Answers to In-Class Questions: March 23, 2023

Rolle's Theorem Checklist:

☐ f is continuous on $[a, b]$

☐ f is differentiable on (a, b)

☐ $f(a) = f(b)$

Mean Value Theorem Checklist:

☐ f is continuous on $[a, b]$

☐ f is differentiable on (a, b)

Recall that we say f is differentiable at $x = c$ if $f'(c)$ exists as a real number.

1. Determine whether **Rolle's Theorem** applies to the given function on the specified interval $[a, b]$. If Rolle's Theorem applies, find all c in the open interval (a, b) which satisfies the conclusion of the theorem. If Rolle's Theorem does not apply, write "Rolle's Theorem does not apply". Provide justification for your conclusions.

(a) $x^2 - 3x - 16$ on $[-2, 5]$

Answer: We go through our checklist:

☑ f is a polynomial so it is continuous everywhere

☑ f is a polynomial so it is differentiable everywhere

☑ $f(-2) = -6 = f(5)$

Thus, Rolle's Theorem **does** apply. We now find all c in $(-2, 5)$ such that $f'(c) = 0$.

$f'(x) = 2x - 3 = 0$ implies that $x = \frac{3}{2}$. Since $\frac{3}{2}$ is indeed in $(-2, 5)$,

$$\boxed{c = \frac{3}{2}}.$$

(b) $2x^3 - 9x^2$ on $[-2, 2]$

Answer: We go through our checklist:

☑ f is a polynomial so it is continuous everywhere

☑ f is a polynomial so it is differentiable everywhere

☒ $f(-2) = -52 \neq -20 = f(2)$

Thus, Rolle's Theorem **does not** apply.

(c) $3x^2 - 2x - 1$ on $[-1, 1]$

Answer: We go through our checklist:

☑ f is a polynomial so it is continuous everywhere

☑ f is a polynomial so it is differentiable everywhere

☒ $f(-1) = 4 \neq 0 = f(1)$

Thus, Rolle's Theorem **does not** apply.

(d) $\dfrac{x^2}{x^2-16}$ on $[-5,5]$

Answer: We go through our checklist:

☒ f is not continuous on $[-5,5]$ as it has discontinuities at $x=-4,4$

☒ f is not differentiable on $(-5,5)$ as it is not differentiable at $x=-4,4$

☑ $f(-5) = \dfrac{25}{9} = f(5)$

Thus, Rolle's Theorem **does not** apply.

(e) $\dfrac{x^2-2x+1}{x^2-2x-24}$ on $[0,2]$

Answer: Before we go through our checklist, consider that

$$f(x) = \frac{x^2-2x+1}{x^2-2x-24} = \frac{(x-1)^2}{(x-6)(x+4)}.$$

We go through our checklist:

☑ f is continuous on $[0,2]$.

☑ f is differentiable on $(0,2)$.

☑ $f(0) = -\dfrac{1}{24} = f(2)$.

Thus, Rolle's Theorem **does** apply. We now find all c in $(0,2)$ such that $f'(c)=0$.

Write

$$\begin{aligned}
f'(x) &= \frac{((x-6)(x+4))(2(x-1)) - (x-1)^2((x-6)+(x+4))}{((x-6)(x+4))^2} \\
&= \frac{2(x-6)(x+4)(x-1) - (x-1)^2(2x-2)}{(x-6)^2(x+4)^2} \\
&= \frac{2(x-1)\left[(x-6)(x+4) - (x-1)^2\right]}{(x-6)^2(x+4)^2} \\
&= \frac{2(x-1)\left[(x^2-2x-24) - (x^2-2x+1)\right]}{(x-6)^2(x+4)^2} \\
&= \frac{2(x-1)\left[x^2-2x-24-x^2+2x-1\right]}{(x-6)^2(x+4)^2}
\end{aligned}$$

$$= \frac{2(x-1)\,[-25]}{(x-6)^2(x+4)^2}$$
$$= \frac{-50(x-1)}{(x-6)^2(x+4)^2}$$

We see that when $f'(x) = 0$, $-50(x-1) = 0 \Rightarrow x = 1$. Since this value is in $(0, 2)$, we conclude

$$\boxed{c = 1}.$$

(f) $|x - 4|$ on $[0, 2]$

Answer: We go through our checklist:

☑ f is continuous on $[0, 2]$.
☑ f is differentiable on $(0, 2)$.
☒ $f(0) = 4 \neq 2 = f(2)$.

Thus, Rolle's Theorem **does not** apply.

(g) $|x^2 - 4|$ on $[-2, 2]$

Answer: We note that $|x^2 - 4| = |(x-2)(x+2)|$ is not differentiable at $x = -2, 2$. Further, when $-2 \leq x \leq 2$,

$$|x^2 - 4| = -(x^2 - 4) = 4 - x^2$$

because $x^2 - 4 \leq 0$ for this choice of x.

We go through our checklist:

☑ f is continuous on $[-2, 2]$.
☑ f is differentiable on $(-2, 2)$.
☑ $f(-2) = 0 = f(2)$.

Thus, Rolle's Theorem **does** apply. We now find all c in $(-2, 2)$ such that $f'(c) = 0$.

Because $f(x) = |x^2 - 4| = 4 - x^2$ when $-2 < x < 2$, we can take the derivative as if it were a polynomial. So, $f'(x) = -2x = 0$ implies $x = 0$. As $x = 0$ is in the interval $(-2, 2)$, we conclude

$$\boxed{c = 0}.$$

2. Determine whether the **Mean Value Theorem (MVT)** applies to the given function on the specified interval $[a, b]$. If MVT applies, find all c in the open interval (a, b) which satisfies the conclusion of the theorem. If MVT does not apply, write "MVT does not apply". Provide justification for your conclusions.

(a) $x^2 + 3x - 4$ on $[0, 2]$

Answer: We go through our checklist:

☑ f is continuous on $[0, 2]$

☑ f is differentiable on $(0, 2)$

Hence, MVT **applies** and we find c in $(0, 2)$ such that

$$f'(c) = \frac{f(2) - f(0)}{2 - 0}.$$

Write

$$\begin{aligned} \frac{f(2) - f(0)}{2 - 0} &= \frac{((2)^2 + 3(2) - 4) - ((0)^2 + 3(0) - 4)}{2} \\ &= \frac{(4 + 6 - 4) + 4}{2} \\ &= \frac{6 + 4}{2} \\ &= \frac{10}{2} \\ &= 5 \end{aligned}$$

Then, $f'(x) = 2x + 3$ implies

$$\begin{aligned} f'(x) &= 5 \\ \Rightarrow \quad 2x + 3 &= 5 \\ \Rightarrow \quad 2x &= 2 \\ \Rightarrow \quad x &= 1 \end{aligned}$$

Since $x = 1$ is in $(0, 2)$, we conclude

$$\boxed{c = 1}.$$

(b) $x^2 + 8x + 12$ on $[-1, 1]$

Answer: We go through our checklist:

☑ f is continuous on $[-1, 1]$

☑ f is differentiable on $(-1, 1)$

Hence, MVT **applies** and we find c in $(-1, 1)$ such that

$$f'(c) = \frac{f(1) - f(-1)}{1 - (-1)}.$$

Write

$$\begin{aligned}\frac{f(1)-f(-1)}{1-(-1)} &= \frac{((1)^2+8(1)+12)-((-1)^2+8(-1)+12)}{2} \\ &= \frac{(1+8+12)-(1-8+12)}{2} \\ &= \frac{21-5}{2} \\ &= \frac{16}{2} \\ &= 8\end{aligned}$$

Then, $f'(x) = 2x + 8$ implies

$$\begin{aligned} f'(x) &= 8 \\ \Rightarrow \quad 2x+8 &= 8 \\ \Rightarrow \quad 2x &= 0 \\ \Rightarrow \quad x &= 0\end{aligned}$$

Since $x = 0$ is in $(-1, 1)$, we conclude

$$\boxed{c = 0}.$$

(c) $-\dfrac{x^2}{x^2-4}$ on $[-3, 3]$

Answer: We go through our checklist:

☒ f is not continuous on $[-3, 3]$ as it is not continuous at $x = -2, 2$

☒ f is not differentiable on $(-3, 3)$ as it is not differentiable at $x = -2, 2$

Hence, MVT does **not** apply.

(d) $3x^2 + 2x - 2$ on $[-1, 0]$

Answer: We go through our checklist:

☑ f is continuous on $[-1, 0]$

☑ f is differentiable on $(-1, 0)$

Hence, MVT **applies** and we find c in $(-1, 0)$ such that

$$f'(c) = \frac{f(0)-f(-1)}{0-(-1)}.$$

Write

$$\begin{aligned}\frac{f(0)-f(-1)}{0-(-1)} &= \frac{(3(0)^2+2(0)-2)-(3(-1)^2+2(-1)-2)}{1} \\ &= (-2)-(3-2-2) \\ &= -2-(-1) \\ &= -1\end{aligned}$$

Then, $f'(x) = 6x + 2$ implies

$$\begin{aligned} f'(x) &= -1 \\ \Rightarrow \quad 6x + 2 &= -1 \\ \Rightarrow \quad 6x &= -3 \\ \Rightarrow \quad x &= -\frac{1}{2} \end{aligned}$$

Since $x = -\frac{1}{2}$ is in $(-1, 0)$, we conclude

$$\boxed{c = -\frac{1}{2}}.$$

(e) $|x^2 - x - 2|$ on $[-1, 3]$

Answer: We note that

$$|x^2 - x - 2| = |(x - 2)(x + 1)|$$

is not differentiable at $x = -1, 2$.

We go through our checklist:

- ☑ f is continuous on $[-1, 3]$
- ☒ f is not differentiable on $(-1, 3)$

Hence, MVT does **not** apply.

(f) $|x^2 - x - 12|$ on $[-3, 4]$

Answer: We note that

$$|x^2 - x - 12| = |(x - 4)(x + 3)|$$

is not differentiable at $x = -3, 4$. Further, when $-3 \leq x \leq 4$,

$$|x^2 - x - 12| = -(x^2 - x - 12) = 12 + x - x^2$$

because $x^2 - x - 12 \leq 0$ for this choice of x.

We go through our checklist:

- ☑ f is continuous on $[-3, 4]$.
- ☑ f is differentiable on $(-3, 4)$.

Hence, MVT **applies** and we find c in $(-3, 4)$ such that

$$f'(c) = \frac{f(4) - f(-3)}{4 - (-3)}.$$

Write

$$\begin{aligned}\frac{f(4)-f(-3)}{4-(-3)} &= \frac{(12+(4)-(4)^2)-(12+(-3)-(-3)^2)}{7} \\ &= \frac{(12+4-16)-(12-3-9)}{7} \\ &= \frac{(0)-(0)}{7} \\ &= 0\end{aligned}$$

Then, $f'(x) = 1 - 2x$ implies

$$\begin{aligned} f'(x) &= 0 \\ \Rightarrow \quad 1-2x &= 0 \\ \Rightarrow \quad -2x &= -1 \\ \Rightarrow \quad x &= \frac{1}{2}\end{aligned}$$

Since $x = \dfrac{1}{2}$ is in $(-3, 4)$, we conclude

$$\boxed{c = \frac{1}{2}}.$$

(g) $\sqrt{16+x^2}$ on $[0, 3]$

Answer: We go through our checklist:

☑ f is continuous on $[0, 3]$.

☑ f is differentiable on $(0, 3)$.

Hence, MVT applies and we find c in $(0, 3)$ such that

$$f'(c) = \frac{f(3)-f(0)}{3-(0)}.$$

Write

$$\begin{aligned}\frac{f(3)-f(0)}{3-(0)} &= \frac{\sqrt{16+(3)^2}-\sqrt{16+(0)^2}}{3} \\ &= \frac{\sqrt{16+9}-\sqrt{16}}{3} \\ &= \frac{\sqrt{25}-\sqrt{16}}{3} \\ &= \frac{5-4}{3} \\ &= \frac{1}{3}\end{aligned}$$

Then, $f'(x) = \dfrac{x}{\sqrt{16+x^2}}$ implies

$$\begin{aligned} f'(x) &= \frac{1}{3} \\ \Rightarrow \quad \frac{x}{\sqrt{16+x^2}} &= \frac{1}{3} \\ \Rightarrow \quad x &= \frac{1}{3}\sqrt{16+x^2} \\ \Rightarrow \quad x^2 &= \frac{1}{9}(16+x^2) \\ \Rightarrow \quad 9x^2 &= 16+x^2 \\ \Rightarrow \quad 8x^2 &= 16 \\ \Rightarrow \quad x^2 &= 2 \\ \Rightarrow \quad x &= \pm\sqrt{2} \end{aligned}$$

Since $x = \sqrt{2}$ is in $(0,3)$, we conclude

$$\boxed{c = \sqrt{2}}.$$

3. Use the Mean Value Theorem to answer each of the following.

(a) If $f(6) = 2$ and $1 \le f'(x) < \infty$ for all x, what is the smallest value of $f(7)$?

Answer: $1 \le f'(x) < \infty$ for all x implies that f is differentiable everywhere. Because differentiability implies continuity, we also see that f is continuous everywhere. Thus, MVT applies to f on $[6,7]$ because f is continuous on $[6,7]$ and differentiable on $(6,7)$.

MVT implies that there exists c in $(6,7)$ such that

$$f'(c) = \frac{f(7)-f(6)}{7-6} = \frac{f(7)-2}{1} = f(7)-2.$$

Because $1 \le f'(x)$ for all x, we see that $1 \le f'(c)$ as well (since c is a specific x-value). This means that

$$1 \le f'(c) = f(7) - 2$$

which implies

$$\begin{aligned} & 1 \le f(7) - 2 \\ \Rightarrow \quad & 3 \le f(7) \end{aligned}$$

We conclude that the smallest value that $f(7)$ can be is 3.

(b) If $f(2) = -3$ and $-4 \leq f'(x) < \infty$ for all x, what is the smallest value of $f(8)$?

Answer: $-4 \leq f'(x) < \infty$ for all x implies that f is differentiable everywhere. Because differentiability implies continuity, we also see that f is continuous everywhere. Thus, MVT applies to f on $[2, 8]$ because f is continuous on $[2, 8]$ and differentiable on $(2, 8)$.

MVT implies that there exists c in $(2, 8)$ such that

$$f'(c) = \frac{f(8) - f(2)}{8 - 2} = \frac{f(8) - (-3)}{6} = \frac{f(8) + 3}{6}.$$

Because $-4 \leq f'(x)$ for all x, we see that $-4 \leq f'(c)$ as well (since c is a specific x-value). This means that

$$-4 \leq f'(c) = \frac{f(8) + 3}{6}$$

which implies

$$\begin{aligned} & -4 \leq \frac{f(8)+3}{6} \\ \Rightarrow \quad & -24 \leq f(8) + 3 \\ \Rightarrow \quad & -27 \leq f(8) \end{aligned}$$

We conclude that the smallest value that $f(8)$ can be is -27.

(c) If $f(-1) = 0$ and $-\infty < f'(x) \leq 5$ for all x, what is the largest value of $f(0)$?

Answer: $-\infty < f'(x) \leq 5$ for all x implies that f is differentiable everywhere. Because differentiability implies continuity, we also see that f is continuous everywhere. Thus, MVT applies to f on $[-1, 0]$ because f is continuous on $[-1, 0]$ and differentiable on $(-1, 0)$.

MVT implies that there exists c in $(-1, 0)$ such that

$$f'(c) = \frac{f(0) - f(-1)}{0 - (-1)} = \frac{f(0) - 0}{1} = f(0).$$

Because $5 \geq f'(x)$ for all x, we see that $5 \geq f'(c)$ as well (since c is a specific x-value). This means that

$$5 \geq f'(c) = f(0)$$

which implies

$$5 \geq f(0).$$

We conclude that the largest value that $f(0)$ can be is 5.

(d) If $f(3) = 5$ and $-\infty < f'(x) \leq -4$ for all x, what is the largest value of $f(6)$?

Answer: $-\infty < f'(x) \leq -4$ for all x implies that f is differentiable everywhere. Because differentiability implies continuity, we also see that f is continuous everywhere. Thus, MVT applies to f on $[3, 6]$ because f is continuous on $[3, 6]$ and differentiable on $(3, 6)$.

MVT implies that there exists c in $(3, 6)$ such that

$$f'(c) = \frac{f(6) - f(3)}{6 - 3} = \frac{f(6) - 5)}{3} = \frac{f(6) - 5}{3}.$$

Because $-4 \geq f'(x)$ for all x, we see that $-4 \geq f'(c)$ as well (since c is a specific x-value). This means that

$$-4 \geq f'(c) = \frac{f(6) - 5}{3}$$

which implies

$$\begin{aligned} & -4 \geq \frac{f(6) - 5}{3} \\ \Rightarrow \quad & -12 \geq f(6) - 5 \\ \Rightarrow \quad & -7 \geq f(6) \end{aligned}$$

We conclude that the largest value that $f(6)$ can be is -7.

Theorem 1 (Rolle's Theorem)**.** If f is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b), and if $f(a) = f(b)$, then there is at least one point c in (a, b) for which $f'(c) = 0$.

Theorem 2 (Mean Value Theorem (MVT))**.** If f is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b), then there is at least one point c in (a, b) for which

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

A **corollary** is a fact which logically follows from a theorem.

Corollary 1. If $f'(x) = 0$ everywhere on an interval, then $f(x)$ is constant on that interval.

Corollary 2. If $f'(x) = g'(x)$ everywhere on an interval, then $f(x) = g(x) + \text{Constant}$.

Corollary 3. If $f'(x) > 0$ everywhere on an interval, then $f(x)$ is increasing on the interval.

If $f'(x) < 0$ everywhere on an interval, then $f(x)$ is decreasing on the interval.