MATH 210
RULES FOR HYPERREALS

Extending the reals to the hyperreals

(1) **The Extension Principle**
- The reals are a subset of the hyperreals, and the order $x < y$ on the reals is a suborder of the order on the hyperreals;
- There is a nonzero infinitesimal;
- Any function f on the reals has a *natural extension* to a function f^* on the hyperreals, with the same number of variables.

(2) **The Transfer Principle.** Any *real statement* about functions f, g, ... on the reals holds for their natural extensions f^*, g^*, ... on the hyperreals.

Notation: We drop the *s to make it easier to read.

A *real statement* is a combination of equalities, inequalities, or statements about whether a function is defined or undefined.

Examples:
- (Commutativity of addition) $x + y = y + x$;
- (Rules for $<$) if $0 < x < y$ then $0 < 1/y < 1/x$;
- (Domains) the cube root funciton is defined everywhere;
- (No division by 0) $x/0$ is never defined;
- (Algebraic identities) $x^2 - y^2 = (x + y)(x - y)$;
- (Trig identities) $\sin^2 x + \cos^2 x = 1$.

Counterexamples:
- There are no nonzero infinitesimals.
- The domain of sin only consists of real numbers;
- Every input to the exponential function is finite.

The algebra of infinitesimal, finite, and infinite numbers

For these, ε and δ (epsilon and delta) are infinitesimal, b and c are finite but non-infinitesimal, and H and K are infinite. And n is a natural number.

(1) **Real Numbers**
 (a) 0 is the only infinitesimal real number;
 (b) Every real number is finite.
(2) **Negatives**
 (a) $-\varepsilon$ is infinitesimal;
 (b) $-b$ is finite and non-infinitesimal;
 (c) $-H$ is infinite.
(3) **Reciprocals**
 (a) $1/\varepsilon$ is infinite (if $\varepsilon \neq 0$);
 (b) $1/b$ is finite;
 (c) $1/H$ is infinitesimal.
(4) **Sums**
 (a) $\varepsilon + \delta$ is infinitesimal;
 (b) $b + \varepsilon$ is finite and non-infinitesimal;
 (c) $b + c$ is finite (possibly infinitesimal);
 (d) $H + \varepsilon$ and $H + b$ are infinite.
(5) **Products**
 (a) $\varepsilon \cdot \delta$ and $\varepsilon \cdot b$ are infinitesimal;
 (b) $b \cdot c$ is finite and non-infinitesimal;
 (c) $b \cdot H$ and $H \cdot K$ are infinite.
(6) **Quotients**
 (a) ε/b, ε/H, and b/H are infinitesimal;
 (b) b/c is finite and non-infinitesimal;
 (c) b/ε, H/ε, and H/B are infinite.
(7) **Powers**
 (a) ε^n is infinitesimal;
 (b) b^n is finite and non-infinitesimal;
 (c) H^n is infinite.
(8) **Roots**
 (a) $\sqrt[n]{\varepsilon}$ is infinitesimal (if ε is positive);
 (b) $\sqrt[n]{b}$ is finite and non-infinitesimal (if b is positive);
 (c) $\sqrt[n]{H}$ is infinite (if H is positive).
(9) **Indeterminate forms**
 (a) ε/δ, H/K, $\varepsilon \cdot H$ and $H + K$ could all be either infinitesimal, finite but non-infinitesimal, or infinite. It depends on the specific values.