses is that if you know how to turn a construction for φ into a construction for 0—that is, if you have a construction for $\varphi \Rightarrow 0$—then you can conclude $\neg\varphi$.

$$\frac{\varphi \qquad \neg\varphi}{0} \qquad (\neg \text{ elimination})$$

The elimination rule expresses the logical rule known as the *law of noncontradiction*, which asserts that it is impossible to have both φ and $\neg\varphi$.

$$\frac{0}{\varphi} \qquad (\text{falsity rule})$$

This rule expresses the *principle of explosion*—from an impossibility you can prove anything. What it expresses is, if you have a construction for 0 then you have a construction for φ. This is vacuously true, because you can never have a construction for 0!

3.5. **Putting it altogether: derivations.** We can combine the rules together to show whether formulas are valid. That is, rather than having to have a complicated construction for a complex formula we can break it up into pieces and understand them individually. These are called *derivations* or *(formal) proofs*. Some authors like to present them in a tree structure, with the consequent of one rule forming an antecedent of another. I personally find that unwieldly so let's be more informal in how we write them.

Write $\vdash \varphi$ to mean there is a derivation of φ. In general, write $A \vdash \varphi$ to mean there is a derivation of φ using A as assumptions. The connection to the syntax for derivation rules is, $A \vdash \varphi$ in an antecedent means that you have a derivation.

Example 4. Let's give a derivation to show that $\vdash \varphi \Rightarrow \neg\neg\varphi$.

To introduce an $\Rightarrow$, we need a derivation showing $\varphi \vdash \neg\neg\varphi$. So let's work on that. That is, we are allowed φ as an assumption and we need to derive $(\varphi \Rightarrow 0) \Rightarrow 0$. This outside IF-THEN will need to use the introduction rule for $\Rightarrow$, by producing a derivation for $(\varphi \Rightarrow 0) \vdash 0$. We can do this using the elimination rule for $\Rightarrow$:

$$\frac{\varphi \qquad \varphi \Rightarrow 0}{0},$$

since we have both φ and $\varphi \Rightarrow 0$ as assumptions. That is, we know $(\varphi \Rightarrow 0) \vdash 0$. We can now apply the introduction rule for $\Rightarrow$:

$$\frac{\varphi \Rightarrow 0 \vdash 0}{(\varphi \Rightarrow 0) \Rightarrow 0}.$$

We have seen that from the assumption of φ we can derive $\neg\neg\varphi$, and thus we use the introduction rule for $\Rightarrow$ one last time

$$\frac{\varphi \vdash (\varphi \Rightarrow 0) \Rightarrow 0}{\varphi \Rightarrow \neg\neg\varphi}$$

and we are done.

I wrote this in a wordy fashion to explain a step at a time. Let's see another example with less talk.

Example 5. Let's give a derivation to show $\vdash \neg\neg(\varphi \vee \neg\varphi)$, synonymously $\vdash ((\varphi \vee (\varphi \Rightarrow 0)) \Rightarrow 0) \Rightarrow 0$.

Lemma: $\varphi,\ (\varphi \vee (\varphi \Rightarrow 0)) \Rightarrow 0 \vdash \varphi \Rightarrow 0$:

$$\frac{(\varphi \vee (\varphi \Rightarrow 0)) \Rightarrow 0 \qquad \dfrac{\varphi}{\varphi \vee (\varphi \Rightarrow 0)}}{0}$$

Note the tree-like structure. In the branch on the right we used the $\vee$ introduction rule. Working our way down we used the elimination rule for $\Rightarrow$.

Using $(\varphi \vee (\varphi \Rightarrow 0)) \Rightarrow 0$ as an assumption for the $\Rightarrow$ introduction rule we immediately get $(\varphi \vee (\varphi \Rightarrow 0)) \Rightarrow 0 \vdash \varphi \Rightarrow 0$:

$$\frac{\varphi \vdash 0}{\varphi \Rightarrow 0}$$

Lemma: $(\varphi \vee (\varphi \Rightarrow 0)) \Rightarrow 0 \vdash 0$:

$$\frac{(\varphi \vee (\varphi \Rightarrow 0)) \Rightarrow 0 \qquad \dfrac{\varphi \Rightarrow 0}{\varphi \vee \varphi \Rightarrow 0}}{0}.$$

This follows a similar pattern as the first lemma. Again we apply the introduction rule for $\Rightarrow$:

$$\frac{(\varphi \vee (\varphi \Rightarrow 0)) \Rightarrow 0 \vdash 0}{((\varphi \vee (\varphi \Rightarrow 0)) \Rightarrow 0) \Rightarrow 0}$$

At this point we have discharged all the assumptions and have shown $\vdash ((\varphi \vee (\varphi \Rightarrow 0)) \Rightarrow 0) \Rightarrow 0$. Done.

Exercises

Exercise 1. Give a construction for $\neg(\varphi \wedge \neg\varphi)$.

Exercise 2. Give a derivation which shows $\vdash \neg(\varphi \wedge \neg\varphi)$.

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