SOLUTIONS TO MATH 321 MIDTERM

1. When roots of integers are rational (60 points total)

Problem 1 (5 points)**.** Prove that if n is a perfect k-power then $\sqrt[k]{n}$ is rational.

Solution. Suppose $n = a^k$ for an integer a. Then $\sqrt[k]{n} = a$ is rational. □

Problem 2 (15 points)**.** Prove that if a^k is a multiple of p^r, where a, k, and r are positive integers and p is prime, then a is a multiple of p.

Solution. Suppose a^k is a multiple of p^r. Then, a^k is a multiple of p. By Euclid's lemma a is a multiple of p. □

Problem 3 (15 points)**.** Suppose p is prime and $r < k$ are positive integers. Prove that $\sqrt[k]{p^r}$ is irrational. Use this to conclude that $\sqrt[k]{p^m}$ is irrational if m is not a multiple of k.

Solution. Suppose toward a contradiction that $\sqrt[k]{p^r}$ is rational. Then, we can write $\sqrt[k]{p^r} = a/b$, where a and b have no common factors. Some algebra gives $a^k = p^r b^k$. That is, a^k is a multiple of p^r. By the previous problem, we can conclude that a is a multiple of p. That is, $a = np$ for some integer n. Substituting this in we get

$$n^k p^k = p^r b^k$$
$$n^k p^{k-r} b^k.$$

Because $r < k$ we have that p^{k-r} is a positive power of p. So we can apply the previous problem again to conclude that b is a multiple of p. Thus, a and b have p as a common factor. This contradicts that they have no common factors, completing the argument that $\sqrt[k]{p^r}$ is irrational.

For the general case, use the Euclidean division lemma to write $m = qk + r$, where $0 < r < k$. We know that $0 < r$ because m is not a multiple of k. Then,

$$\sqrt[k]{p^m} = \sqrt[k]{p^{qk} \cdot p^r} = p^q \sqrt[k]{p^r}$$

is a product of a nonzero integer and an irrational number. By problem 1 of homework 2, we conclude it is irrational. □

Problem 4 (20 points)**.** Suppose p and q are primes, and k, m, and n are positive integers so that either m is a not a multiple of k or n is not a multiple of k. Prove that $\sqrt[k]{p^m q^n}$ is irrational.

Solution. Without loss of generality suppose that m is not a multiple of k. If n is a multiple of k, then $\sqrt[k]{p^m q^n} = \sqrt[k]{p^m} \cdot \sqrt[k]{q^n}$ is a product of an irrational number and an integer, hence irrational. Now consider the other case, where n is not a multiple of k. Observe that it suffices to prove the case where $m, n < k$, as otherwise we can use the same trick as from the previous problem to write $\sqrt[k]{p^m q^n}$ as a product of a nonzero integer and a kth root where the exponents inside are both less than k.

Suppose toward a contradiction that $\sqrt[k]{p^m q^n} = a/b$, where a and b have no common factors. Rewrite this equation as $a^k = b^k p^m q^n$. Then a is a multiple of both p and q, by Problem 2. Since p and q are distinct primes, a is a multiple of pq. Write $a = pqc$ for some integer c. Substituting this in gives

$$p^k q^k c^k = b^k p^m q^n$$
$$p^{k-m} q^{k-n} c^k = b^k.$$

Both p^{k-m} and q^{k-n} are positive powers of their prime base, by the assumption that $m, n < k$. Again applying Problem 2 we get that b is a multiple of both p and q. So pq is a common factor of a and b, contradicting that they have no common factors. □

Problem 5 (Extra credit, up to +5)**.** Suppose $p_1, p_2, \ldots, p_\ell$ is a list of distinct primes, and that $k, m_1, m_2, \ldots, m_\ell$ are positive integers so that at least one m_i is not a multiple of k. Prove that

$$\sqrt[k]{p_1^{m_1} p_2^{m_2} \cdots p_\ell^{m_\ell}}$$

is irrational.

Solution. Like with the previous problem, we can reduce this to the case where each $m_i < k$. Suppose toward a contradiction that

$$\sqrt[k]{p_1^{m_1} p_2^{m_2} \cdots p_\ell^{m_\ell}} = \frac{a}{b}$$

for integers a and b with no common factors. Then, we get

$$a^k = b^k p_1^{m_1} p_2^{m_2} \cdots p_\ell^{m_\ell}.$$

By Problem 2, a is a multiple of p_i for each i. Since the p_i are distinct primes, we get that a is a multiple of $p_1 p_2 \cdots p_\ell$, write $a = p_1 p_2 \cdots p_\ell c$. Substituting and rearranging similar to the previous problem gives

$$b^k = c^k p_1^{k-m_1} p_2^{k-m_2} \cdots p_\ell^{k-m_\ell}.$$

Each of the exponents on the right is positive, since $k > m_i$ for each i. So again using Problem 2 we get that b is a multiple of each p_i, contradicting that a and b have no common factors. □

Problem 6 (5 points)**.** Explain how the previous problems together constitute a proof of the theorem.

Theorem. *$\sqrt[k]{n}$ is rational if and only if n is a perfect k-power, where n is a perfect k-power if there is an integer a so that $n = a^k$.*

Solution. Problem 1 establishes the backward direction of the if and only if. For the forward direction, if n is not a perfect k power then, applying the fundamental theorem of arithmetic, $n = p_1^{m_1} \cdots p_2^{m_2} \cdots p_\ell^{m_\ell}$ for primes p_i where at least one exponent m_i is not a multiple of k. Problem 5 then establishes that $\sqrt[k]{n}$ is irrational. □

2. Induction proofs (40 points total)

Problem 7 (20 points)**.** Let f_n denote the n-th Fibonacci number. Prove that, for $n > 0$,

$$f_{n-1} f_{n+1} - {f_n}^2 = (-1)^n.$$

Solution. The base case $n = 1$ is the equation $0 \cdot 1 - 1^2 = -1$. For the inductive step, fix n and assume that $f_{n-1} f_{n+1} - {f_n}^2 = (-1)^n$. Then,

$$\begin{aligned} f_n f_{n+2} - {f_{n+1}}^2 &= f_n(f_n + f_{n+1}) - {f_{n+1}}^2 \\ &= {f_n}^2 - f_{n+1}(f_{n+1} - f_n) \\ &= {f_n}^2 - f_{n+1} f_{n-1} \\ &= -(-1)^n \\ &= (-1)^{n+1}. \end{aligned}$$

□

Problem 8 (20 points)**.** Suppose you have finitely many real numbers $a_1, a_2, \ldots, a_n$. Prove that

$$|a_1 + a_2 + \cdots + a_n| \leq |a_1| + |a_2| + \cdots + |a_n|.$$

Solution. First, we check that $|a + b| \leq |a| + |b|$. We do this by cases. First consider the case where a and b are both positive. Then the inequality becomes $a + b \leq a + b$, which is obviously true. Next consider the case where a and b are both negative. Then, the inequality becomes $-(a + b) \leq -a - b$ which is also obviously true. Finally, consider the case where one is positive and one is negative. Then either $|a + b| < |a|$ or $|a + b| < |b|$, depending on which has the larger magnitude. Thus, $|a + b| < |a| + |b|$, as desired.

With this fact, we are ready to prove this result by induction. The base case $n = 1$ is the trivial inequality $|a_1| \leq |a_1|$. For the inductive step fix n and suppose $|A| \leq |a_1| + \cdots + |a_n|$, where $A = a_1 + \cdots a_n$. Then, by the previous paragraph,

$$|A + a_{n+1}| \leq |A| + |a_{n+1}| \leq |a_1| + \cdots + |a_n| + |a_{n+1}|.$$

□

Problem 9 (20 points)**.** Suppose that x is a real number satisfying that $x + \frac{1}{x}$ is an integer. Prove that

$$x^n + \frac{1}{x^n}$$

is an integer for every natural number n.

Solution. We check the first two cases as base cases. The base case $n = 0$ gives $x^n + \frac{1}{x^n} = 1 + 1 = 2$ is an integer and the base case $n = 1$ is true by assumption. Now fix $n > 0$ and suppose $x^n + \frac{1}{x^n}$ and $x^{n-1} + \frac{1}{x^{n-1}}$ are integers. Then

$$\left(x^n + \frac{1}{x^n}\right)\left(x + \frac{1}{x}\right) = x^{n+1} + x^{n-1} + \frac{1}{x^{n-1}} + \frac{1}{x^{n+1}}$$

is also an integer. Since, by inductive hypothesis, $x^{n-1} + \frac{1}{x^{n-1}}$ is an integer, we conclude that $x^{n+1} + \frac{1}{x^{n+1}}$ is also an integer. □