Math 211 Midterm 2

Name: Answer Key

This is the midterm for unit 2.

Carefully read each question and understand what is being asked before you start to solve the problem. Please show your work in an orderly fashion, and circle or mark in some way your final answers.

No calculators nor other electronic devices are allowed.

1. (15 points) (a) Evaluate $\int x\cos(2x)\,dx$.

10

$u = x \qquad v = \frac{\sin(2x)}{2}$

$du = dx \qquad dv = \cos(2x)\,dx$

$$= \frac{x\sin(2x)}{2} - \int \frac{\sin(2x)}{2}\,dx + C$$

$$= \frac{x\sin(2x)}{2} + \frac{\cos(2x)}{4} + C$$

5

(b) Evaluate $\int_0^\pi x\cos(2x) + 4\,dx$.

$$= \left(\frac{x\sin(2x)}{2} + \frac{\cos(2x)}{4}\right)\Big|_0^\pi + \left(4x\right|_0^\pi$$

$$= \frac{\pi\sin(2\pi)}{2} + \frac{\cos(2\pi)}{4} - 0 - \frac{\cos 0}{4} + (4\pi - 0)$$

$$= 0 + \frac{1}{4} - \frac{1}{4} + 4\pi$$

$$= \underline{4\pi}$$

2. (10 points) Evaluate two of the three integrals on this page. **If you attempt all three, cross out the one you don't want me to grade.**

$$\int \sin x \cos^2 x \, dx$$

$u = \cos x \quad du = -\sin x \, dx$

$= -\int u^2 du$

$= -\frac{\cos^3 x}{3} + C$

$$\int \cot^3 x \csc^3 x \, dx$$

$u = \cancel{\cot x}\ \csc x \quad du = -\cot x \csc x \, dx$

$= \int \cot x \csc x (\csc^2 x - 1) \csc^2 x \, dx$

$= -\int u^2 (u^2 - 1) du$

$= \int u^2 - u^4 du$

$= \frac{\csc^3 x}{3} - \frac{\csc^5 x}{5} + C$

$$\int \sin^2 x \, dx$$

$= \int \frac{1}{2} - \frac{\cos(2x)}{2} dx = \frac{x}{2} - \frac{\sin(2x)}{4} + C$

3. (15 points) (*a*) Give the partial fraction decomposition of

$$\frac{x-2}{x^2-x}. = \frac{A}{x-1} + \frac{B}{x}$$

$x-2 = Ax + Bx - B$

$1 \cancel{x} = A + B \quad A = 1$

$-2 = -B \quad \Rightarrow \quad B = 2$

$$\boxed{= \frac{2}{x} - \frac{1}{x-1}}$$

(*b*) Use your partial fraction decomposition from part (*a*) to evaluate

$$\int \frac{x-2}{x^2-x}\,dx.$$

$$= \int \frac{2}{x} - \frac{1}{x-1}\,dx$$

$$= \underline{2\ln|x| - \ln|x-1| + C}$$

4. (15 points) Evaluate $\int_0^\infty \frac{x}{3} \cdot e^{-x^2}\, dx$.

$$\int \frac{x}{3} \cdot e^{-x^2} du \qquad u = -x^2 \quad du = -2x\,dx$$

$$= -\frac{1}{6}\int e^u du = \underline{-\frac{e^{-x^2}}{6} + C}$$

$$\int_0^\infty \frac{x}{3} \cdot e^{-x^2} dx = \lim_{t\to\infty} \int_0^t \frac{x}{3} \cdot e^{-x^2} dx = \lim_{t\to\infty} \left(-\frac{e^{-t^2}}{6}\right)\Big|_0^t$$

$$= \lim_{t\to\infty}\left(\underbrace{\frac{1}{6}}_{\text{constant}} - \underbrace{\frac{e^{-t^2}}{6}}_{\to 0}\right)$$

$$= \underline{\frac{1}{6}}$$

5. (20 points) Determine whether this series converges or diverges. **Give two different explanations for why.**

$$\sum_{n=0}^{\infty} \frac{n}{e^{n^2}}$$

Explanation 1:

Integral Test: $\int_0^{\infty} \frac{x}{e^{x^2}}\,dx$ is 3 times the integral in #4. It was finite, so this is finite, so the series converges.

Root Test: $\lim_{n\to\infty} \frac{\sqrt[n]{n}}{\sqrt[n]{e^{n^2}}} = \lim_{n\to\infty} \frac{\sqrt[n]{n}}{e^{n}} \text{ "=" } \frac{1}{\infty} = 0$

this limit is $0<1$, so by the root test the series converges

Ratio Test: Explanation 2: $\lim_{n\to\infty} \frac{n+1}{e^{(n+1)^2}} \Big/ \left(\frac{n}{e^{n^2}}\right) = \lim_{n\to\infty} \frac{n+1}{n} \cdot \frac{e^{n^2}}{e^{n^2+2n+1}}$

$= \lim_{n\to\infty} \frac{n+1}{n} \cdot \frac{1}{e^{2n+1}} = 1 \cdot 0 = 0$

This limit is $0<1$, so converges by ratio test.

Limit Comparison: Compare to $\sum_{n=0}^{\infty} \frac{1}{e^n}$. This is a geometric series, which converges because its radix is <1.

Easier: Compare to $\sum \frac{1}{n^2}$

$\lim_{n\to\infty} \frac{n/e^{n^2}}{1/e^n} = \lim_{n\to\infty} \frac{n}{e^{n^2-n}} = 0$. Limit is 0, so series also converges.

Direct Comparison: Compare to $\sum_{n=0}^{\infty} \frac{1}{e^{n}}$. Note that $\frac{n}{e^{n^2}} < \frac{1}{e^{n}} \iff ne^{n} < e^{n^2}$

$\iff n < e^{n^2-n}$.

Easier: Compare to $\sum \frac{1}{n^2}$

The RHS of inequality increases faster than the LHS, so if true for $n=N$ it's true for $n>N$. But $2<e^2 \approx 8\ldots$

So the series is smaller than a convergent geometric series so it also converges.

6. (10 points) Explain why this series converges conditionally. [Hint: there are two things you need to check.]

$$\sum_{n=17}^{\infty} \frac{(-1)^n}{\sqrt{n}}$$

It converges by alternating series test: $\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$.

But it doesn't converge absolutely: $\sum \frac{1}{\sqrt{n}}$ is a p-series with $p = 1/2 < 1$, which diverges.

So the series converges conditionally.

7. (15 points) Determine the Maclaurin series (equivalently, the Taylor series centered at $x = 0$) for the function $c(x) = 3\cos(\sqrt{x})$. Write your answer in sigma notation. Determine the first four terms (namely, the constant through x^3 terms) of the Maclaurin series for the antiderivative of $c(x)$.

$$\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

$$c(x) = \sum_{n=0}^{\infty} 3\cdot(-1)^n \cdot \frac{(\sqrt{x})^{2n}}{(2n)!} = \sum_{n=0}^{\infty} 3\cdot(-1)^n \cdot \frac{x^n}{(2n)!}$$

10

$$= 3 - \frac{3x}{2} + \frac{3x^2}{4!} \mp \cdots$$

So antiderivative $= C + 3x - \frac{3x^2}{4} + \frac{x^3}{4!} \mp \cdots$ 5

8. **Extra credit** (up to +5) Explain why a real number with a decimal expansion which eventually repeats must be a rational number (that is, can be written as a fraction of two integers).

Suppose $x = \underbrace{\overset{\text{Integer}}{\text{Part } K}}, \underbrace{a_1 \cdots a_m}_{\text{digits}} \underbrace{b_1 \cdots b_n, b_1 \cdots b_n, \cdots}_{\text{Repeating digits}}$

So $x = K + 0, a_1 \cdots a_m + 0.\underbrace{0 \cdots 0}_{m \text{ many}} \underbrace{b_1 \cdots b_n}_{\text{repeating}} \rightarrow$

$$= K + \frac{a_1 \cdots a_m}{10^m} + \frac{0.\underbrace{b_1 \cdots b_n}_{\text{repeating}} \rightarrow}{10^m}$$

$$= K + \frac{a_1 \cdots a_m}{10^m} + \frac{1}{10^m}\left(\sum_{k=1}^{\infty} \frac{b_1 \cdots b_n}{(10^n)^k}\right)$$

$$= \underbrace{K}_{\text{rational}} + \underbrace{\frac{a_1 \cdots a_m}{10^m}}_{\text{rational}} + \underbrace{\frac{1}{10^m}}_{\text{rational}} \cdot \left(\underbrace{\frac{b_1 \cdots b_n}{1 - 1/10^n}}_{\text{rational}} - \underbrace{b_1 \cdots b_n}_{\text{rational}}\right)$$

As a sum/product of rational numbers, x is also rational.

(Extra space. Please clearly label which problem the work is for.)

$$\frac{n/e^{n^2}}{1/e^n} = \frac{n}{e^n}$$

$$\frac{n}{e^{n^2}} \text{ vs } \frac{n}{e^{n^3}}$$