(A) Velocity & Acceleration

1) $\frac{100}{3}\frac{\text{rev}}{\text{min}} = \frac{100}{3}\frac{\text{rev}}{\text{min}} \cdot \frac{2\pi\ \text{rad/rev}}{60\ \text{sec/min}} = \frac{10\pi}{9}$ rad/sec

This is $\frac{d\theta}{dt}$ angular velocity.

2) 

$s = r\cdot\theta$, by the definition of radians. (This is why radians are a better measure of angles!)

So $\frac{ds}{dt} = r\cdot\frac{d\theta}{dt}$. For a point on the edge, $r=6$ inch

So $\frac{ds}{dt} = 6\cdot\frac{10\pi}{9} = \frac{20\pi}{3}$ inch/sec

3) If $r=3$ $\frac{ds}{dt} = 3\cdot\frac{10\pi}{9} = \frac{10\pi}{3}$ inch/sec. Even though the whole disk has the same angular velocity, linear velocity differs — a point farther from the center has to travel farther to go the same angle as a point closer to the center.

If $r=0$ $\frac{ds}{dt} = 0$ inch/sec.

4) Suppose $\frac{d\theta}{dt} = \frac{10\pi}{9} - \frac{10\pi}{9}e^{-t}$.

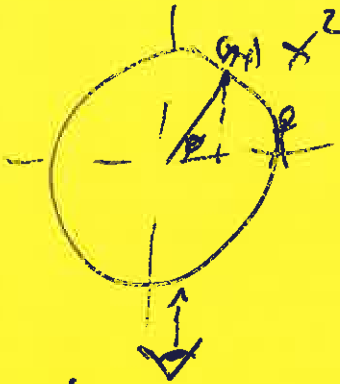
$\lim_{t\to\infty}\frac{d\theta}{dt} = \frac{10\pi}{9} - 0 = \frac{10\pi}{9}$. at $t=0$, $\frac{d\theta}{dt} = \frac{10\pi}{9} - \frac{10\pi}{9} = 0$.

angular velocity starts at 0, increases to the standard 33 1/3 rpm of a 12" LP.

5) Angular acceleration $= \frac{d^2\theta}{dt^2} = \frac{10\pi}{9}e^{-t}$. (Differentiate $\frac{d\theta}{dt}$!)

$\lim_{t\to\infty}\frac{d^2\theta}{dt^2} = 0$. Makes sense — it's approaching a constant speed so accel. should go to 0.

6) (x,y) $x^2+y^2=36$ radius 6



$x = 6\cos\theta$

$y = 6\sin\theta$

7)
8) $\frac{dx}{dt} = -6\sin\theta \cdot \frac{d\theta}{dt}$ (chain rule!)

OR: if $\theta = \frac{10\pi}{9}t$, $x = 6\cos\left(\frac{10\pi}{9}t\right)$ so $\frac{dx}{dt} = -6 \cdot \frac{10\pi}{9}\sin\left(\frac{10\pi}{9}t\right)$

$= -\frac{20\pi}{3}\sin\left(\frac{10\pi}{9}t\right)$

$\frac{dx}{dt} = 0$ at the left & right endpoints, where $\sin\theta = 0$

>0 in Quadrants 3 & 4, where $\sin\theta < 0$

<0 in Quadrants 1 & 2, where $\sin\theta > 0$.

If record rotates counterclockwise, this makes sense. The figure appears to move left when on far side, right on near side, and is momentarily not moving when it reaches either end.

9) $\frac{d^2x}{dt^2} = -6\cos\theta \cdot \left(\frac{d\theta}{dt}\right)^2$ OR: $\frac{d^2x}{dt^2} = -\frac{20\pi}{3} \cdot \frac{10\pi}{9}\cos\left(\frac{10\pi}{9}t\right) = -\frac{200\pi}{27}\cos\left(\frac{10\pi}{9}t\right)$

accel $= 0$ at top & bottom

>0 in Quadrants 1 & 4

<0 in Quadrants 2 & 3

Similar meaning as for velocity.

B) Population Change

$P(0) \approx 1300$ bees $\quad P(30) \approx 1600$ bees

1) Use exponential growth $\quad P = P_0 \cdot e^{kt} \quad P_0 =$ initial population

a) Need to solve for k at $t=30$: $\quad 1600 = 1300\, e^{30k}$

$$30k = \ln\left(\tfrac{16}{13}\right)$$

$$k = \ln\left(\tfrac{16}{13}\right)/30 \approx 0.0069.$$

$$P = P_0 \cdot e^{0.0069t}$$

b) $\frac{dP}{dt} = \underbrace{P_0 \cdot k \cdot e^{kt}}_{= P}$ (Chain rule!) is the instantaneous change in the population.

$= k \cdot P$

This is an approximation. Of course the population is a whole number. But modeling it as a continuous quantity allows you to use calculus. And since you're approximating anyway, that's okay.

c) $P(60) = 1300\, e^{60k} \approx 2000$ bees, rounded up from 1969

d) $P(t) = 2000 = 1300 e^{kt} \Rightarrow kt = \ln\left(\frac{20}{13}\right) \Rightarrow t = \frac{\ln(20/13)}{k} \approx 62$ days

e) $P(365) = 1300 e^{365k} \approx 16{,}000$ bees

f) $P(3650) = 1300\, e^{3650k} \approx 120$ trillion bees

g) Bad long term model because it assumes population will always keep growing at same rate proportional to size. But if the population is too large there's not enough resources so it shouldn't grow so fast.

2) Use logistic growth $P = \dfrac{L}{1+\frac{L-P_0}{P_0}e^{-kt}}$, P_0 = initial population, L = carrying capacity

a) Suppose $L=3000$.

at $t=30$ $P=1600$: $1600 = \dfrac{3000}{1+\frac{1700}{1300}e^{-k30}} \Rightarrow \dfrac{3000}{1600} = 1+\frac{17}{13}e^{-k30}$

$\Rightarrow \frac{30}{16} - \frac{16}{16} = \frac{17}{13}e^{-k30} \Rightarrow \frac{14}{16}\cdot\frac{13}{17} = e^{-k30} \Rightarrow -k30 = \ln\left(\frac{14}{16}\cdot\frac{13}{17}\right)$

$\Rightarrow k = -\dfrac{\ln\left(\frac{14}{16}\cdot\frac{13}{17}\right)}{30} \approx 0.013$. Plug into formula then P & t are only unknowns.

b) $\dfrac{dP}{dt} = \dfrac{L\cdot k\cdot e^{-kt}\cdot\frac{L-P_0}{P_0}}{\left(1+\frac{L-P_0}{P_0}e^{-kt}\right)^2}$ (Quotient Rule!)

$= P$

$= kP\cdot\left(\dfrac{e^{-kt}\cdot\frac{L-P_0}{P_0}}{1+\frac{L-P_0}{P_0}e^{-kt}}\right)$.

Should be $1-\frac{P}{L}$. Let's check.

$1-\frac{P}{L} = 1-\dfrac{L/L}{1+\frac{L-P_0}{P_0}e^{-kt}} = \dfrac{1+\frac{L-P_0}{P_0}e^{-kt}-1}{1+\frac{L-P_0}{P_0}e^{-kt}}$

$= \dfrac{\frac{L-P_0}{P_0}e^{-kt}}{1+\frac{L-P_0}{P_0}e^{-kt}}$ ✓

c) $P(60) \Rightarrow$ Plug in $t=60 \Rightarrow P(60) \approx 1900$ bees

d) $P = 2000 = \dfrac{L}{1+\frac{L-P_0}{P_0}e^{-kt}}$ Solve for t. Similar to how (a) was done.

$t \approx 72$ days

e) $\displaystyle\lim_{t\to\infty} P = \lim_{t\to\infty} \frac{L}{\underbrace{1+\frac{L-P_0}{P_0}\underbrace{e^{-kt}}_{\to 0}}_{\to 1}} = \frac{L}{1} = L.$ As time goes on, population approaches the carrying capacity

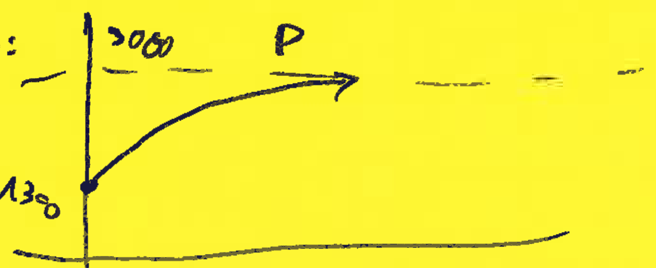
f) $\displaystyle\lim_{t\to\infty} \frac{dP}{dt} = \lim_{t\to\infty} \frac{\overbrace{kL\cdot\frac{L-P_0}{P_0}\overbrace{e^{-kt}}^{\to 0}}^{\to 0}}{\underbrace{\left(1+\frac{L-P_0}{P_0}\underbrace{e^{-kt}}_{\to 0}\right)^2}_{\to 1}} = \frac{0}{1} = 0.$ As population approaches its maximum the change approaches 0.

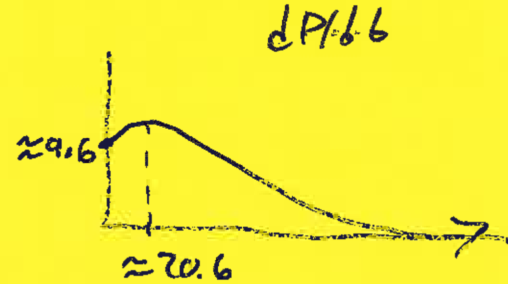
OR:

$$\lim_{t\to\infty} \frac{dP}{dt} = \lim_{t\to\infty} kP\left(1-\frac{P}{L}\right) = kL\left(1-\frac{L}{L}\right) = 0.$$

By (e), $P \to L$

Graphs:





Note: max of $\frac{dP}{dt}$ is when $P = 1500 = L/2$.

g) A couple limitations (there are more!)

i) Population should be cyclical throughout year, not constantly increasing. Include a periodic piece with period 365 days

ii) Model doesn't take into account predation. Larger bee populations means predators — birds, wasps, etc. — have more food available so can themselves support a higher population.