MATH 321: HOMEWORK 4 SOLUTION

Problem 5 (Exercise 4.11 from the textbook)**.** Prove that every natural number has a unique base-3 representation.

Solution. A base-3 representation for a number is just a compact way to represent a number n as a sum of the form

$$n = a_1 3^{m_1} + \cdots + a_k 3^{m_k},$$

where each coefficient a_i is either 1 or 2 and $m_1 < \cdots < m_k$. So what we need to see is that every natural number can we written uniquely as such a sum. I will abuse notation and also refer to this sum as a base-3 representation. Let's prove the $n = 0$ case and then separately prove the $n > 0$ case by strong induction. For the $n = 0$ case, observe that the only way to write 0 as a sum of positive integers is as the empty sum, which gives the unique base-3 representation 0.

Now let's do the $n > 0$ case. Assume that each natural number $k < n$ has a unique base-3 representation. Let m be the largest integer so that $3^m \leq n$. I claim that $n = a \cdot 3^m + r$ for unique a and r satisfying that a is either 1 or 2 and $r < 3^m$. That such a and r with $r < 3^m$ are unique is an instance of the Euclidean division lemma, so we just have to see that a is either 1 or 2. First, observe that $a \geq 3$ is impossible, as in that case $3^{m+1} \leq a \cdot 3^m \leq n$, contradicting the leastness of m. And $r < 3^m$ implies that a cannot be 0, as if $a = 0$ that would be saying that $n = r < 3^m \leq n$, which is absurd.

By inductive hypothesis, r can be uniquely written in a base-3 representation. Adding $a \cdot 3^m$ to the base-3 representation for r thus gives a base-3 representation for n. It remains only to see that this representation is unique. It must contain a copy of 3^m, as the sum $\sum_{i=0}^{m-1} 2 \cdot 3^i = 3^m - 1$ is strictly less than n. If $a = 1$, it cannot contain two copies of 3^m, as in this case $2 \cdot 3^m > a \cdot 3^m + r = n$. If $a = 2$, then it must contain two copies of 3^m, as in this case $3^m + \sum_{i=0}^{m-1} 2 \cdot 3^i = 2 \cdot 3^m - 1$ is strictly less than n. So we have seen that in either case the base-3 representation for m must contain $a \cdot 3^m$. With this fact established, the uniqueness for n now follows from the uniqueness for r; there is only one way to write $r = n - a \cdot 3^m$ as a base-3 representation, and that's exactly what we must add to get the representation for n. ☐