MATH 211: 10-30 WORKSHEET

Tests for convergence or divergence

- If $\lim_{n\to\infty} a_n \neq 0$ or does not exist then $\sum_{n=0}^{\infty} a_n$ diverges.
- (Integral test) Consider a series $\sum_{n=0}^{\infty} a_n$ with positive terms. Suppose there is a continuous decreasing function $f(x)$ with $f(n) = a_n$ for all $n \geq N$. Then the series converges if and only if the integral $\int_N^{\infty} f(x)\,\mathrm{d}x$ converges.
- The p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \leq 1$.
- (Comparison test) Suppose $0 \leq a_n \leq b_n$ for all but finitely many n. If $\sum_{n=0}^{\infty} b_n$ converges then $\sum_{n=0}^{\infty} a_n$ converges.
- (Comparison test) Suppose $a_n \geq b_n \geq 0$ for all but finitely many n. If $\sum_{n=0}^{\infty} b_n$ diverges then $\sum_{n=0}^{\infty} a_n$ diverges.
- (Limit comparison test) Suppose $a_n, b_n \geq 0$ for all but finitely many n.
  - If $\lim_{n\to\infty} \frac{a_n}{b_n}$ exists and is nonzero, then the series $\sum_{n=0}^{\infty} a_n$ converges if and only if $\sum_{n=0}^{\infty} b_n$ converges.
  - If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ and $\sum_{n=0}^{\infty} b_n$ converges then $\sum_{n=0}^{\infty} a_n$ converges.
  - If $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ and $\sum_{n=0}^{\infty} b_n$ diverges then $\sum_{n=0}^{\infty} a_n$ diverges.

(1) Show that $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ converges.
(2) Show that $\sum_{n=1}^{\infty} \frac{1}{n-\sqrt{n}}$ diverges.
(3) Does $\sum_{n=0}^{\infty} \frac{1}{n!}$ converge or diverge? Explain why.
[Reminder: $n! = n(n-1)\cdot(n-2)\cdot\cdots\cdot 2\cdot 1$ is the *factorial* of n.]
(4) As you saw on a previous worksheet, writing a number as an infinite decimal expansion is shorthand for an infinite series. Specifically,

$$0.d_1d_2d_3d_4\ldots = \sum_{n=1}^{\infty} \frac{d_n}{10^n}.$$

Confirm that this series always converges no matter what the sequence of digits $\{d_n\}$ is.
(5) Draw a picture that describes the comparison test. Why does this picture explain why the test is valid?
(6) Check the rule for convergence of p-series by using the integral test.