Math 1420 Exam 1

Wednesday, Feb 8

Name: Answer Key

This is the first midterm exam.

Carefully read each question and understand what is being asked before you start to solve the problem. **Show your work in an orderly fashion, and circle or mark in some way your final answers.**

No calculators nor other electronic devices are allowed.

Learning Objective	**Grade**
Conceptual Understanding	
Formal Understanding	
Rules for Calculations	
Approximations and Applications	

Conceptual Understanding

5 each, no partial credit

1. (40 points) Use the graph of the function $f(x)$ below to determine the following limits. Write "DNE" if the limit does not exist. (It is okay to eyeball it and give an approximate answer; just by looking at a graph it can be hard to get a precise value.)

$\lim_{x\to-\infty} f(x) =$ 0

$\lim_{x\to 0} f(x) =$ 1

$\lim_{x\to 2} f(x) =$ DNE

$\lim_{x\to 4} f(x) =$ DNE

$\lim_{x\to 6} f(x) =$ $+\infty$

$\lim_{x\to 8} f(x) =$ 3

$\lim_{x\to 10} f(x) =$ 3 (maybe 4?)

$\lim_{x\to\infty} f(x) =$ $+\infty$

5 each, no partial credit

2. (25 points) Use the graph of $f(x)$ to identify which kind of discontinuity (removable, jump, or infinite) occurs at $f(x)$ for each of the values of x.

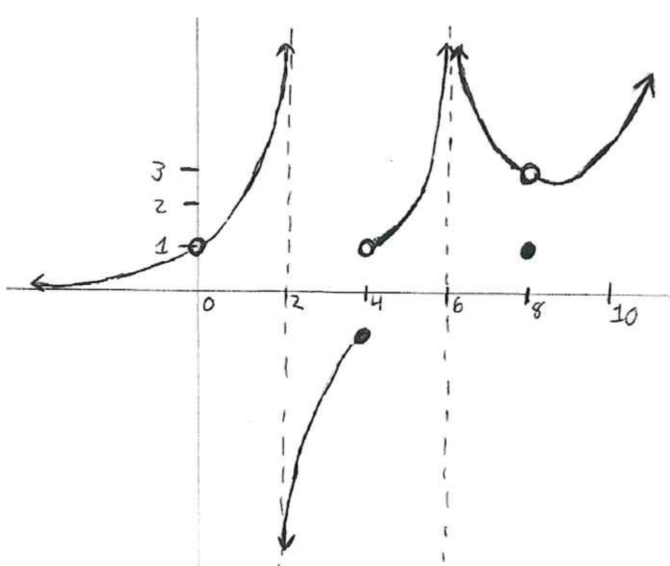
$x = 0$: Removable

$x = 2$: Infinite

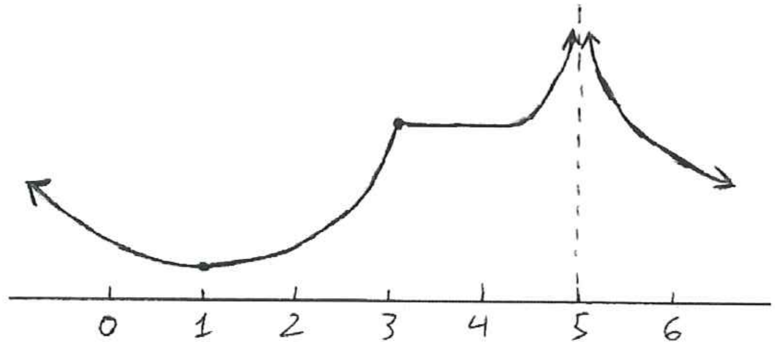
$x = 4$: Jump

$x = 6$: Infinite

$x = 8$: Removable



graph of $f(x)$



graph of $g(x)$

5 each, no partial credit

3. (35 points) Use the graph of the function $g(x)$ to answer whether $g'(x)$ is positive, negative, zero, or undefined at each value for x.

$x = 0$: Negative

$x = 1$: Zero

$x = 2$: Positive

$x = 3$: Undefined

$x = 4$: Zero

$x = 5$: Undefined

$x = 6$: Negative

Formal Understanding

4. (20 points) Consider the piecewise-defined function

$$f(x) = \begin{cases} 2^{x-1} & \text{if } x < 2 \\ 5 - x^2 & \text{if } x \geq 2 \end{cases}$$

Calculate the two one-sided limits

$$\lim_{x \to 2^-} f(x) \qquad \text{and} \qquad \lim_{x \to 2^+} f(x)$$

and use these to determine whether $f(x)$ has a jump discontinuity at $x = 2$. Write a sentence justifying your answer.

$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} 2^{x-1} = 2^{2-1} = 2$

$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} 5 - x^2 = 5 - 2^2 = 1$

There is a jump discontinuity because the two one-sided limits differ.

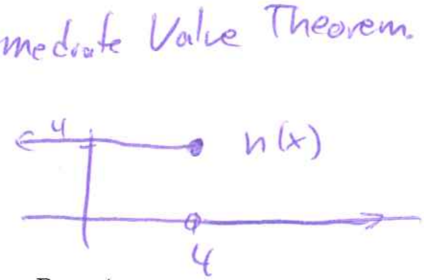
5. (20 points) (*a*) You know that the function $g(x)$ is continuous everywhere, that $g(0) = -2$, and $g(3) = 3$. Can you conclude there is an input x where $g(x) = 0$? Write a sentence justifying your answer.

Yes. $g(0) \leq 0 \leq g(3)$, so by the intermediate value theorem $g(x) = 0$ for some x between 0 and 3.

(*b*) You know that the function $h(x)$ is continuous everywhere except at $x = 4$. You also know that $h(0) = 4$ and $h(10) = 0$. Can you conclude there is an input x where $h(x) = 1$? Write a sentence justifying your answer.

No. Because $h(x)$ is not continuous on $[0, 10]$, we cannot apply the Intermediate Value Theorem.

Here is a counter example:



Either the counterexample or noting why IVT doesn't hold is enough. They don't need both.

6. (60 points) Consider the function $j(x) = x^2 - 2x$.

(*a*) Use the definition of the derivative to calculate $j'(1)$, the derivative of $j(x)$ at $x = 1$. Please show all steps.

$$j'(1) = \lim_{h\to 0} \frac{j(1+h) - j(1)}{h} = \lim_{h\to 0} \frac{(1+h)^2 - 2(1+h) - (1^2 - 2\cdot 1)}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{1} + \cancel{2h} + h^2 - \cancel{2} - \cancel{2h} - \cancel{1} + \cancel{2}}{h} = \lim_{h\to 0} \frac{h^2}{h} = \lim_{h\to 0} h = \boxed{0}$$

also fine to use formula from (b)

(*b*) Use the definition of the derivative to calculate $j'(x)$, the derivative of $j(x)$ for any input x. Please show all steps.

$$j'(x) = \lim_{h\to 0} \frac{j(x+h) - j(x)}{h} = \lim_{h\to 0} \frac{(x+h)^2 - 2(x+h) - (x^2 - 2x)}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{x^2} + 2xh + h^2 - \cancel{2x} - 2h - \cancel{x^2} + \cancel{2x}}{h} = \lim_{h\to 0} \frac{2xh + h^2 - 2h}{h}$$

$$= \lim_{h\to 0} 2x + h - 2 = \boxed{2x - 2}$$

Rules for Calculations

7. (35 points) Consider the function

$$f(x) = \frac{3x(x-1)^4}{x(x+2)}.$$

Calculate each of the following limits. Write "DNE" if the limit does not exist.

$\lim_{x\to 1} f(x) =$ $\underline{0}$

$\frac{0}{3}$ ✓

$\lim_{x\to 0} f(x) =$ $\lim_{x\to 0} \frac{3(x-1)^4}{\cancel{x}(x+2)} = \boxed{\frac{3}{2}}$

$\frac{0}{0}$ indeterminant form.

$\lim_{x\to\infty} f(x) =$ $\lim_{x\to\infty} \frac{3x^5}{x^2} = \lim_{x\to\infty} 3x^3 = \underline{+\infty}$

$\frac{\infty}{\infty}$

$\lim_{x\to-\infty} f(x) =$ $\lim_{x\to\infty} \frac{3x^5}{x^2} = \lim_{x\to-\infty} 3x^3 = \underline{-\infty}$

$\frac{-\infty}{\infty}$

(Problem 7, continued)

$$f(x) = \frac{3x(x-1)^4}{x(x+2)}.$$

$\lim_{x \to -2^-} f(x) =$ DNE

$\frac{K}{0} = \frac{3^5}{0}$, so $\pm\infty$ asymptote

locally looks like $\frac{3^5}{x+2}$

so opposite directions, so limit DNE

Swap answers

$\lim_{x \to -2^+} f(x) =$ $+\infty$ from graph

Alt: check sign: ~~...~~

$f(-1) = \frac{-3 \cdot 2^4}{-1 \cdot 1} > 0$, so positive

$\lim_{x \to -2} f(x) =$ $-\infty$ from graph

Alt: check sign: $f(-3) = \frac{-3^2 \cdot 4^4}{-3(-5)} < 0$, so negative

8. (40 points) (*a*) Calculate the limit

$$\lim_{t\to 0} \frac{\sqrt{t+4}-2}{t} = \frac{0}{0} \text{ indeterminate form}$$

Please show all steps. [Hint: multiply by the conjugate!]

$$\lim_{t\to 0} \frac{\sqrt{t+4}-2}{t} \cdot \frac{\sqrt{t+4}+2}{\sqrt{t+4}+2} = \lim_{t\to 0} \frac{t+4-4}{t(\sqrt{t+4}+2)} = \lim_{t\to 0} \frac{t}{t(\sqrt{t+4}+2)}$$

$$= \lim_{t\to 0} \frac{1}{\sqrt{t+4}+2} = \frac{1}{\sqrt{4}+2} = \boxed{\frac{1}{4}}$$

(*b*) Calculate the limit

$$\lim_{t\to 0} e^{4(\sqrt{t+4}-2)/t} =$$

already calc. limit for this, it's $\frac{1}{4}$

Please show all steps.

$$\lim_{t\to 0} e^{4\left(\frac{\sqrt{t+4}-2}{t}\right)} = e^{\left[4\cdot \lim_{t\to 0} \frac{\sqrt{t+4}-2}{t}\right]} = e^{4\cdot 1/4} = e^{1}$$

$$= \underline{e}$$

9. (25 points) Use the squeeze theorem to calculate

$$\lim_{x \to -\infty} e^x \sin x =$$

Please show all steps, and clearly identify which functions you are using for the upper and lower bounds.

$-1 \leq \sin x \leq 1$

e^x:

$$\underbrace{-e^x}_{\text{lower limit}} \leq e^x \sin x \leq \underbrace{e^x}_{\text{upper limit}}$$

$-e^x$

$$\lim_{x \to -\infty} -e^x = 0$$

$$\lim_{x \to \infty} e^x = 0$$

So by squeeze theorem,

$$\lim_{x \to -\infty} e^x \sin x = 0$$

Approximations and Applications

10. (25 points) The following table gives some of the values of a function $f(x)$. Assuming the pattern in the table holds, use it to estimate the limit

$$\lim_{x \to 0} f(x). = 3$$

x	$f(x)$
-1.000	3.317
-0.100	3.104
-0.010	3.041
-0.001	3.002
0.000	Undefined
0.001	2.995
0.010	2.987
0.100	2.795
1.000	2.661

$\rightarrow 3$

11. (25 points) The following table gives some of the values of a function $g(x)$. Assuming the pattern in the table holds, use it to say whether $g(x)$ is continuous at $x = 1$. Write a sentence to explain your answer.

x	$g(x)$
0.000	2.000
0.900	1.100
0.990	1.010
0.999	1.001
1.000	2.000
1.001	1.001
1.010	1.010
1.100	1.100
2.000	2.000

$g(x)$ is <u>not</u> continuous at $x=1$ because $g(1)=2$ but $\lim_{x\to 1} g(x) = 1$.

12. (50 points) The following table gives some of the values of a function $j(x)$. Assuming the pattern in the table holds, use it to estimate $j'(2)$, the derivative of $j(x)$ at $x = 2$. Use $h = 0.001$ for your approximation.

x	$j(x)$
1.000	1.870
1.900	3.741
1.990	3.975
1.999	3.998
2.000	4.000
2.001	4.002
2.010	4.022
2.100	4.256
3.000	6.312

$$j'(2) \approx \frac{j(2+h)-j(2)}{h} = \frac{4.002-4.000}{0.001}$$

$$= \frac{0.002}{0.001} = \underline{2}$$