MATH 217M
DEFINITIONS BY RECURSION AND PROOFS BY INDUCTION

Often in mathematics an object is defined *recursively*. Rather than being given an explicit formula, it's given in terms of one or more base cases and rules explaining how later cases are defined in terms of previous ones.

The most famous example is the *Fibonacci sequence*:

$$f_0 = 0, \quad f_1 = 1, \quad f_{n+2} = f_n + f_{n+1}.$$

Whenever an object is defined by recursion, induction is a natural proof technique to use when studying it. The base case(s) of the definition become the base case(s) of the induction, and the recursive rule is used for the inductive step.

(1) Consider the sequence defined by $a_0 = 1$ and $a_{n+1} = 3a_n$. Prove that $a_n = 3^n$.
(2) Consider the sequence defined by $b_0 = 4$ and $a_{n+1} = 3b_n$. Determine a closed form expression for b_n and prove that it holds.
(3) Consider the sequence defined by $c_0 = 4$ and $c_{n+1} = 5c_n$. Determine a closed form expression for c_n and prove that it holds.
(4) Generalize this. Fix numbers x and y and define a sequence as $d_0 = y$ and $d_{n+1} = xd_n$. Determine a closed form expression for d_n and prove that it holds.

When proving things about the Fibonacci sequence there are two base cases of the recursion, so your induction proof needs two base cases.

(1) Prove $f_n < 2^n$ for every n.
(2) Prove that
$$f_0{}^2 + f_1{}^2 + \cdots + f_n{}^2 = f_n \cdot f_{n+1}.$$
(If you get stuck, this is Theorem 25 in the textbook.)
(3) Prove that
$$f_n = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}}{\sqrt{5}}.$$

You can think of multiplication of natural numbers as being defined recursively: fix $x \in \mathbb{N}$ and define $x \cdot n$ as: $x \cdot 0 = 0$ and $x \cdot (n+1) = x \cdot n + x$. Given this recursive definition of multiplication, induction is the way to prove basic properties of multiplication.

(1) Prove by induction that $0 \cdot n = 0$ for any n.
(2) Prove by induction that $1 \cdot n = n$ for any n.
(3) Prove by induction that multiplication is commutative: $x \cdot n = n \cdot x$. *[Hint 1: There are two variables here. You want to fix x and do induction on n; Hint 2: It would be circular to use the commutativity of multiplication in your proof, but you should feel free using other basic arithmetic facts.]*