Math 1420 Exam 3

Monday, Apr 10

Name: Answer Key

This is the second midterm exam.

Carefully read each question and understand what is being asked before you start to solve the problem. **Show your work in an orderly fashion, and circle or mark in some way your final answers.**

No calculators nor other electronic devices are allowed.

Learning Objective	**Grade**
Conceptual Understanding	
Formal Understanding	
Rules for Calculations	
Approximations and Applications	

Conceptual Understanding

1. (30 points) A sprinter runs the 100 meter dash with a time of 10.6 seconds, for an average speed of approximately 9.43 meters per second (about 21.1 miles per hour). Was there a time during the race when her instaneous speed was exactly equal to her average speed? Justify your answer with a short explanation.

Yes. Her position is a continuous function of time, and it is differentiable because its derivative—her speed—is always defined. So the MVT applies.

2. (35 points) Your friend insists that the extreme value theorem implies that the function

$$f(x) = \frac{x^2 - 16}{x - 4}$$

has an absolute maximum on the interval $[-4, 4]$. Are they correct? If yes, explain why. If no, explain why not.

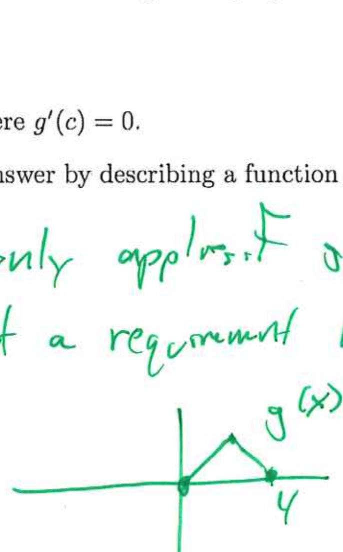
No. $f(x)$ is not continuous on $[-4, 4]$, because it has a discontinuity at $x=4$.

3. (35 points) Is there a function $g(x)$ satisfying the following three properties?

   1. $g(0) = g(4) = 0$;
   2. $g(x)$ is continuous on $[0, 4]$; and
   3. There is *no* point c between 0 and 4 where $g'(c) = 0$.

   If no, explain why not. If yes, support your answer by describing a function with these three properties.

Yes. Rolles' theorem only applies if $g(x)$ is differentiable on $(0,4)$, but that's not a requirement here. And it suggests a solution:



Formal Understanding

4. (30 points) Find the absolute maximum and minimum of $a(x) = \frac{x^3}{3} - 3x^2 + 8x - 10$ on the interval $[0, 3]$.

   On the interval $[0, 3]$:

   - $a(x)$ has an absolute maximum of $-10/3$ at $x =$ 2.
   - $a(x)$ has an absolute minimum of -10 at $x =$ 0.

$$a'(x) = x^2 - 6x + 8$$

$$= (x-4)(x-2)$$

$$a'(x) = 0 \iff x = 2 \text{ or } x = 4$$

x	$a(x)$
0	-10
2	$-\frac{10}{3} \approx -3.3$
3	-4

$$a(0) = -10$$

$$a(2) = \frac{8}{3} - 12 + 16 - 10 = \frac{8}{3} - 6 = -\frac{10}{3}$$

$$a(3) = 9 - 27 + 24 - 10 = -4$$

5. (10 points) Compute the first and second derivatives of $b(x) = x^2 e^x$.

$$b'(x) = x^2 e^x + 2x e^x = \underline{e^x(x^2 + 2x)}$$

$$b''(x) = e^x(x^2 + 2x) + e^x(2x + 2) = \underline{e^x(x^2 + 4x + 2)}$$

6. (30 points) Find the locations of all local maximums, local minimums, and critical points of $b(x) = x^2e^x$.

- $b(x)$ has local maximum(s) at $x =$ $\underline{-2}$.
- $b(x)$ has local minimum(s) at $x =$ $\underline{0}$.
- $b(x)$ has critical point(s) at $x =$ $\underline{-2 \pm \sqrt{2}}$.

$$b'(x) = e^x(x^2+2x)=0$$
$$x^2+2x=0$$
$$x(x+2)=0$$
$$x=0,-2$$

$$b''(x) = e^x(x^2+4x+2)=0$$
$$x^2+4x+2=0$$
$$x = \frac{-4 \pm \sqrt{16-8}}{2} = -2 \pm \frac{\sqrt{8}}{2}$$
$$= -2 \pm \sqrt{2}$$

7. (30 points) State the intervals on which $b(x) = x^2e^x$ is increasing, decreasing, concave up, and concave down. **Write your answers in interval notation.**

- $b(x)$ is increasing on $\underline{(-\infty,-2) \cup (0,\infty)}$.
- $b(x)$ is decreasing on $\underline{(-2,0)}$.
- $b(x)$ is concave up on $\underline{(-\infty,-2-\sqrt{2}) \cup (-2+\sqrt{2},\infty)}$.
- $b(x)$ is concave down on $\underline{(-2-\sqrt{2},-2+\sqrt{2})}$.

$$b'(-3) = e^{-3}(9-6) > 0$$
$$b'(-1) = e^{-1}(1-2) < 0$$
$$b'(1) = e(1+2) > 0$$

$$b''(-5) = e^{-5}(25-20+2) > 0$$
$$b''(-2) = e^{-2}(4-8+2) < 0$$
$$b''(0) = 2e^0 > 0$$

+ 0 − 0 + $b'(x)$

−2 0

+ 0 − 0 + $b''(x)$

$-2-\sqrt{2}$ $-2+\sqrt{2}$

Rules for Calculations

8. (20 points) Calculate the indefinite integral

$$\int 3\cos x - 2\sin x + 5\, dx.$$

[Hint: don't forget the +C!]

$= +3\sin x + 2\cos x + 5x + C$

$7x^{5/2}$

9. (30 points) Find the antiderivative $F(x)$ of

$$f(x) = 2e^x + 7\sqrt{x^5} - 6x^2$$

which satisfies the initial condition $F(0) = 3$.

$F(x) = 2e^x + \frac{7x^{7/2}}{7/2} - \frac{6x^3}{3} + C$

$= 2e^x + 2\sqrt{x^7} - 2x^3 + C$

$3 = 2e^0 + 0 - 0 + C$

$= 2 + C$

$C = 1$

$F(x) = 2e^x + 2\sqrt{x^7} - 2x^3 + 1$

10. (20 points) Use L'Hôpital's rule to calculate the limit

$$\lim_{x\to 0} \frac{3\sin x}{x}. \qquad \frac{0}{0}$$

$$= \lim_{x\to 0} \frac{3\cos x}{1} = \frac{3}{1} = \underline{3}$$

11. (30 points) Use L'Hôpital's rule to calculate the limit

$$\lim_{x\to 1^+} (x-1)\ln(x-1). \qquad 0 \cdot \infty$$

$$= \lim_{x\to 1^+} \frac{\ln(x-1)}{1/(x-1)} = \lim_{x\to 1^+} \frac{1/(x-1)}{-1/(x-1)^2} = \lim_{x\to 1^+} -(x-1) = \underline{0}$$

$$\frac{\infty}{\infty}$$

Approximations and Applications

12. (40 points) A portion of race track for dirt bikes is shaped like the curve $3xy^2 - 6 = 0$, for $1 \leq y \leq 10$. You are at the coordinates $(4, 4)$. Which location on the track is closest to your position? Assume the width of the track is negligible, so that it can be modeled as a curve with no width. [Hint: the distance between two points (x_0, y_0) and (x_1, y_1) is given by the Euclidean formula $\sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2}$.]

    Determine a function in a single variable which gives the value you are trying to optimize, and say whether you want to find a minimum or a maximum. **Do not compute the derivative of this function nor find its critical points. I am only asking you to set it up.**

$d(x,y) = \sqrt{(x-4)^2 + (y-4)^2}$

$3xy^3 6 = 0$

$3xy^2 = 6$

$x = \frac{2}{y^2}$

optimize

$d(y) = \sqrt{\left(\frac{2}{y^2} - 4\right)^2 + (y-4)^2}$

Find a minimum.

13. (60 points) You are building a fenced-in outdoors area for your dog to have a place to run around. You've decided to make the area a rectangle with the side of your house forming one side of the rectangle. If you want the enclosed area to be 5,000 square feet, what is the minimum length of fencing you need to buy? [Hint: Draw a picture to start!]

$F(x,y) = 2x + y$ $\qquad 5000 = xy$

$F(x) = 2x + \frac{5000}{x}$ on $(0,\infty)$. $\qquad y = \frac{5000}{x}$

$F'(x) = 2 - \frac{5000}{x^2}$ $\qquad F''(x) = \frac{10,000}{x^3}$

$F'(x) = 0$

$\downarrow$

$2 - \frac{5000}{x^2} = 0$ $\qquad F''(50) > 0$

$2x^2 = 5000$

$x^2 = 2500$

$\underline{x = \pm 50}$ $\qquad$ concave up, so a min.

$y = \frac{5000}{50} = 100$.

So total fencing is $F(50, 100) = 100 + 100$

$\underline{= 200 \text{ ft}}$