MATH 210: 11-1 WORKSHEET
4.4 INTEGRATION BY CHANGE OF VARIABLES

Integration by change of variables, also called *integration by substitution*, is a powerful technique that amounts to the backward version of the substitution rule.

Before seeing the general form of the technique, let's do some warmup. Compute each of the following indefinite integrals by recognizing that the integrand came from applying the chain rule and determining what you had to differentiate to get it.

(1) $\int 2xe^{x^2}\,\mathrm{d}x$

(2) $\int e^x \sec^2(e^x)\,\mathrm{d}x$

(3) $\int 3\cos(x)\sin^2(x)\,\mathrm{d}x$

(4) $\int \frac{5x^4}{2\sqrt{x^5+1}}\,\mathrm{d}x$

In general, when seeing whether change of variables can be used you're looking for whether the integral has the form

$$\int f(u(x))\,\underbrace{u'(x)\,\mathrm{d}x}_{=\mathrm{d}u} = \int f(u)\,\mathrm{d}u,$$

where the change of variables from the x domain to the u domain is given by

$$u = u(x) \qquad\qquad \mathrm{d}u = u'(x)\,\mathrm{d}x.$$

Once you make the substitution, the problem reduces to finding an antiderivative $F(u)$ for $f(u)$, and then you substitute back to the x domain to get the antiderivative $F(u(x))$.

Calculate the following indefinite integrals by using change of variables. First do the substitution to rewrite the integral in the u domain, then find an antiderivative.

(1) $\int 18x^2(x^3+3)^5 \, \mathrm{d}x$
(2) $\int \sqrt{\ln x + 4} \cdot \frac{1}{x} \, \mathrm{d}x$
(3) $\int xe^{x^2} \, \mathrm{d}x$. [Hint: your choice of u should yield $\mathrm{d}u = 2x \, \mathrm{d}x$. To find the missing 2 multiply the integrand by $\frac{2}{2}$.]
(4) $\int \sin(7x-1) \, \mathrm{d}x$. [Hint: again you are missing a constant.]
(5) $\int \sin x \sqrt[3]{\cos x} \, \mathrm{d}x$
(6) $\int \csc^2 x \cot x \, \mathrm{d}x$

Sometimes you can avoid needing change of variables by doing some algebra.

(1) Calculate $\int 2x(x^2+1)^2 \, \mathrm{d}x$ by first expanding out the multiplication of the terms into a polynomial, then using the power rule.
(2) Calculate $\int 2x(x^2+1)^2 \, \mathrm{d}x$ by using substitution. Which method do you prefer?
(3) Calculate $\int x(x^2-4)^4 \, \mathrm{d}x$. Which method do you want to use?

Change of variables is a powerful method but it doesn't always apply. Sometimes, your choice for u will leave leftover x's so that you can't translate the integral to be entirely in terms of u.

(1) Why can't you use integration by substitution to integrate $\int e^{-x^2} \, \mathrm{d}x$?
(2) Why can't you use integration by substitution to integrate $\int x^2 e^{x^2} \, \mathrm{d}x$?
(3) Can you use change of variables to integrate $\int 2x \tan(x^2) \, \mathrm{d}x$?