MATH 211: 9-8 WORKSHEET

(A) Together we saw how to use the method of cylindrical shells to calculate the volume of the unit sphere. Here's how we did it.

(1) We think of the sphere as being composed of infinitely thin vertical cylinders, each of them centered on the y-axis. We will get the volume of the sphere by adding up the volumes of these infinitely thin cylinders. As usual, we accomplish this by integrating.

(2) Each cylinder has a radius x, ranging from the left bound $\ell = 0$ to the right bound $r = 1$.

(3) Each cylinder has height

$$t(x) - b(x) = 2\sqrt{1-x^2}.$$

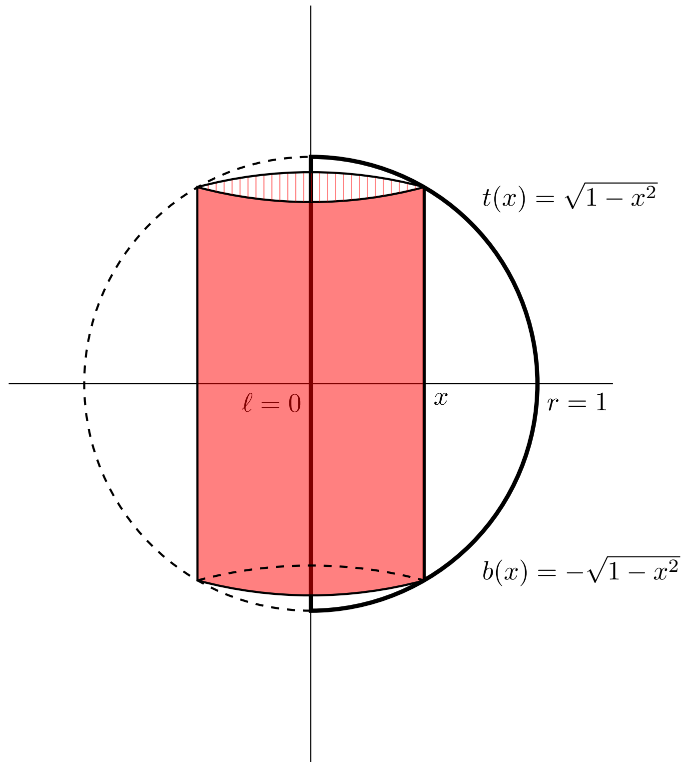
(4) So each cylinder has volume

$$\underbrace{2\pi \cdot x}_{\text{circum.}} \cdot \underbrace{2\sqrt{1-x^2}}_{\text{height}} \underbrace{\mathrm{d}x}_{\textit{thickness}}.$$

(5) Finally, we get the volume of the sphere by calculating the integral

$$\int_0^1 2\pi \cdot x \cdot 2\sqrt{1-x^2}\,\mathrm{d}x.$$

This integral can be calculated by the method of substitution.



radius $= x$

height $= t(x) - b(x)$

In general, suppose you have a region between $x = \ell$ and $x = r$ bounded above by $y = t(x)$ and bounded below by $y = b(x)$, and you rotate it around the y-axis to make a solid of revolution. You can think of this solid as being formed from infinitely thin cylindical shells. Its volume can be calculated as the integral of the volumes of these cylinders:

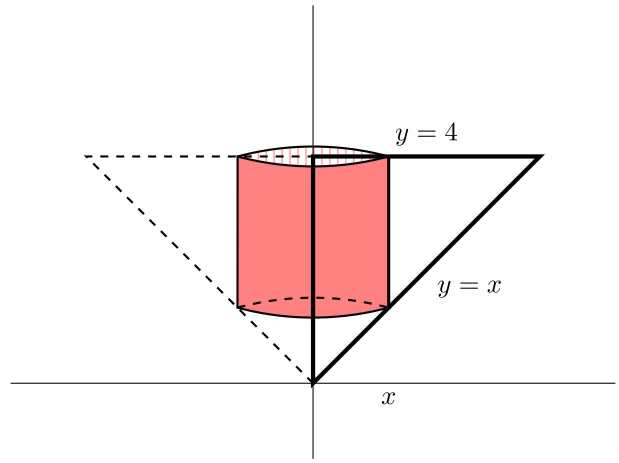
$$\int_\ell^r \underbrace{2\pi \cdot x}_{\text{circum.}} \cdot \underbrace{(t(x) - b(x))}_{\text{height}} \underbrace{\mathrm{d}x}_{\textit{thickness}}.$$

(B) Now try a problem yourself.

The triangle bounded by the lines $x = 0$, $y = 4$, and $y = x$ is rotated around the y-axis to form a cone. Use the method of cylindrical shells to calculate the volume of the cone.

(1) You need to determine the left ℓ and right r bound for x
(2) Also find the top $t(x)$ and bottom $b(x)$ bounds for y.
(3) Put these together to set up the integral which gives the volume. Evaluate this integral to finish finding the volume.

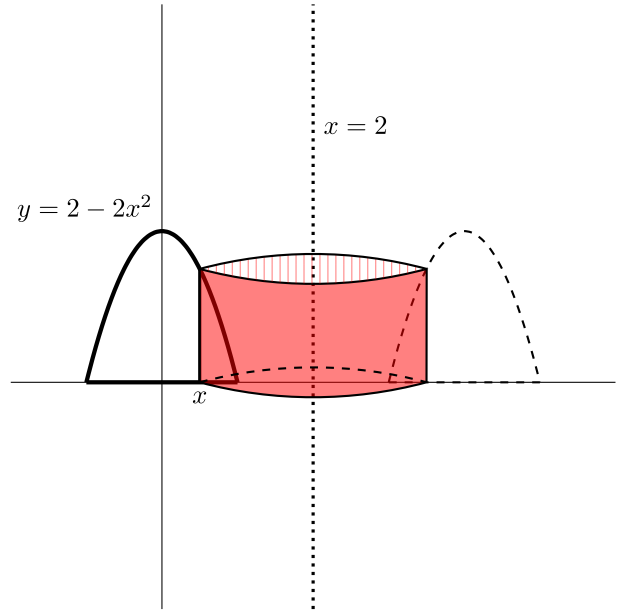
You can check your work using the formula $V = \frac{1}{3}\pi H R^2$ for the volume of a cone, where H is the height of the cone and R is the radius at the base.



(C) You can also rotate around other vertical lines. Do this for a bundt cake shape.

The region bounded above by the parabola $y = 2 - 2x^2$ and below by the x-axis is rotated around the vertical line $x = 2$ to form a solid of revolution.

(1) The process for finding the left/right bounds for x and the top/bottom bounds for y are the same as before. Figure out these bounds and get an expression that describes the height of the cylinder at x.
(2) What's different is what the radius is for each value for x. Determine an expression for the radius at x, and use that to get an expression for the circumference of the cylinder at x.
(3) Combine your work from the previous two parts to set up an integral that gives the volume of this solid. *Do not evaluate the integral.*
(4) What if instead of rotating the region around the line $x = 2$, you rotated it around the line $x = 4$? Set up *but do not evaluate* an integral which gives the volume of this region.
(5) What if instead you rotated the region around the vetical line $x = -3$? Set up *but do not evaluate* an integral which gives the volume of this region.
(6) Can you give a general rule for what to do for any vertical line?



(D) You can use the same method for a solid formed by rotating a region around a horizontal line, such as the x-axis, to get horizontal cylindrical shells. For this, you swap the roles of x and y from the method for vertical cylinders.

The region bounded by the lines $y = 0$, $x = 2$, and the curve $x = \sqrt{y}$ is rotated around the x-axis to get a solid of revolution.

(1) Figure out the bottom b and top t bounds for y. These should both be numbers! What is the radius of the cylinder at y?
(2) Figure out the left $l(y)$ and right $r(y)$ bounds for x to get an expression for the height of the cylinder. This should be an expression that depends upon y!
(3) Combine your work from the previous two parts to get an expression for the volume of the infinitely thin cylinder at y. Set up *but do not evaluate* an integral which gives the volume of the solid. [Hint: you should be integrating with respect to y.]
(4) This volume could have alternatively be calculated using the disk method. Set up *but do not evaluate* an integral using the disk method which gives the volume. Which method do you think is easier for this solid? [Hint: this integral should be with respect to x.]
(5) Draw your own region for a solid of revolution whose volume you can calculate using either the cylindrical shell method or the disk/washer methods. Which method do you think is easier for your solid?