MATH 210: 10-9 WORKSHEET (RATES OF CHANGE)

If Q is a quantity that changes over time t, then $\frac{\mathrm{d}Q}{\mathrm{d}t}$ is the *instantaneous rate of change* of Q. The *averange rate of change* of Q over a period of time is $\frac{\Delta Q}{\Delta t}$.

(A) Velocity and acceleration

Velocity is the instaneous change in position and *acceleration* is the instaneous change in velocity. *Speed* is velocity but without any directional information; when working in one dimension it is the absolute value of velocity. When looking at a rotating object, it's often more helpful to look at how the angle is changing. *Angular velocity* is the instaneous change in angle and *angular acceleration* is the instaneous change in angular velocity.

You have acquired a record player, and are spinning the dorm room prog rock classic King Crimson's *In the Court of the Crimson King*. Under the influence of some classic dorm room substances you are staring at the spinning record and thinking deeply about its movement.

(1) The record has a diameter of 12 inches and is spinning at a constant $33+\frac{1}{3}=\frac{100}{3}$ revolutions per minute. Determine the angular velocity of the record in radians per second.

(2) Determine the speed of a point on the edge of the record, in inches per second. [Hint: figure out the relationship between the angle θ rotated and the distance s traveled. If you differentiate both sides you get an equation relating $\frac{\mathrm{d}\theta}{\mathrm{d}t}$ and $\frac{\mathrm{d}s}{\mathrm{d}t}$.]

(3) Determine the speed of a point on the record r inches from the center. What is the speed of a point 3 inches from the center? What is the speed of the point at the very center? Interpret the real-world meaning of these calculations.

(4) Side A ends and you flip the record over to side B. As the record player starts up again, the rotation increases from an initial angular velocity of 0 to an angular velocity of $\frac{10\pi}{9}$ radians per second. You model this increasing angular velocity with the formula

$$\frac{\mathrm{d}\theta}{\mathrm{d}t}=\frac{10\pi}{9}-\frac{10\pi}{9}e^{-t}.$$

Compute the value of $\frac{\mathrm{d}\theta}{\mathrm{d}t}$ at $t=0$ and the limit $\lim_{t\to\infty}\frac{\mathrm{d}\theta}{\mathrm{d}t}$ and explain their meanings.

(5) Determine a formula for the angular acceleration of the record, and compute the limit of the angular acceleration as $t\to\infty$. Explain what this calculation tells you about the spinning record.

(6) After the record gets up to speed, you crouch down so that the record is at eye level, so that when you imagine a little figure standing on the edge of the record it looks to be moving linearly back and forth. Model the edge of the record as the circle $x^2+y^2=36$. What is the radius for the circle? Determine a formula for the x-coordinate at angle θ.

(7) Because the record is spinning at a constant rate of $\frac{10\pi}{9}$ radians per second, the angle at time t is given by $\theta=\theta_0+\frac{10\pi}{9}t$, where θ_0 is the initial angle. Use this to deterimne a formula for the x-coordinate of the imaginary little figure in terms of time t, assuming $\theta_0=0$.

(8) Determine the velocity $\frac{\mathrm{d}x}{\mathrm{d}t}$ of the imaginary figure in the x-direction. When is $\frac{\mathrm{d}x}{\mathrm{d}t}$ equal to 0? When is it positive? When is it negative? Explain the meaning of these in terms of the rotating record.

(9) Determine the acceleration $\frac{\mathrm{d}^2x}{\mathrm{d}t^2}$ of the imaginary figure in the x-direction. When is it equal to 0? When is it positive? When is it negative? Explain the meaning of these in terms of the rotating record.

(B) Population change

You are modeling the bee population on the campus farm. When you first start, you measure that there are approximately 1.3 thousand bees across multiple hives. It is now 30 days later and that population has grown to approximately 1.6 thousand bees.

(1) An *exponential growth model* is one given by an equation $P = P_0 e^{kt}$, where P_0 is the initial quantity and k is a growth constant.
  (a) Determine an exponential growth model for your bee population. Give k, and all further values, rounded to two significant figures.
  (b) Use this model to determine a formula for the instaneous rate of change of the bee population and check that it satisfies the equation

$$\frac{\mathrm{d}P}{\mathrm{d}t} = kP.$$

  What is the real world meaning of $\frac{\mathrm{d}P}{\mathrm{d}t}$?
  (c) Use this model to predict what the bee population will be after 60 days from the start.
  (d) Use this model to predict when the bee population will reach 2 thousand.
  (e) Use this model to predict the bee population after 1 year.
  (f) Use this model to predict the bee population after 10 years.
  (g) The previous two questions should convince you that this is a bad model for long-term forecasting. Explain the limitation of the model and why it gives bad answers long-term.

(2) A better model for long-term population forecasting is the *logistic growth* model. It's given by the equation

$$P = \frac{L}{1 + \frac{L - P_0}{P_0} e^{-kt}},$$

where P_0 is the initial population, the *carrying capacity* L is the largest population that can be supported, and k is a growth constant.
  (a) You estimate that the carrying capacity is 3 thousand. Using this, determine a logistic growth model for your bee population, determining the value k.
  (b) Determine a formula for the instaneous rate of change of the bee population, and check that it satisfies the equation

$$\frac{\mathrm{d}P}{\mathrm{d}t} = kP\left(1 - \frac{P}{L}\right).$$

  (c) Use the model to estimate the bee population after 60 days.
  (d) Use the model to estimate when the population will reach 2 thousand.
  (e) Compute the limit $\lim_{t\to\infty} P$. Explain what this calculation tells you about the population model.
  (f) Calculate the limit $\lim_{t\to\infty} \frac{\mathrm{d}P}{\mathrm{d}t}$. What does this calculation tell you about the population model?
  (g) While it's better for modeling population growth than the exponenital model, the logistic model still has limitations. Give one example of where the model gives bad predictions, and sketch an idea of how you might modify the model to address this limitation.